Acióly-Scott Interval Categories

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Abstract. *In this work, from the category sight, we provide a generalization for the real interval theory. This generalization allows us to study generic properties of data which are "intervals" of another data, providing a categorical foundation of intervals as a parametric data type. In doing so we obtain some properties which holds for real intervals, complex intervals, interval vectors, interval matrices, and so on. For this purpose we introduce a categorical interval constructor on* **POSET** *based on the information order introduced by Dana Scott and used by Benedito Acióly to provide a computational foundation of interval mathematics. We study the categorical properties which this constructor satis£es in order to de£ne the notion of Acióly-Scott interval category. We prove also that several subcategories of* **POSET** *are Acióly-Scott interval categories and we show also that the quasi-metric spaces category, which is important from a computational point of view and is not a subcategories of* **POSET***, is an Acióly-Scott interval category. .*

Resumo. *Neste trabalho, desde a ótica de categorias, proporcionaremos uma generalização da teoria dos intervalos reais. Esta generalização nos permitirá estudar propriedades genéricas de dados que são "intervalos" de outros dados, proporcionando uma fundamentação categórica de intervalos como um tipo de dados paramétrico. Desta forma, nós obtemos algumas propriedades que são válidas para intervalos reais, intervalos complexos, vetores do intervalo, matrizes do intervalo, e assim por diante. Com este propósito, introduziremos um construtor intervalar categórico sobre* **POSET** *baseado na ordem de informação introduzida por Dana Scott e usada por Benedito Acióly para prover uma fundamentação computacional da matemática intervalar. Para de£nir a noção de categoria intervalar de Acióly-Scott, estudamos as propriedades categóricas que este construtor satisfaz. Provamos também que diversas subcategorias de* **POSET** *são categorias intervalares de Acióly-Scott e mostramos que a categoria dos espaços quase-métricos, a qual é importante do ponto de vista computacional e não é uma subcategoria de* **POSET***, é uma categoria intervalar de Acióly-Scott.*

1. Introduction

R.A. Moore in [Moore, 1959], [Moore and Yang, 1959] and [Moore, 1979] developed a mathematical theory for closed real intervals, extending the usual mathematical notions on real numbers to real intervals. The interval mathematics also consider complex intervals [Boche, 1966], matrix and arrays of real and complex intervals [Kulisch and Miranker, 1981]. The main goal of Moore for introducing this theory was to provide an automatic control of the computational errors resulting from numeric computations involving real numbers. In order to perform interval computations[1] in a more clear and ef£cient way, several programming languages have been extended considering computational representation of these intervals as primitive data types. These extensions are denominated XSC (eXtension for Scienti£c Computation) [Kearfott, 1996]. It is reasonable to expect that in the future will be developed XSC languages with a parametric interval data type. This motivate to generalize the interval mathematics in order to provide a theoretical foundation for these parametric interval data type, in such a way that intervals of "any" possible kind be allowed.

Since real intervals are de£ned through the order on the real set, we might de£ne intervals on any partially order set. The real interval itself are also a poset. In fact, there exists four usual partial orders on real intervals. In [Callejas-Bedregal and Bedregal, 2001a] was introduced an interval constructor based on the Kulisch-Miranker order on the real intervals [Kulisch and Miranker, 1981], which was generalized for other kind of categories (not necessarily subcategories of **POSET**) such as the category of topological spaces with continuous functions. These order is important from a mathematical point of view, since it is compatible with the cartesian product. However, from an information or approximation point of view[2] the information order introduced by Scott to the real intervals [Scott, 1970] and used by Acióly in [Acióly, 1991] to provide a computational foundation to interval mathematics is more interesting. In [Callejas-Bedregal and Bedregal, 2001b] was de£ned an interval constructor on posets based on this information order. In this work, we will study the class of Acióly-Scott categories, that is, categories having and interval constructor compatible with the interval constructor based on the information order. We will prove also that the category of quasi-metric spaces and some subcategories of **POSET** are Acióly-Scott interval categories. This is an interesting result since the category of quasi-metric spaces is not a subcategory of **POSET** and it is an important category for computer science.

2. The Interval Mathematics

Let $r, s \in \mathbb{R}$ such that $r \leq s$. The set $\{x \in \mathbb{R} \ : \ r \leq x \leq s\}$ is a closed real interval or simply a real interval and is denoted by the order pair $[r, s]$. The set $\{[r, s] \ : \ r, s \in \mathbb{R}$ and $r \leq s\}$ called the real interval set is denoted by $\mathbb{I}(\mathbb{R})$. A main characteristic of interval

[1]Numeric computations using computational interval representations (interval of ¤oat points) for real numbers.

