A Semantical System of Finite Information for the Real Numbers

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Abstract

In this work, we see a rational interval as a finite information about of each real number belonging to it. It allow us to introduce a finite information logic for the real numbers and we relate both (real numbers and their finite information) using a satisfaction relation, defining, in this way, a semantical system of finite information for the real numbers. From it we define severals notions, such as: information content, models, theory of a real number, logical consequence, information content order, etc.

Keywords: Real numbers, rational intervals, finite information, semantical system, consequence relations.

1 Introduction

Traditionally, logic is understood as a consequence relation between structures (sets, multisets, sequences, etc.) of formulae which can be seen as sentences of a formal language. Unfortunately, the notion of consequence relation has, depending of the structure used, several meanings in the literature, which are similar, but not necessarely identic [Gab94]. But the consensus is that it must be reflexive and transitive and in the most case, also monotonic. This way to see logic excludes an important element of a logical system: the semantics.

Semantical systems are structures composed by a formal language, a class of models or set of objects and a satisfaction relation which relates models with the sentences of the formal language [Par83, Car90, BL94]. If we understand the sentences of a language as information about the objects that it satisfies with respect to the satisfaction relation, then the satisfaction relation indicates when a sentence is a true information about an object. From a semantical system it will always be possible to canonically define a consequence relation, which indicates when an information may be derived from a set of information.

On the other hand, because we are in a computational framework, the information about the (perhaps infinite) objects must be finite, in the sense that must be possible to represent it finitely and to determine their truthfullness with a finite amount of time and work.

In [Aci91] a rational interval is consider as an approximation or as a partial object of a real number. But, because, it is always possible to determine in a finite amount of time and work when a real number belongs to a rational interval, also we consider a rational interval as a finite information about any real number in its range. Thus, in this work, we will give a semantical system to the real numbers whose formal language has as atomic formulas the rational intervals, and where the satisfaction relation will be obtained from the idea that a real number satisfies a rational interval if it belongs to the interval. From this semantical system, we will introduce several notions of logical interest, such as: information content, models, theory of a real number, logical consequence, information content order, etc.

2 Language of Finite Information

An information is said to be finite, if on the hypoteses of its correctness, we can verify it in a finite amount of time and work. For example, consider the information "there exists a blue crow", seemingly false, to verify its truthfulness, we need to observe the assumed crow and determinate its blueness, which will be made in a finite amount of time. But, the information "all the crows are black", seemingly true, can not be confirmed in a finite amount of time, because we need to observe all crows (in the past, present and future). The logical operators that preserve the finiteness of information are the conjunction (finite) and disjunction (perhaps infinite). We, also consider two canonical information: the always true information and the always false information.

On the other hand, let $\mathbb{R}$ be the set of the real numbers, $\mathbb{Q}$ the set of the rational numbers and $\mathbb{I}(\mathbb{Q})$ the set of the intervals with rational numbers as end point, or simplely rational intervals, i.e., $\mathbb{I}(\mathbb{Q}) = \{[p,q] \mid p,q \in \mathbb{Q} \text{ with } p < q\}$.

Since, it is always possible to determine in a finite amount of time and work when a real number r belongs to a rational interval $[p,q]$, the rational interval $[p,q]$ can be seen as finite information about r. Thus, we can define a finite information language based on rational intervals as atomic information.

Definition 2.1 *The finite information language of $\mathbb{I}(\mathbb{Q})$, denoted by $\mathcal{L}(\mathbb{I}(\mathbb{Q}))$, is defined recursively as follows:*

1. **The atomic information:**

   (a) if $[p,q] \in \mathbb{I}(\mathbb{Q})$, then $[p,q] \in \mathcal{L}(\mathbb{I}(\mathbb{Q}))$;

(b) $[-\infty, +\infty]$, $[\ ] \in \mathcal{L}(\mathbb{I}(\mathbb{Q}))$, *where* $[-\infty, +\infty]$ *and* $[\ ]$ *are the always true and always false information, respectively, about real numbers;*

2. *if* $\alpha, \beta \in \mathcal{L}(\mathbb{I}(\mathbb{Q}))$, *then* $\alpha \vee \beta \in \mathcal{L}(\mathbb{I}(\mathbb{Q}))$ *and* $\alpha \wedge \beta \in \mathcal{L}(\mathbb{I}(\mathbb{Q}))$*;*

3. *if* $\Gamma \subseteq \mathcal{L}(\mathbb{I}(\mathbb{Q}))$, *then* $\bigvee_{\alpha \in \Gamma} \alpha \in \mathcal{L}(\mathbb{I}(\mathbb{Q}))$.

