A Quasi-Metric Topology Compatible with Inclusion Monotonicity on Interval Space

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Abstract.
It is well known that the metric topology [Moo66] on interval space is not compatible with the interval inclusion monotonicity property in the sense that it may exist monotonic functions which are not continuous and conversely. This paper provides a quasi-metric topology for the interval space consistent with the real line topology and whose continuous function are exactly the monotonic ones. The provided quasi-metric is not a metric only because it fails to satisfy the symmetrical property. The quoted title is due to the fact that except to the Hausdorff property of the metric - which does not fit for our point of view - the other good metric properties remain.

Key words: interval space, inclusion monotonicity, quasi-metric topology

1. Introduction

In [Moo66] the set of intervals is endowed with a metric topology such that when restricted to the degenerate intervals the induced relative topology coincides with the usual one on the real line. Also the inclusion order is defined on this set, which is quite useful from the computational point of view. Thinking of intervals as **approximations of real numbers**, the *singleton intervals* being **"exact" approximations** it is possible seeing the reverse of the inclusion as an error order, and the interval as the measure of the exact real number which it contains. Now an approximation would be seen as a **generalized real number** or a **partial real number** and therefore the singleton intervals must be identified with the real numbers. Even more it would be interesting that the topology defined on the interval set took into account this view. For that be possible it is necessary that every continuous function regard to this topology be monotonic according to that order. In this case we say that the topology is **compatible** with the order. As we will see in the next section the metric topology by Moore is not compatible with the inclusion order being therefore unable to suport this new point of view. In [Aci91] was defined on the set of intervals a topology-the Scott topology-where this viewpoint is suported. In that work the order defined on the interval set is the reverse of the inclusion, see also [Aci97]. The defined topology is consistent with the idea that each real number - a degenerate interval- is the supreme (in the order sense) or the limit (in the topological sense) of a chain of closed rational intervals better and better accurated. In order to conciliate the qualitative aspect of the Scott topology on the real intervals with the quantitative aspect of the Moore topology, we propose, in this paper, a quasi-metric for this set. This quasi-metric on the real intervals is interesting mathematically and computationally. Mathematically because it allows us to develop a theory of Cauchy sequence, limits, continuity, etc. Computationally because it provides a mean of endowing the interval space with the Scott topology whose computational characteristic is well known.

2. Moore Topology is not compatible

In this section we will see that the induced metric topology on the interval space is not compatible with the inclusion order as found in [Moo66]. For that in what follows we will present some elementary fact about topology.

A *topology* on a set X is a collection of subsets of X which is closed under finite intersections and arbitrary unions, including the empty set and the set X itself. A set X together with a topology $\mathcal{T}$ on X is a topological space denoted by $\langle X, \mathcal{T} \rangle$. The elements of $\mathcal{T}$ are the open sets of the space [Dun66].

A topology $\mathcal{T}$ on a set X is sometimes specified in terms of some convenient subset of $\mathcal{T}$. Namely a *basis* of $\mathcal{T}$ is a subset B of $\mathcal{T}$ s.t. every open set is the union of elements of B.

Let $\mathbb{I}(\mathbb{R})$ be the set of the reals intervals[1] $X = [\underline{x}, \overline{x}]$, where $\underline{x}, \overline{x} \in \mathbb{R}$, denote the inferior and superior extremes, respectively. We identify the degenerate intervals $[a, a]$ with the reals $a \in \mathbb{R}$. Define on $\mathbb{I}(\mathbb{R})$ the function $d(X, Y) = max\{|\underline{x} - \underline{y}|, |\overline{x} - \overline{y}|\}$. It is easy to see that $d(X, Y)$ is a *distance* or *metric* on $\mathbb{I}(\mathbb{R})$, i.e. $\forall X, Y, Z \in \mathbb{I}(\mathbb{R})$, d satisfies the following properties of a metric:

1. $d(X, Y) = 0 \Leftrightarrow X = Y$
2. $d(X, Y) = d(Y, X)$
3. $d(X, Z) \leq d(X, Y) + d(Y, Z)$

[1] A real or rational interval is an interval with real or rational end points, respectively.