[2]An interval can be see as an information or approximation of the real numbers belong to it.

mathematics is to guaranty that all interesting constructions on intervals can be obtained through their extremes, that is we can see intervals as ordered pairs. For example,

$$\begin{aligned}
[r,s]+[t,u] &= \{x+y \ : \ x \in [r,s] \text{ and } y \in [t,u]\} \\
&= [r+t, s+u] \\
[r,s]\cdot[t,u] &= \{xy \ : \ x \in [r,s] \text{ and } y \in [t,u]\} \\
&= [Min\{rt,ru,st,su\}, Max\{rt,ru,st,su\}]
\end{aligned}$$

It is possible to de£ne several partial order on the $\mathbb{I}(\mathbb{R})$ which extend the usual order on the real set [Callejas-Bedregal and Bedregal, 2001a]. In this work we will take into account the information order ($[a,b] \sqsubseteq [c,d] \Leftrightarrow [c,d] \subseteq [a,b]$) introduced by Scott in [Scott, 1970] since it is adequate to provide a computational foundation to interval mathematics [Acióly, 1991] and more speci£cally a computability theory on the interval spaces [Bedregal, 1996], [Bedregal and Acióly, 1997], [A. Edalat, 1999] and [Bedregal and Acióly, 1999].

3. The Interval Constructor on POSET

Observe that the real intervals as much as the order on $\mathbb{I}(\mathbb{R})$ depend upon the usual real order. We can generalize this constructions by considering any partially order set, instead of the real set with its usual order. So, we can think of intervals as a constructor on the category **POSET**.

De£nition 1 *Let* $\mathbf{D} = (D, \leq)$ *be a poset. The poset* $I(\mathbf{D}) = (I(D), \sqsubseteq)$*, where*

- $I(D) = \{[a,b] \ : \ a,b \in D$ *and* $a \leq b\}$
- $[a,b] \sqsubseteq [c,d] \Leftrightarrow a \leq c$ *and* $d \leq b$

is called the **poset of intervals of** $\mathbf{D}$ *[Callejas-Bedregal and Bedregal, 2001b].*

There are two natural functions from $I(D)$ to D, which are the left and right projections $l : I(D) \longrightarrow D$ and $r : I(D) \longrightarrow D$ respectively, de£ned by

$$l([a,b]) = a \text{ and } r([a,b]) = b.$$

The function l is monotonic and therefore it is a morphism from the poset $I(\mathbf{D})$ to the poset $\mathbf{D}$, but the function r is not monotonic. Nevertheless, reverting the order on $\mathbf{D}$, this could be overcome.

De£nition 2 *Let* $\mathbf{D} = (D, \leq)$ *be a poset. The* **opposite poset** *of* $\mathbf{D}$*, denoted by* $\mathbf{D}^{op}$*, is the pair* $\mathbf{D}^{op} = (D^{op}, \leq_{op})$*, where* $D^{op} = D$ *and* $x \leq_{op} y \Leftrightarrow y \leq x$.

Thus, every poset $\mathbf{D}$ has a opposite poset and the functions l and r are monotonic.

Let $\mathbf{D}$ and $\mathbf{E}$ posets. If $f : D \longrightarrow E$ is a monotonic function w.r.t. $\mathbf{D}$ and $\mathbf{E}$, then the function $f^{op} : D^{op} \longrightarrow E^{op}$ de£ned by $f^{op}(x) = f(x)$ is a monotonic function w.r.t. $\mathbf{D}^{op}$ and $\mathbf{E}^{op}$. Thus, ^{op} is a covariant functor from **POSET** into **POSET**.

Proposition 1 *Let* $\mathbf{D} = (D, \leq)$ *be a poset. Then*

1. $(\mathbf{D}^{op})^{op} = \mathbf{D}$ *and*
2. $I(\mathbf{D}^{op})$ *and* $I(\mathbf{D})$ *are order isomorphic.*

Proof: 1. Is trivial. 2. Let $inv : I(D^{op}) \longrightarrow I(D)$ be the function de£ned by $inv([a,b]) = [b,a]$. Clearly inv is well de£ned, monotonic and bijective and therefore is an isomorphism. ■

It is well know that **POSET** is a cartesian closed category (for example see [Asperti and Longo, 1991]). Let $\mathbf{D}_1 = (D_1, \leq_1)$ and $\mathbf{D}_2 = (D_2, \leq_2)$ be posets. The cartesian product between $\mathbf{D}_1$ and $\mathbf{D}_2$, denoted by $\mathbf{D}_1 \times \mathbf{D}_2$, is de£ned by $\mathbf{D}_1 \times \mathbf{D}_2 = (D_1 \times D_2, \leq)$, where $(a,b) \leq (c,d) \Leftrightarrow a \leq_1 c$ and $b \leq_2 d$.