3 Semantical System of $\mathbb{R}$

A **semantical system** is a structure $\langle A, Mod, \models \rangle$, where A is a set, Mod is a class of models (i.e. a set of objects) and $\models$ is a relation between sentences and models, i.e. $\models \subseteq Mod \times A$, denominated satisfaction relation [Par83, Car90, BL94, Gab94].

The sentences in $\mathcal{L}(\mathbb{I}(\mathbb{Q}))$ can be seen as information on the elements of $\mathbb{R}$ and conversely, we can see the real numbers as models of the information in $\mathcal{L}(\mathbb{I}(\mathbb{Q}))$. Thus, we can introduce the relation "$\models$" between the elements of $\mathbb{R}$ and the information in $\mathcal{L}(\mathbb{I}(\mathbb{Q}))$. The relation $\models \subseteq \mathbb{R} \times \mathcal{L}(\mathbb{I}(\mathbb{Q}))$ is defined recursively as follows:

1. $r \models [-\infty, +\infty]$;
2. $r \models [p, q]$ iff $p < r < q$, where $[p, q] \in \mathbb{I}(\mathbb{Q})$;
3. $r \models \alpha \wedge \beta$ iff $r \models \alpha$ and $r \models \beta$.
4. $r \models \alpha \vee \beta$ iff $r \models \alpha$ or $r \models \beta$;
5. $r \models \bigvee \Gamma$ iff $r \models \alpha$ for some $\alpha \in \Gamma$;

The system $\mathcal{R} = \langle \mathbb{R}, \models, \mathcal{L}(\mathbb{I}(\mathbb{Q})) \rangle$ is called **Semantical System of** $\mathbb{R}$.

According to Shannon [SW49], information reduces the uncertainty, which is determined by a probabilistic measure [SW49], while for us "reduction of uncertainty" is done by the reduction of the spectrum of elements. Thus, the **contents of an information** in $\mathcal{R}$ is associated with the set of their non models [Car90], i.e.

$$Cont(\alpha) = \mathbb{R} - \{r \in \mathbb{R} \mid r \models \alpha\}.$$

The following proposition establishes a calculus of information contents to $\mathcal{R}$.

Proposition 3.1 *If* $\alpha, \beta \in \mathcal{L}(\mathbb{I}(\mathbb{Q}))$ *and* $\Gamma \subseteq \mathcal{L}(\mathbb{I}(\mathbb{Q}))$, *then, the following are valid:*

1. $Cont([p, q]) = \{r \in \mathbb{R} \mid r \leq p \text{ or } q \leq r\}$;
2. $Cont([-\infty, +\infty]) = \emptyset$;
3. $Cont([\ ]) = \mathbb{R}$;
4. $Cont(\alpha \wedge \beta) = Cont(\alpha) \cup Cont(\beta)$;
5. $Cont(\alpha \vee \beta) = Cont(\alpha) \cap Cont(\beta)$;
6. $Cont(\bigvee \Gamma) = \bigcap_{\alpha \in \Gamma} Cont(\alpha)$.

The property 2 says that the always true information has an empty content in the system, in other words this information is *unrelevant*. Analogously, the property 3 states that the always false information has a greatest content, or in other words is *absurd* or *inconsistent*.

The dual notion of information content is the notion of **models of an information**. It is the set of objects in $\mathbb{R}$ which satisfy an information, and can be seen as the function $Mod : \mathcal{L}(\mathbb{I}(\mathbb{Q})) \longrightarrow \mathcal{P}(\mathbb{R})$ defined by

$$Mod(\alpha) = \{r \in \mathbb{R} \mid r \models \alpha\}.$$

Note that, $Mod([p,q] = \{r \in \mathbb{R} \mid p < r < q\}$, $Mod(\alpha) \cup Cont(\alpha) = \mathbb{R}$ and $Mod(\alpha) \cap Cont(\alpha) = \emptyset$.

We can introduce a consistence predicate on $\mathcal{L}(\mathbb{I}(\mathbb{Q}))$ indicating when a finite set of information is consistent. A subset $\Gamma \subseteq^{fin} \mathcal{L}(\mathbb{I}(\mathbb{Q}))$ is **consistent** if $Mod(\Gamma) \neq \emptyset$, (i.e. if there exists an element in $\mathbb{R}$ satisfying all the information in Γ).

The inverse notion of models determines all the information in the system about a real number, it is called **theory of the real number** in $\mathcal{R}$, which can be seen as its specification in $\mathcal{R}$. Thus, the theory of the real numbers in $\mathcal{R}$, is the function $Theo : \mathbb{R} \longrightarrow \mathcal{L}(\mathbb{I}(\mathbb{Q}))$ defined by

$$Theo(r) = \{\alpha \in \mathcal{L}(\mathbb{I}(\mathbb{Q})) \mid r \models \alpha\}.$$

The following proposition states that two diferent real numbers satisfy diferent sets of information in $\mathcal{R}$.