$\mathbb{I}(\mathbb{R})$ endowed with a metric d, denoted $\langle \mathbb{I}(\mathbb{R}), d\rangle$, is called a *metric space.*

A metric d on a set D induces a topology on D whose basic open sets are the open balls, i.e. $B_r(x) = \{y \in D : d(y,x) < r\}$. Therefore, $\mathbb{I}(\mathbb{R})$ becomes a topological space, whose topology is induced by the metric above defined.

If X, Y are topological spaces, a map $f : X \longrightarrow Y$ is *continuous* if the inverse image, $f^{-1}(\mathcal{O})$, of each open set in Y is open in X.

A *poset* is a set X endowed with a partial order $\sqsubseteq$, i.e. a relation on X which is reflexive, anti-symmetric and transitive. $\mathbb{I}(\mathbb{R})$ with the inclusion order, $\sqsubseteq$, between intervals is a poset.

The arithmetic operations on intervals are defined as follows:

$[a,b] + [c,d] = [a+c, b+d]$
$[a,b] - [c,d] = [a-d, b-c]$
$[a,b] \cdot [c,d] = [min\{ac, bc, ad, bd\}, max\{ac, bc, ad, bd\}]$
$[a,b]/[c,d] = [a,b] \cdot [\frac{1}{d}, \frac{1}{c}]$ if $0 \notin [c,d]$

This arithmetic satisfies the inclusion monotonicity property, see [Moo66], i.e. if $I, J, K, L \in \mathbb{I}(\mathbb{R})$, $I \subseteq K$ and $J \subseteq L$, then $I + J \subseteq K + L$, $I - J \subseteq K - L$, $I \cdot J \subseteq K \cdot L$ and $I/J \subseteq K/L$ (if $0 \notin J, L$).

The arithmetic operations are simultaneously continuous and monotonic functions. They are also natural extentions of the real arithmetic operations, see [Moo79]. This suggests us to isolate the interval functions which are simultaneously monotonic and natural interval extensions of continuous real fuctions. In other words we are asking for a topology compatible with the inclusion monotonicity property.

Regard to this definition of compatibility the metric topology defined on the interval set is not compatible with the inclusion order defined on it. In fact, the interval function

$$F(X) = m(X) + \frac{1}{2}(X - m(X)),$$

where m(X) is the midpoint function as defined in [Moo79] pg. 10, is continuous but not monotonic, see [Moo79] pg. 21 for more details. Observe that the function $m(X)$ is not a natural extension of a real continuous function. In what follows we will see that this phenomenon was possible because the topology is metric.

3. No metric topology is compatible

A topological spaces X is said to be T_0 if, for any two distinct points $x, y \in X$, there is an open set which contains one, but not both points i.e. if $\forall x, y \in X$ $(\forall \mathcal{O} \in \mathcal{T}$ $x \in \mathcal{O}$ iff $y \in \mathcal{O}) \Rightarrow$ $x = y$. A space is T_1 if, for any pair of distinct points x, y there is an open set which contains x but not y. Finally, A space is T_2 (*Hausdorff*) if, whenever x, y are distinct points, there are two disjoint open sets containing x, y respectively. Fact: $T_2 \Rightarrow T_1 \Rightarrow T_0$ [Dun66]. For any topological space $\langle X, \mathcal{T} \rangle$, we have the *specialization order*, $\sqsubseteq_{\mathcal{T}}$, induced by $\mathcal{T}$ on X defined by: $x, y \in X$, $x \sqsubseteq_{\mathcal{T}} y$ iff $\forall \mathcal{O} \in \mathcal{T}.\ x \in \mathcal{O} \Rightarrow y \in \mathcal{O}$. It easy to see that the space is T_0 iff the specialization relation of X is a partial order and X is T_1 iff it is the identity. For example, the reflexive and the transitive properties of $\sqsubseteq_{\mathcal{T}}$ are obvious and the anti-symetrical property is exactly the T_0 condition of X. On the other hand, X is T_1 if $\forall x, y \in X$, either $x = y$ or x, y are uncomparable. Notice that any topology induced from a metric is Hausdorff, hence T_1. Therefore, the specialization order on a Hausdorff space is the identity.