Lemma 1 *Let* $\mathbf{D}_1 = (D_1, \leq_1)$ *and* $\mathbf{D}_2 = (D_2, \leq_2)$ *be posets. Then* $I(\mathbf{D}_1 \times \mathbf{D}_2)$ *is order isomorphic to* $I(\mathbf{D}_1) \times I(\mathbf{D}_2)$[3].

Proof: Let $f : I(D_1 \times D_2) \longrightarrow I(D_1) \times I(D_2)$ be de£ned by $f([(a,b),(c,d)]) = ([a,c],[b,d])$.

Since,

$$\begin{aligned}
[(a_1,b_1),(a_2,b_2)] \in I(D_1 \times D_2) &\Leftrightarrow (a_1,b_1) \leq_\times (a_2,b_2) \\
&\Leftrightarrow a_1 \leq_1 a_2 \text{ and } b_1 \leq_2 b_2 \\
[(c_1,d_1),(c_2,d_2)] \in I(D_1 \times D_2) &\Leftrightarrow (c_1,d_1) \leq_\times (c_2,d_2) \\
&\Leftrightarrow c_1 \leq_1 c_2 \text{ and } d_1 \leq_2 d_2
\end{aligned}$$

then,

$$\begin{aligned}
[(a_1,b_1),(a_2,b_2)] \sqsubseteq [(c_1,d_1),(c_2,d_2)] &\Leftrightarrow ((a_1,b_1) \leq_\times (c_1,d_1) \text{ and } (c_2,d_2) \leq_\times (a_2,b_2) \\
&\Leftrightarrow a_1 \leq_1 c_1 \text{ and } b_1 \leq_2 d_1 \text{ and } c_2 \leq_1 a_2 \text{ and } d_2 \leq_2 b_2 \\
&\Leftrightarrow a_1 \leq_1 c_1 \leq_1 c_2 \leq_1 a_2 \text{ and } b_1 \leq_2 d_1 \leq_2 d_2 \leq_2 b_2 \\
&\Leftrightarrow [a_1,a_2] \sqsubseteq_1 [c_1,c_2] \text{ and } [b_1,b_2] \sqsubseteq [d_1,d_2] \\
&\Leftrightarrow ([a_1,a_2],[b_1,b_2]) \sqsubseteq_\times ([c_1,c_2],[d_1,d_2]) \\
&\Leftrightarrow f([(a_1,b_1),(a_2,b_2)]) \sqsubseteq_\times f([(c_1,d_1),(c_2,d_2)])
\end{aligned}$$

So f is monotonic. Since f is bijective we have that it is an isomorphism. ■

The cartesian construction provide the introduction of the arrays and matrix of some type. Thus, this lemma states that an array (or matrix) of intervals of some type can be see as an interval of an array (or matrix) of these types.

Lemma 2 *Let* $\mathbf{D} = (D, \leq)$ *be a poset. There is a unique monomorphism* $m(\mathbf{D}) : I(D) \longrightarrow D \times D^{op}$ *which makes the following diagram*

[3]Consider in the proof $\leq_\times$, $\sqsubseteq$, $\sqsubseteq_\times$, $\sqsubseteq_1$ and $\sqsubseteq_2$ as the orders of $\mathbf{D}_1 \times \mathbf{D}_2$, $I(\mathbf{D}_1 \times \mathbf{D}_2)$, $I(\mathbf{D}_1) \times I(\mathbf{D}_2)$, $I(\mathbf{D}_1)$ and $I(\mathbf{D}_2)$, respectively.

$$\begin{tikzcd}
& I(D) \arrow[dl, "l"'] \arrow[d, "m(\mathbf{D})"] \arrow[dr, "r"] & \\
D & D \times D^{op} \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] & D^{op}
\end{tikzcd}$$

commutative.

Proof: Follows from the universal property of the cartesian product. ■

Proposition 2 *Let $\mathbf{D}_1 = (D_1, \leq_1)$ and $\mathbf{D}_2 = (D_2, \leq_2)$ be posets. Let $f : D_1 \longrightarrow D_2$ be a monotonic function. The function $I(f) : I(D_1) \longrightarrow I(D_2)$, de£ned by $I(f)([a, b]) = [f(a), f^{op}(b)]$, is the unique monotonic function which makes the following diagram*

$$\begin{tikzcd}
D_1 \arrow[d, "f"'] & I(D_1) \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "I(f)"] & D_1^{op} \arrow[d, "f^{op}"] \\
D_2 & I(D_2) \arrow[l, "l"'] \arrow[r, "r"] & D_2^{op}
\end{tikzcd}$$

commutative.

Proof: Clearly $I(f)$ is monotonic and makes the above diagram commutative.