Proposition 3.2 *If $x, y \in \mathbb{R}$ and $x \neq y$ then $theo(x) \neq theo(y)$.*

This functions can be extended to a set of information (Γ) or objects (X), as follows:

$$Cont(\Gamma) = \bigcup_{\alpha\in\Gamma} Cont(\alpha) \qquad Mod(\Gamma) = \bigcap_{\alpha\in\Gamma} Mod(\alpha) \qquad Theo(X) = \bigcap_{x\in X} Theo(x).$$

Proposition 3.3 [*Par83*] *The functions Mod and $Theo$ constitutes a Galois connection, for if $\Gamma, \Delta \subseteq \mathcal{L}(\mathbb{I}(\mathbb{Q}))$ and $X, Y \subseteq \mathbb{R}$, then*
1. *If $\Gamma \subseteq \Delta$ then $Mod(\Delta) \subseteq Mod(\Gamma)$;*
2. *If $X \subseteq Y$ then $Theo(Y) \subseteq Theo(X)$;*
3. *$\Gamma \subseteq Theo(Mod(\Gamma))$ and $X \subseteq Mod(Theo(X))$.*

Let $\Gamma \subseteq \mathcal{L}(\mathbb{I}(\mathbb{Q}))$. The set of information Γ is **complete** iff Γ has an unique model, i.e. $Mod(\Gamma) = \{r\}$ for some $r \in \mathbb{R}$. Γ is **maximally complete** iff Γ is complete and it becomes inconsistent with the addition of any new information to Γ, i.e. to each $\alpha \in \mathcal{L}(\mathbb{I}(\mathbb{Q}))$, if $\alpha \notin \Gamma$ then $Mod(\Gamma \cup \{\alpha\}) = \mathbb{R}$. Γ is **minimally complete** iff Γ is complete and there does not exist any proper subset $\Delta \subset \Gamma$ such that $Mod(\Delta) = Mod(\Gamma) = \{r\}$. So, Γ is the information set which is sufficient and necessary to specify the real number r.

Proposition 3.4 *For each $r \in \mathbb{R}$ the set $\{[p,q] \in \mathbb{I}(\mathbb{Q}) \mid p < r < q\}$ is maximally complete.*

An important concept, which arises when we see $\mathcal{R}$ as a logical theory, is that of logical consequence for information.

Definition 3.5 *The function* $CnI : \mathcal{L}(\mathbb{I}(\mathbb{Q})) \longrightarrow \mathcal{P}(\mathcal{L}(\mathbb{I}(\mathbb{Q})))$ *defined by* $CnI(\alpha) = Theo(Mod(\alpha))$, *determine the information which can be deduced from* α, *i.e. their* **logical consequences**.

Proposition 3.6 *Let* $\Gamma, \Delta \subseteq \mathcal{L}(\mathbb{I}(\mathbb{Q}))$. *Then,* $CnI(\Gamma \cup \Delta) = CnI(\Gamma) \cup CnI(\Delta)$.[1]

The notion of logical consequence can be seen as a relation, $\mid\!\sim\, \subseteq \mathcal{P}(\mathcal{L}(\mathbb{I}(\mathbb{Q}))) \times \mathcal{L}(\mathbb{I}(\mathbb{Q}))$, where $\Gamma \mid\!\sim \alpha$ if $\alpha \in CnI(\Gamma)$. If $\alpha \in CnI(\emptyset)$ (if $\emptyset \mid\!\sim \alpha$), then α is a **theorem**, denoted by $\mid\!\sim \alpha$. When $\Gamma = \{\alpha_1, \cdots, \alpha_n\}$, we will write $\alpha_1, \cdots, \alpha_n \mid\!\sim \alpha$ instead of $\{\alpha_1, \cdots, \alpha_n\} \mid\!\sim \alpha$