Now let $\langle X, \sqsubseteq \rangle$ be a poset. $\mathcal{O} \subseteq X$ is said to be *Alexandroff* open if whenever $x \in \mathcal{O}$ and $x \sqsubseteq y$ then $y \in \mathcal{O}$. Fact: Let $\langle X_1, \sqsubseteq_1 \rangle$ and $\langle X_2, \sqsubseteq_2 \rangle$ be posets and $f : X_1 \longrightarrow X_2$ a function. f is monotonic, i.e. $\forall x, y \in X_1$, $x \sqsubseteq_1 y \Rightarrow f(x) \sqsubseteq_2 f(y)$, iff it is Alexandroff-continuous. Indeed if $\langle X, \sqsubseteq \rangle$ is a poset and $\mathcal{T}$ is the Alexandroff topology over X then the set $U_x = \{z \in X /\ z \not\sqsubseteq x\}$ is Alexandroff open. Now suppose that f is continuous, $x \sqsubseteq_1 y$ and $f(x) \not\sqsubseteq_2 f(y)$. Then, because $U_{f(y)}$ is open, $f(x) \in U_{f(y)}$. Hence $x \in f^{-1}(U_{f(y)})$ is open in X_1. Therefore, $y \in f^{-1}(U_{f(y)})$, i.e. $f(y) \in U_{f(y)}$, contradiction. Thus, f is monotonic. Conversely, suppose that f is monotonic. Let $\mathcal{O}$ be open in X_2. We will show that $f^{-1}(\mathcal{O})$ is open in X_1. In fact, if $x \in f^{-1}(\mathcal{O})$ and $x \sqsubseteq_1 y$ then because f is monotonic and $\mathcal{O}$ is open in X_2, $f(y) \in \mathcal{O}$ hence $y \in f^{-1}(\mathcal{O})$. Therefore, $f^{-1}(\mathcal{O})$ is open in X_1.

According to the above argument a topology compatible with a order have to be Alexandroff. But the Alexandroff topology on a poset whose order is not the identity is a T_0 not T_1 topology. Therefore the Hausdorff topology is not compatible with a nonidentity order. In other words no metric topology is compatible with a order.

4. A quasi-metric topology is compatible

In this section we will see the possibility of defining on the interval set a function which although it is not a metric preserves its interesting properties. On other hand its induced topology - the Scott topology - is compatible with the inclusion monotonicity property.

Definition 4.1. [Smy92] Let X be a non empty set and $d : X \times X \longrightarrow \mathbb{R}^+$ a function. The pair $\langle X, d \rangle$ is said to be a **quasi-metric space** if for each x, y and $z \in X$, d satisfies the following axioms.

i) $d(x,x) = 0$
ii) $d(x,z) \leq d(x,y) + d(y,z)$
iii) $d(x,y) = d(y,x) = 0 \Rightarrow x = y$

In this case d is called a **quasi-metric**.

The dropping of the symmetry axiom is not as bizarre as it may seem. Think of the distance from x to y as a measure of the effort involved in going from x to y. Then think of hilly terrain... [Smy92]. Another intuition is to see a quasi-metric as a measure of the distance between informations. Then a distance of an information x to y is zero if x is more complete than y, but the distance from y to x is greater than zero.

From this definition clearly every metric space is also a quasi-metric one. Let $\langle X, d\rangle$ be a quasi-metric space. Like metric space the standard topology induced on X by a quasi-metric d is defined by taking a set $\mathcal{O} \subseteq X$ as being open if, and only if for any $x \in X$, $x \in \mathcal{O}$ implies that $B_\epsilon(x) \subseteq \mathcal{O}$ for some $\epsilon > 0$, where $B_\epsilon(x) = \{y \in X : \ d(y,x) < \epsilon\}$. In that case, the open ϵ-balls $B_\epsilon(x)$ constitute evidently a base of open sets for a topology on X.