If $G : I(D_1) \longrightarrow I(D_2)$ is another monotonic function such that $l \circ G = f \circ l$ and $r \circ G = f^{op} \circ r$ then,

$$\begin{aligned}
G([a, b]) = [c, d] \;\; &\Leftrightarrow l(G([a, b])) = l([c, d]) \text{ and } r(G([a, b])) = r([c, d]) \\
&\quad \text{(since } m(\mathbf{D}_2) : I(D_2) \longrightarrow D_2 \times D_2^{op} \text{ is injective)} \\
&\Leftrightarrow f(l([a, b])) = l([c, d]) \text{ and } f^{op}(r([a, b])) = r([c, d]) \\
&\quad \text{(by commutativity)} \\
&\Leftrightarrow f(a) = c \text{ and } f^{op}(b) = d
\end{aligned}$$

Thus $G([a, b]) = [f(a), f^{op}(b)] = I(f)([a, b])$. Therefore $I(f)$ is unique. ■

Remark: The above proposition guarantees that we have a covariant functor $I : \mathbf{POSET} \longrightarrow \mathbf{POSET}$ and provide a generalization of the best interval representation notion [R.H.N. Santiago and Acióly, 2004] for monotonic functions.

Lemma 3 *Let $\mathcal{C}$ be a category with cartesian product. Let $f : A \longrightarrow B$ and $g : C \longrightarrow D$ be morphisms of C. Then there is a unique morphism*

$$f \times g : A \times C \longrightarrow B \times D$$

which makes commutative the following diagram

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times C \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] \arrow[d, "f \times g"] & C \arrow[d, "g"] \\
B & B \times D \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] & D.
\end{tikzcd}$$

Proof: Follows from the universal property of the cartesian product. ■

Remark: The lemma 3 guarantees that if $F : \mathcal{C} \longrightarrow \mathcal{C}$ is a covariant functor, then

$$Prod_F : \mathcal{C} \longrightarrow \mathcal{C}$$

is a covariant functor, where $Prod_F$ is de£ned by $Prod_F(A) = A \times F(A)$ for each object A of $\mathcal{C}$ and if $f : A \longrightarrow B$ is a morphism then

$$Prod_F(f) = f \times F(f) : A \times F(A) \longrightarrow B \times F(B)$$

Lemma 4 *The collection*

$$m = \{m(\mathbf{D}) : I(D) \longrightarrow Prod_{op}(D) \ : \ \mathbf{D} \textit{ is a poset } \}$$

of morphisms is a natural transformation from I : $\mathbf{POSET} \longrightarrow \mathbf{POSET}$ to $Prod_{op}$: $\mathbf{POSET} \longrightarrow \mathbf{POSET}$.

Proof: Let $f : D_1 \longrightarrow D_2$ be a monotonic function of posets. We must prove that the following diagram

$$\begin{tikzcd}
I(D_1) \arrow[r, "m(\mathbf{D}_1)"] \arrow[d, "I(f)"'] & D_1 \times D_1^{op} \arrow[d, "f \times f^{op}"] \\
I(D_2) \arrow[r, "m(\mathbf{D}_2)"] & D_2 \times D_2^{op}
\end{tikzcd}$$

commutes.

In fact, if $[a,b] \in I(D_1)$ then

$$\begin{aligned} f \times f^{op}(m(\mathbf{D}_1)([a,b])) &= f \times f^{op}([a,b]) \\ &= (f(a), f^{op}(b)) \\ &= m(\mathbf{D}_2)([f(a), f^{op}(b)]) \\ &= m(\mathbf{D}_2)(I(f)([a,b])) \end{aligned}$$

Thus, $(f \times f^{op}) \circ m(\mathbf{D}_1) = m(\mathbf{D}_2) \circ I(f)$. ■

4. Acióly-Scott Interval Categories

De£nition 3 *An* **Acióly-Scott interval category** *is a quadruple* $(\mathcal{C}, {}^{op}, \mathbb{I}, m)$ *such that*