Proposition 3.7 *The relation* $\mid\!\sim$ *satisfies the follows properties:*

1. $[p_1, q_1] \mid\!\sim [p_2, q_2]$ *iff* $p_2 \leq p_1$ *and* $q_1 \leq q_2$;
2. *if* $\alpha \in \Gamma$ *then* $\Gamma \mid\!\sim \alpha$;
3. *if* $\Gamma \mid\!\sim \alpha$, *to each* $\alpha \in \Delta$, *and* $\Delta \mid\!\sim \beta$ *then* $\Gamma \mid\!\sim \beta$;
4. *if* $\Gamma \subseteq \Delta$ *and* $\Gamma \mid\!\sim \alpha$ *then* $\Delta \mid\!\sim \alpha$;
5. *if* $\Gamma \mid\!\sim \alpha$ *then* $\Gamma \cup \{\alpha\}$ *is consistent*;
6. *if* Γ *is not consistent then* $\Gamma \mid\!\sim \alpha$ *to each* $\alpha \in \mathcal{L}(\mathbb{I}(\mathbb{Q}))$;
7. $\alpha_1, \cdots, \alpha_n \mid\!\sim \alpha$ *iff* $\alpha_1 \wedge \cdots \wedge \alpha_n \mid\!\sim \alpha$;
8. *if* $\beta \vee \gamma \mid\!\sim \alpha$ *then* $\beta \mid\!\sim \alpha$ *and* $\gamma \mid\!\sim \alpha$;
9. *if* $\beta \mid\!\sim \alpha$ *and* $\gamma \mid\!\sim \alpha$ *then* $\beta \wedge \gamma \mid\!\sim \alpha$;
10. *if* $\Gamma \mid\!\sim \alpha$ *and* $\Gamma \mid\!\sim \beta$ *then* $\Gamma \mid\!\sim \alpha \wedge \beta$.

The properties 2, 3 and 4 of $\mid\!\sim$ are knowns as reflexivity, transitivity and monotonicity, respectively. So, $\mid\!\sim$ is a **consequence relation** in $\mathcal{L}(\mathbb{I}(\mathbb{Q}))$ [Gab94].

From the definition of information content we establish some interesting relations on information.

Definition 3.8 *Let* $\mathcal{R}$ *and* $\alpha, \beta \in \mathcal{L}(\mathbb{I}(\mathbb{Q}))$. *The information* α *is* **greater** *than or* **more exact** *than the information* β, *denoted by* $\beta \sqsubseteq \alpha$ *iff* $Cont(\beta) \subseteq Cont(\alpha)$. *Analogously,* α *and* β *are* **equivalent information**, *denoted by* $\alpha \equiv \beta$, *iff* $Cont(\alpha) = Cont(\beta)$.

The *information content order* $\sqsubseteq$ is an intensional order. On the other hand, the dual order, $\sqsupseteq$, model-based, is an extensional information content order in the sense of [Bun74].

Proposition 3.9 *Let* $\alpha, \beta \in \mathcal{L}(\mathbb{I}(\mathbb{Q}))$ *and* $\Gamma \subseteq \mathcal{L}(\mathbb{I}(\mathbb{Q}))$. *Then*

1. $\alpha \vee \beta \equiv \beta \vee \alpha$ *and* $\alpha \wedge \beta \equiv \beta \wedge \alpha$;
2. $\alpha \wedge \bigvee_{\beta \in \Gamma} \beta \equiv \bigvee_{\beta \in \Gamma} (\alpha \wedge \beta)$;
3. $\alpha \equiv \beta$ *iff* $\alpha \sqsubseteq \beta$ *and* $\beta \sqsubseteq \alpha$;
4. $\alpha \wedge \beta \sqsubseteq \alpha$;
5. $\alpha \sqsubseteq \alpha \vee \beta$;
6. $\alpha \mid\!\sim \beta$ *iff* $\beta \sqsubseteq \alpha$;
7. $[p_1, q_1] \sqsubseteq [p_2, q_2]$ *iff* $p_1 \leq p_2$ *and* $q_2 \leq q_1$;
8. $[-\infty, +\infty] \sqsubseteq \alpha$ *and* $\alpha \sqsubseteq [\ ]$;
9. *if* $q_1 < p_2$ *then* $[p_1, q_1 \wedge [p_2, q_2] \equiv [\ ]$;
10. *If* $p_2 \leq q_1$ *then* $[p_1, q_1] \vee [p_2, q_2] \equiv [p_1, q_2]$;
11. *If* $p_2 \leq q_1$ *then* $[p_1, q_1] \wedge [p_2, q_2] \equiv [p_2, q_1]$.

[1] The expression $CnI(\Gamma)$ is an abreviation of $\bigcup_{\alpha \in \Gamma} CnI(\alpha)$.

4 Conclusion

In this work, we have cosidered a rational interval as a finite information of the real numbers in its range, which allows us to obtain a language of finite information of rational intervals, in a similar way as Vickers [Vic89] introduces the logic of observable properties. We then provided a connection between real numbers and their finite information language based on rational intervals, using a satisfaction notion which extends the idea that each rational interval is an information about all real numbers which it approximates. This notion, reveals several logical concepts, such as: consequence relation, information content, models of an information, theory of an element, theorems, information content order, equivalence between information, etc.

Thus, this work allows us to introduce a semantical system or, in other words, a logical system, where it is possible to deal with partial information about the real numbers and where it is possible to especify each real number in terms of its partial information, i.e. as a set of rational intervals.

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