Continuous and uniformely continuous functions for quasi-metric spaces are defined analogously to metric spaces.

Observe too that a metric space $\langle X, d^*\rangle$ is obtained from a quasi-metric space $\langle X, d\rangle$ by taking $d^*(x,y) = max\{d(x,y), d(y,x)\}$ (the associated metric).

Proposition 4.2. Define $q : \mathbb{I}(\mathbb{R}) \times \mathbb{I}(\mathbb{R}) \longrightarrow \mathbb{R}^+$ by

$$q([r,s],[t,u]) = \begin{cases} 0 & \text{if } [t,u] \sqsubseteq [r,s] \\ max\{t-r, s-u\} & \text{otherwise} \end{cases}$$

Then q is a quasi-metric on $\mathbb{I}(\mathbb{R})$.

PROOF: The axioms $i)$ and $iii)$ of quasi-metric are trivial, the only one which deserves consideration is the axiom $ii)$.

If $[t,u] \sqsubseteq [r,s]$ and $[v,w] \not\sqsubseteq [t,u]$ then $q([r,s],[t,u]) + q([t,u],[v,w]) = 0 + q([t,u],[v,w]) = max\{v-t, u-w\}$. Since $t \leq r$ and $s \leq u$ then $v - r \leq v - t$ and $s - w \leq u - w$. So, $q([r,s],[v,w]) \leq q([r,s],[t,u]) + q([t,u],[v,w])$. If $[t,u] \sqsubseteq [r,s]$ and $[v,w] \sqsubseteq [t,u]$ then $[v,w] \sqsubseteq [r,s]$ and therefore $q([r,s],[v,w]) = 0$. So, $q([r,s],[v,w]) \leq q([r,s],[t,u]) + q([t,u],[v,w])$.

The case of $[v,w] \sqsubseteq [t,u]$ is similar to previous one.

If $[t,u] \not\sqsubseteq [r,s]$ and $[v,w] \not\sqsubseteq [t,u]$ then $q([r,s],[t,u]) + q([t,u],[v,w]) = max\{t-r, s-u\} + max\{v-t, u-w\}$ and $q([r,s],[v,w]) = max\{v -$

$r, s-w\}$. Suppose that $s-w \leq v-r$, i.e. $q([r,s],[v,w]) = v-r$. We have four possibilities which, by symmetry, can be reduced to the following two cases.

1. If $max\{t-r, s-u\} = t-r$ and $max\{v-t, u-w\} = v-t$ then, by the triangular inequality of the Euclidian distance, $v - r \leq (t-r) + (v-t)$. Therefore, $max\{v-r, s-w\} \leq max\{t-r, s-u\} + max\{v-t, u-w\}$.

2. If $max\{t-r, s-u\} = t-r$ and $max\{v-t, u-w\} = u-w$ then $(t-r)+(v-t) \leq (v-t)+(u-w)$. By the triangular inequality of the Euclidian distance, $v-r \leq (t-r)+(v-t)$. So, by transitivity, $v-r \leq (t-r)+(u-w)$. Therefore, $max\{v-r, s-w\} \leq max\{t-r, s-u\} + max\{v-t, u-w\}$. ■

Corollary 4.3. The Moore metric can be derived from the quasi-metric q.

PROOF: In fact, the associate metric q^* to the quasi-metric q is defined by

$$\begin{aligned} q^*([r,s],[t,u]) &= max\{q([r,s],[t,u]), q([t,u],[r,s]), 0\} \\ &= max\{t-r, s-u, r-t, u-s, 0\} \\ &= max\{|r-t|, |u-s|, 0\} \\ &= max\{|r-t|, |u-s|\}. \blacksquare \end{aligned}$$

A poset $\langle X, \sqsubseteq \rangle$ is a dcpo (directed complete partial order) if every directed set $S \subseteq X$ (S is directed if given $x, y \in S$ there is $z \in S$ s.t. $x \sqsubseteq z$ and $y \sqsubseteq z$) has supreme in X.