1. $\mathcal{C}$ *is a category with cartesian product*
2. ${}^{op} : \mathcal{C} \longrightarrow \mathcal{C}$ *is a covariant functor such that* $(A^{op})^{op} = A$ *for all object* A *of* $\mathcal{C}$
3. $\mathbb{I} : \mathcal{C} \longrightarrow \mathcal{C}$ *is a covariant functor such that*
   (a) $\mathbb{I}(A \times B)$ *is isomorphic to* $\mathbb{I}(A) \times \mathbb{I}(B)$ *for all pair of objects* A *and* B *of* $\mathcal{C}$ *and*
   (b) $\mathbb{I}(A^{op})$ *is isomorphic to* $\mathbb{I}(A)$ *for all object* A *of* $\mathcal{C}$
4. m *is an injective natural transformation from* $\mathbb{I} : \mathcal{C} \longrightarrow \mathcal{C}$ *to* $Prod_{op} : \mathcal{C} \longrightarrow \mathcal{C}$
5. *There exists a covariant functor* $F : \mathcal{C} \longrightarrow$ **POSET** *such that for each* $A, B, C, D \in Obj_{\mathcal{C}}$ *and for each* $f : A \longrightarrow C$ *and* $g : B \longrightarrow D$ *morphisms we have that*
   (a) $F(A^{op}) = F(A)^{op}$,
   (b) $F(f^{op}) = F(f)^{op}$,
   (c) $F(\mathbb{I}(A)) = I(F(A))$,
   (d) $F(\mathbb{I}(f)) = I(F(f))$,
   (e) $F(A \times B) = F(A) \times F(B)$,
   (f) $F(f \times g) = F(f) \times F(g)$ *and*
   (g) $F(m(A)) = m(F(A))$.

Notice that the functor ^{op} and the natural transformation m of the left and right sides of the equations of de£nition 3 are different, because the left sides are de£ned w.r.t. the category $\mathcal{C}$ whenever the right sides are de£ned w.r.t. the **POSET** category. Analogously, $\mathbb{I}$ and I are different.

Lemma 5 *Let* $(\mathcal{C}, {}^{op}, \mathbb{I}, m)$ *be an Acióly-Scott interval category and* A *be an object of* $\mathcal{C}$. *There are unique morphisms* $l : \mathbb{I}(A) \longrightarrow A$ *and* $r : \mathbb{I}(A) \longrightarrow A^{op}$ *which makes the following diagram*

$$\begin{tikzcd} & \mathbb{I}(A) \arrow[dl, "l"'] \arrow[d, "m(A)"] \arrow[dr, "r"] & \\ A & A \times A^{op} \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] & A^{op} \end{tikzcd}$$

commutative.

Proof: Follows from the universal property of the cartesian product. ■

Proposition 3 *Let $(\mathcal{C}, {}^{op}, \mathbb{I}, m)$ be an Aciòly-Scott interval category, A and B be objects of $\mathcal{C}$ and $f : A \longrightarrow B$ be a morphism. There is a unique morphism $\mathbb{I}(f) : \mathbb{I}(A) \longrightarrow \mathbb{I}(B)$ which makes the following diagram*

$$\begin{tikzcd}
A \arrow[d, "f"'] & \mathbb{I}(A) \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "\mathbb{I}(f)"] & A^{op} \arrow[d, "f^{op}"] \\
B & \mathbb{I}(B) \arrow[l, "l"'] \arrow[r, "r"] & B^{op}
\end{tikzcd}$$

commutative.

Proof: The morphism $\mathbb{I}(f) = (f \circ l) \times (f \circ r)$. ■

5. Some Trivial Interval Categories

In this section we study several categories which are trivially Aciòly-Scott interval categories.

Proposition 4 $(\mathbf{POSET}, {}^{op}, I, m)$ *is an Aciòly-Scott interval category.*

Proof: Properties 1), 2), 3) and 4)of the above de£nition follows from what we had discussed before. To prove property 5) it is enough to take $F : \mathbf{POSET} \longrightarrow \mathbf{POSET}$ as the identity functor. ■

Proposition 5 *Let $(\mathcal{C}, {}^{op}, \mathbb{I}, m)$ be an Aciòly-Scott interval category. If $\mathcal{D}$ is a subcategory of $\mathcal{C}$ with cartesian product, which is closed under the functors $\mathbb{I}$ and ^{op} then $(\mathcal{D}, {}^{Op'}, \mathbb{I}', m')$ is an interval category, where ${}^{Op'}$, $\mathbb{I}'$ and m' are the restrictions of ^{op}, $\mathbb{I}$ and m to $\mathcal{D}$, respectively.*

Proof: Straightforward. ■

Corollary 1 *The following Categories are Aciòly-Scott interval categories:*

1. **biDCPO** *with dcpos whose reverse poset is also a dcpo as objects and continuous functions as morphisms.*

2. **biCCDCPO** *with consistently complete dcpos whose reverse poset is also a consistently complete dcpos as objects and continuous functions as morphisms.*
3. **biADCPO** *with algebraic dcpos whose reverse poset is also an algebraic dcpo as objects and continuous functions as morphisms.*
4. **biSDom** *with Scott domains whose reverse poset is also a Scott domain as objects and continuous functions as morphisms.*