Now if $\langle X, \sqsubseteq \rangle$ is a dcpo $\mathcal{O} \subseteq X$ is said to be *Scott open* if it is Alexandroff open and besides whenever $S \subseteq X$ is directed and $Sup\ S \in \mathcal{O}$ then there exists $s \in S$ such that $s \in \mathcal{O}$. So every Scott continuous function is also monotonic.

While for posets the adequate topology is the Alexandroff one, for dcpo's, because of the sup condition, the Scott topology is the most adequate. Now in dcpo's D_1, D_2 with the Scott topology a function $f : D_1 \longrightarrow D_2$ is topologically continuous if and only if it is order continuous, i.e. preserves supremes of directed set [SLG94]. Moreover if D_1 is an ω-continuous dcpo, i.e. D_1 has a denumerable basis B_1, then every monotonic function $f : B_1 \longrightarrow D_2$ has a unique continuous extension [Aci97]. In this sense the continuous functions $f : D_1 \longrightarrow D_2$ are essentially those ones monotonic on the basis.

Define on $\mathbb{I}(\mathbb{R})$ the following relations:

$$[a,b] \sqsubseteq [c,d] \text{ iff } a \leq c \leq d \leq b \text{ and } [a,b] \ll [c,d] \text{ iff } a < c \leq d < b.$$

The first is an order relation on $\mathbb{I}(\mathbb{R})$ while the other is a transitive relation satisfying some conditions (For example, $a \ll b \Rightarrow a \sqsubseteq b$ and $a \ll c \Rightarrow \exists b$ s.t. $a \ll b \ll c$) [AbJ94]. We will use $\Uparrow[r,s]$ to denote the set $\{[t,u] \in \mathbb{I}(\mathbb{R}) :\ [r,s] \ll [t,u]\}$.

$\mathcal{R} = \langle \mathbb{I}(\mathbb{R}), \sqsubseteq \rangle$ is a ω-continuous dcpo, where $\mathbb{I}(\mathbb{Q})$ - the set of rational intervals - is a denumerable *basis*, that is, every chain in $\mathbb{I}(\mathbb{R})$ converges and conversely every interval is the supreme or limit of a chain of rational intervals. Because in $\mathbb{I}(\mathbb{R})$ all bounded set has supreme and infimum, if we add to $\mathbb{I}(\mathbb{R})$ the bottom and the top, $\mathcal{R}$ becomes a ω-continuous lattice. On the other hand, the category of ω-continuous lattices and continuous functions is a model of the typed and untyped lambda calculus [AbJ94]. Therefore the interval space with the Scott topology, $\langle \mathbb{I}(\mathbb{R}), \Omega_S(\mathcal{R})\rangle$, is a nice environment to do computation in the functional language sense. A denumerable basis for this topology is the set $\{\Uparrow[p,q] :\ [p,q] \in \mathbb{I}(\mathbb{Q})\}$.

Any topological space endowed with a Scott topology gives rise to an information theory whose non-total elements are seen as partial ones. The order compatible with this topology is an information order which allows us to do qualitative comparison between the elements of the space. Therefore in these spaces the mathematics developed are essentially a qualitative one differently from that derived from a Hausdorff topology whose mathematics is mainly quantitative. This does not mean that we cannot develop quantitative mathematics in dcpo's.

A topological space $\langle X, \mathcal{T}\rangle$ is said to be **quasi-metrizable** (**metrizable**) if a quasi-metric (metric) can be defined on X such that $\mathcal{T}$ is the topology induced by it [Smy92].

Proposition 4.4. [Bed96] The topological space induced by the quasi-metric q is $\langle \mathbb{I}(\mathbb{R}), \Omega_S(\mathcal{R})\rangle$, where $\Omega_S(\mathcal{R})$ is the Scott topology on $\mathbb{I}(\mathbb{R})$.

PROOF: It is suficient to show that the base of the standard topology induced by q (the open ϵ-balls $B_\epsilon(x)$) is a basis (or equivalently contain a basis) of $\Omega_S(\mathcal{R})$.

Since we know that $\{\Uparrow[p,q] :\ [p,q] \in \mathbb{I}(\mathbb{Q})\}$ is a basis of $\Omega_S(\mathcal{R})$, we will show that $\Uparrow[p,q]$ is open ϵ-ball. Let $B_{\frac{q-p}{3}}([p+\frac{q-p}{3}, p+\frac{q-p}{3}])$. This open ϵ-ball contain each real interval $[r,s]$ such that $q([r,s],[p+\frac{q-p}{3}, p+\frac{q-p}{3}]) < \frac{q-p}{3}$, i.e. $p+\frac{q-p}{3} - r < \frac{q-p}{3}$ and $s - p + \frac{q-p}{3} < \frac{q-p}{3}$. So, $p < r$ and $s < q$. Therefore, $[p,q] \ll [r,s]$, i.e. $[r,s] \in \Uparrow[p,q]$. ■

Corollary 4.5. [Bed96] The topological space $\langle \mathbb{I}(\mathbb{R}), \Omega_S(\mathcal{R})\rangle$ is quasi-metrizable. ■

5. The Theory of $\mathbb{I}(\mathbb{R})$ as Quasi-Metric Spaces

In this section we just start for lack of space the interval quasi-metric theory similar to the one developed for metric space.

Let $\langle M, d\rangle$ be a quasi-metric space, $S \subseteq M$ and $x \in M$. x is an **adherent point of S** if to each $\epsilon > 0$, $B_\epsilon(x)$ containts points of S (i.e. $B_\epsilon(x) \cap S \neq \emptyset$). The set of all adherent points of S is called the closure of S and it is denoted by $\overline{S}$. A set $S \subseteq M$ is **dense** in $\langle M, d\rangle$ if $\overline{S} = M$.

Proposition 5.1. The set $\mathbb{I}(\mathbb{Q})$ is dense in $\langle \mathbb{I}(\mathbb{R}), q\rangle$. ■

A sequence in M is a mapping $x : \mathbb{N} \longrightarrow M$. x_n denotes the point $x(n)$ and (x_n) denotes this sequence.

There exists, in the literature [RSV82, Doi88], several distinct notions of Cauchy sequence for quasi-metric spaces. Some of them requires reference to extraneous points of the space [dOS94]. Thus, we have just three reasonable internal notions.

Definition 5.2. A sequence (x_n) in the quasi-metric space $\langle X, d\rangle$ is said to be

1. *forward Cauchy* if $\forall\epsilon > 0.\ \exists k.\ k \leq l \leq m \Rightarrow d(x_l, x_m) \leq \epsilon$
2. *backward Cauchy* if $\forall\epsilon > 0.\ \exists k.\ k \leq l \leq m \Rightarrow d(x_m, x_l) \leq \epsilon$
3. *bi-Cauchy* $\forall\epsilon > 0.\ \exists k.\ k \leq l, m \Rightarrow d(x_l, x_m) \leq \epsilon$

Clearly a sequence is bi-Cauchy if it is a Cauchy sequence in the metric space $\langle X, d^*\rangle$. On the other hand, a sequence is bi-Cauchy if it is both forward and backward Cauchy sequence.

Definition 5.3. Let $\langle M, d\rangle$ be a quasi-metric space. A sequence (x_n) **converge** to $a \in M$, denoted by $x_n \to a$, if $\forall\epsilon > 0\ \exists n_0$ such that $n \geq n_0$ implies that $x_n \in B_\epsilon(a)$, i.e. $n \geq n_0 \Rightarrow d(x_n, a) < \epsilon$.