Proof: In [Callejas-Bedregal and Bedregal, 2001a] was proved that the interval functor I is closed under this kind of posets and if $f : D \longrightarrow E$ is continuous then $I(f)$ is also continuous. By de£nition, the ^{op} functor is also closed under this kind of posets. So, all those categories are closed under the functors I and ^{op}. Since, trivially, those categories are subcategories of **POSET** (all dcpo is a poset and all continuous function is monotonic) then, by propositions 4 and 5, all those categories are Acióly-Scott categories. ∎

Another trivial way to obtain an Acióly-Scott interval category is to consider structured set categories with functions as morphism, the functor F mapping objects (A, Γ), where A is the set and Γ the structure, in the poset $(A, =)$, the functor ^{op} as the identity and the $\mathbb{I}$ functor mapping an object (A, Γ) in $(\mathbb{I}(A), \Gamma')$ where $\mathbb{I}(A) = \{\{a\} : a \in A\}$ and Γ' is the natural structure obtained from Γ. For example, if A is a numerical set and $\Gamma = \{+, \leq\}$ with the usual meaning, then $\Gamma' = \{+', \leq'\}$ with the follows means: $\{x\} +' \{y\} = \{x + y\}$ and $\{x\} \leq' \{y\} \Leftrightarrow x \leq y$.

In the next section we will give a non-trivial example of an Acióly-Scott interval category, in the sense that neither is a subcategory of **POSET** nor the functor F mapped objects always into a poset with the equality as order.

6. Quasi-metric Spaces as an Acióly-Scott Interval Category

In [Moore, 1979] the set of real intervals is endowed with a metric topology such that when restricted to the degenerate intervals the induced relative topology coincides with the usual one on the real line. It is well known that the metric topology [Moore, 1979] on an interval space is not compatible with the interval inclusion monotonicity property in the sense that may exist monotonic functions which are not continuous and conversely. Quasi-metric spaces is a generalization of metric spaces. In [Acióly and Bedregal, 1997] a quasi-metric topology for the interval space was provided which is consistent with the real line topology and whose continuous functions are monotonic. This quasi-metric is not a metric since it fails to satis£es the symmetrical property. Thus, except for the Hausdorff property of the metric - which does not £t with our point of view - the other good metric properties remains true. Lately [Trindade, 2002] used this quasi-metric to provide some results in order to develop a Scott interval analysis as alternative to Moore interval analysis. Quasi-metric spaces and their relation to domain theory have turned into a £eld of intensive research recently, see for example [Smyth, 1992] and [Acióly and Bedregal, 1997]. Another application of quasi-metric spaces are in the study of hyperspaces, function spaces, fuzzy topology and £xed point theory [J. Cao and Romaguera, 1998], [Gregori and Romaguera, 2000] and [J.S. Jung and Choi, 2000].

De£nition 4 *[Smyth, 1992] Let X be a non empty set and $q : X \times X \longrightarrow \mathbb{R}^+$ be a function. The pair (X, q) is said to be a* **quasi-metric space** *if for each x, y and z in X, q satis£es the following axioms.*

i) $q(x, x) = 0$
ii) $q(x, z) \leq q(x, y) + q(y, z)$
iii) $q(x, y) = q(y, x) = 0 \ \Rightarrow \ x = y$

In this case q is called a **quasi-metric***.*

The weakening of the symmetry axiom is not as absurd as it may seem. For example, the distance from a point x to a point y could be thought as a measure of the effort performed in going from x to y. Thus when x is at the basis of a hill and y at the top, the distance (effort) in going from x to y is considerably greater than the effort in going from y to x [Smyth, 1992]. Another intuition is to see a quasi-metric as a measure of the distance between information. Thus the distance of an information x to y is zero if x is more complete or is equal to y, otherwise it is greater than zero.

From this de£nition every metric space is also a quasi-metric one. Let $\mathbf{X} = (X, q)$ be a quasi-metric space. As in the case of metric spaces the standard topology induced on X by a quasi-metric q is de£ned by taking a set $\mathcal{O} \subseteq X$ as being open if, and only if, for any $x \in X$, $x \in \mathcal{O}$ implies that $B_\epsilon(x) \subseteq \mathcal{O}$ for some $\epsilon > 0$, where $B_\epsilon(x) = \{y \in X \ : q(x, y) < \epsilon\}$. In that case, the open ϵ-balls $B_\epsilon(x)$ constitute evidently a base of open sets for a topology on X.

Continuous and uniformly continuous functions for quasi-metric spaces are de£ned analogously to metric spaces.

De£nition 5 *Let $\mathbf{X} = (X, q)$ and $\mathbf{Y} = (Y, q')$ be quasi-metric spaces. $f : X \longrightarrow Y$ is* **uniformly continuous** *if*

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x, y \in X. \ q(x, y) \leq \delta \Rightarrow q'(f(x), f(y)) \leq \epsilon$$

The idea of uniform continuity is that if a pair of points is closed in $\mathbf{X}$ then their image must be closed in $\mathbf{Y}$, uniformly across the space $\mathbf{X}$ [Smyth, 1992]. If f is uniformly continuous then f is continuous [Smyth, 1992].

Proposition 6 *Let $\mathbf{X} = (X, q)$ and $\mathbf{Y} = (Y, q')$ be quasi-metric spaces. If $f : X \longrightarrow Y$ is uniformly continuous and $q(x, y) = 0$ then $q'(f(x), f(y)) = 0$.*

Proof: Suppose that $q(x, y) = 0$ and $q'(f(x), f(y)) \neq 0$. Then, there exists ϵ such that $0 < \epsilon < q'(f(x), f(y))$. By de£nition of uniformly continuous function, there exists $\delta > 0$ such that $q(x, y) \leq \delta \Rightarrow q'(f(x), f(y)) \leq \epsilon$. Since, by hypothesis, $q(x, y) = 0$

then $q(x,y) \leq \delta$. So, $q'(f(x), f(y)) \leq \epsilon$, which is a contradiction. ■

Quasi-metric spaces with uniformly continuous functions form a category, denoted by **QMS**.

Observe that a metric space (X, q^*) could be obtained from a quasi-metric space (X, q) by taking $q^*(x,y) = max\{q(x,y), q(y,x)\}$ (the associated metric).

Proposition 7 *[Acióly and Bedregal, 1997] De£ne* $q : \mathbb{I}(\mathbb{R}) \times \mathbb{I}(\mathbb{R}) \longrightarrow \mathbb{R}^+$ *by* $q([r,s],[t,u]) = max\{0, t-r, s-u\}$. *Then* q *is a quasi-metric on* $\mathbb{I}(\mathbb{R})$.

This quasi-metric on the real intervals is interesting mathematically and computationally. Mathematically because it allows us to develop a theory of Cauchy sequence, limits, continuity, etc. Computationally because it provides a mean of endowing the interval space with the Scott topology whose computational characteristic is well known [Scott, 1972], [Smyth, 1992] and [Mislove, 1998].

In what follows we will show that **QMS** is an Acióly-Scott interval category.

6.1. The functor ^{op}

Lemma 6 *Let* $\mathbf{X} = (X, q)$ *be a quasi-metric space. Then* $q^{op} : X^{op} \times X^{op} \longrightarrow \mathbb{R}^+$ *de£ned by* $q^{op}(x,y) = q(y,x)$[4], *where* $X^{op} = X$ *is a quasi-metric on* X^{op}.

Proof: We will prove that the three conditions of de£nition 4 are satis£ed.

1. $q^{op}(x,x) = q(x,x) = 0$.
2. $$\begin{aligned} q^{op}(x,z) &= q(z,x) \\ &\leq q(z,y) + q(y,x) \\ &= q(y,x) + q(z,y) \\ &= q^{op}(x,y) + q^{op}(y,z) \end{aligned}$$
3. If $q^{op}(x,y) = q^{op}(y,x) = 0$ then $q(y,x) = q(x,y) = 0$. So, $x = y$.

■

Theorem 1 *Let* $\mathbf{X} = (X, q)$ *be a quasi-metric space. Then* $\mathbf{X}^{op} = (X^{op}, q^{op})$ *is a quasi-metric space.*

Proof: Straightforward from lemma 6. ■

Let $\mathbf{X} = (X, q)$ and $\mathbf{Y} = (Y, q')$ be quasi-metric spaces and $f : X \longrightarrow Y$ be a uniformly continuous function. De£ne $f^{op} : X^{op} \longrightarrow Y^{op}$ by $f^{op}(x) = f(x)$.

Theorem 2 f^{op} *is a uniformly continuous function.*

Proof: Trivially f^{op} is a well de£ned function. Let $\epsilon > 0$ then, since f is uniformly continuous, there exists $\delta > 0$ such that if $q(x,y) \leq \delta$ then $q'(f(x), f(y)) \leq \epsilon$. That is, if $q^{op}(y,x) \leq \delta$ then $(q')^{op}(f^{op}(y), f^{op}(x)) \leq \epsilon$. So, f^{op} is uniformly continuous. ■

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[4] q^{op} is known as the *conjugate* of q [Smyth, 1992].

6.2. The functor $\mathbb{I}$

Let $\mathbf{X} = (X, q)$ be a quasi-metric space. We de£ne the set $\mathbb{I}(X)$ by

$$\mathbb{I}(X) = \{[x, y] \ : \ x, y \in X \text{ and } q(y, x) = 0\},$$

and de£ne the function $\mathbb{I}(q) : \mathbb{I}(X) \longrightarrow \mathbb{R}^+$ by $\mathbb{I}(q)([r, s], [t, u]) = max\{0, q(r, t), q(u, s)$.