Proposition 5.4. If (I_n) is a bi-Cauchy sequence of the quasi-metric space $\langle \mathbb{I}(\mathbb{R}), q\rangle$ then $I_n \to I$ to some $I \in \mathbb{I}(\mathbb{R})$. ■

Notice that this result does not hold for all quasi-metric space. Furthermore limits of Cauchy sequences in $\langle \mathbb{I}(\mathbb{R}), q\rangle$ are uniques.

Theorem 5.5. For each $[r, s] \in \mathbb{I}(\mathbb{R})$ there exists a bi-Cauchy sequence (I_n) of rational intervals in the quasi-metric space $\langle \mathbb{I}(\mathbb{R}), q\rangle$ such that $(I_n) \to [r, s]$.

PROOF: Let the sequence $([r_i, s_i + 1 \cdot 10^{-i}])$ where r_i (analogously s_i) is r trunqued in the i^{th} decimal digit. Clearly, $([r_i, s_i + 1 \cdot 10^{-i}])$ is a bi-Cauchy sequence of rational intervals converging to $[r, s]$. ∎

Thus, we can say that in the quasi-metric space $\langle \mathbb{I}(\mathbb{R}), q\rangle$ the three notions of Cauchy sequence coincide in power, i.e. each real intervals is the limit of a forward, backward and bi Cauchy sequence.

Let $\langle X_1, d_1\rangle$, $\langle X_2, d_2\rangle$, $\cdots$, $\langle X_n, d_n\rangle$ be arbitrary quasimetric spaces. In the following we will see how to define, through the quasi-metrics $d_1, \cdots, d_n$, a quasi-metric on the set $X = X_1 \times X_2 \times \cdots \times X_n$. Let $x = (x_1, x_2, \cdots, x_n$ and $y = (y_1, y_2, \cdots, y_n)$ be two arbitrary elements of X. Define $D, D_1, D_2 : X \longrightarrow \mathbb{R}^+$ by

$$\begin{aligned} D_1(x,y) &= \sqrt{d_1(x_1,y_1)^2 + \cdots + d_n(x_n,y_n)^2} \\ D_2(x,y) &= d_1(x_1,y_1)^2 + \cdots + d_n(x_n,y_n) \\ D_3(x,y) &= max\{d_1(x_1,y_1)^2, \cdots, d_n(x_n,y_n)\} \end{aligned}$$

Trivially, the above function are all quasi-metrics on X.

Proposition 5.6. Let $\mathbb{I}(\mathbb{R})^n = \underbrace{\mathbb{I}(\mathbb{R}) \times \cdots \mathbb{I}(\mathbb{R})}_{n-times}$ and q be our quasi-metric on $\mathbb{I}(\mathbb{R})$. The three quasi-metric above to $\mathbb{I}(\mathbb{R})^n$ are equivalent[2]. ∎

Proposition 5.7. Let $\langle \mathbb{I}(\mathbb{R}) \times \mathbb{I}(\mathbb{R}), Q\rangle$ be the quasi-metric space where Q is defined as any one of the above quasi-metrics. The interval arithmetics are uniformely continuous. ∎

6. Conclusion

There are several advantages of endowing the interval set $\mathbb{I}(\mathbb{R})$ with the proposed quasi-metric q namely:

- It is compatible with the interval monotonicity property. This characteristic permits us to develop a mathematical theory for interval space in such a way that errors are took into account from the topological point of view as well as from the order, being both viewpoints compatible in the above defined sense.
- The interesting properties of the metric are preserved. Important notions such as continuity, sequence convergences, uniforme continuity etc. are quite similar to the metric ones.

[2] two quasi-metric space are equivalent if the induced relative topology are the same

- The topology induced by the quasi-metric - the Scott topology - is an informational topology, allowing the development of a qualitative theory on interval mathematics and consequently the development of a specification theory for the real numbers;
- It becomes a model of the typed and untyped lambda calculus and hence it is a functional language environment;

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