Lemma 7 *The pair* $(\mathbb{I}(X), \mathbb{I}(q))$ *is a quasi-metric space.*

Proof: Axioms $i)$ and $iii)$ in the de£nition of quasi-metric spaces are trivially veri£ed. Let us, verify axiom $ii)$:

If $\mathbb{I}(q)([r, s], [v, w]) = 0$ then trivially for any $[t, u]$, $\mathbb{I}(q)([r, s], [v, w]) \leq \mathbb{I}(q)([r, s], [t, u]) + \mathbb{I}(q)([t, u], [v, w])$.

If $\mathbb{I}(q)([r, s], [v, w]) > 0$ then $\mathbb{I}(q)([r, s], [v, w]) = max\{q(r, v), q(w, s)\}$. But $q(r, v) \leq q(r, t) + q(t, v)$ and $q(w, s) \leq q(w, u) + q(u, s)$. So,

$$\begin{aligned} max\{q(r, v), q(w, s)\} &\leq max\{q(r, t) + q(t, v), q(w, u) + q(u, s)\} \\ &= max\{q(r, t), q(w, u)\} + max\{q(t, v), q(u, s)\} \end{aligned}$$

and therefore, $\mathbb{I}(q)([r, s], [v, w]) \leq \mathbb{I}(q)([r, s], [t, u]) + \mathbb{I}(q)([t, u], [v, w])$. ■

Let $\mathbf{X} = (X, q)$ and $\mathbf{Y} = (Y, q')$ be quasi-metric spaces and $f : X \longrightarrow Y$ be a uniformly continuous function. De£ne $\mathbb{I}(f) : \mathbb{I}(X) \longrightarrow \mathbb{I}(Y)$ by

$$\mathbb{I}(f)([x, y]) = [f(x), f^{op}(y)].$$

Theorem 3 $\mathbb{I}(f)$ *is a uniformly continuous function.*

Proof: First, we will prove that $\mathbb{I}(f)$ is a well de£ned function. If $[x, y] \in \mathbb{I}(X)$ then $q(y, x) = 0$. So, by proposition 6 $q'(f(y), f(x)) = 0$. Therefore, $[f(x), f(y)] = [f(x), f^{op}(y)] \in \mathbb{I}(Y)$.

We will prove next that $\mathbb{I}(f) : \mathbb{I}(X) \longrightarrow \mathbb{I}(Y)$ is uniformly continuous.

Let $\epsilon > 0$ and $[r, s], [t, u] \in \mathbb{I}(X)$. Then by de£nition 5, there exists $\delta > 0$ such that if $q(r, t) \leq \delta$ then $q'(f(r), f(t)) \leq \epsilon$ and if $q(u, s) \leq \delta$ then $q'(f(u), f(s)) \leq \epsilon$.

$\mathbb{I}(q)([r, s], [t, u]) = 0$ or $\mathbb{I}(q)([r, s], [t, u]) = max\{q(r, t), q(u, s) \leq \delta$.

If $\mathbb{I}(q)([r, s], [t, u]) = 0$, then $q(t, r) = 0$ and $q(s, u) = 0$, and therefore $q'(f(t), f(r)) = 0$ and $q'(f(s), f(u)) = 0$. So

$$\mathbb{I}(q)([f(r), f(s)], [f(t), f(u)]) = \mathbb{I}(q)(\mathbb{I}(f)([r, s]), \mathbb{I}(f)([t, u])) = 0.$$

Therefore $\mathbb{I}(q)(\mathbb{I}(f)([r, s]), \mathbb{I}(f)([t, u])) \leq \epsilon$.

If $\mathbb{I}(q)([r, s], [t, u]) = max\{q(r, t), q(u, s)\}$ then $q(r, t) \leq \delta$ and $q(u, s) \leq \delta$. Hence, $q'(f(r), f(t)) \leq \epsilon$ and $q'(f(u), f(s)) \leq \epsilon$. So,

$max\{q'(f(r), f(t)), q'(f(u), f(s))\} \leq \epsilon$, i.e. $\mathbb{I}(q)(\mathbb{I}(f)([r, s]), \mathbb{I}(f)([t, u])) \leq \epsilon$. ■

What we had proved so far is that $\mathbb{I} : \mathbf{QMS} \longrightarrow \mathbf{QMS}$ is a covariant functor.

Lemma 8 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be quasi-metric spaces. Then* $\mathbb{I}(X \times Y)$ *is isomorphic to* $\mathbb{I}(X) \times \mathbb{I}(Y)$.

Proof: Let $f : \mathbb{I}(X \times Y) \longrightarrow \mathbb{I}(X) \times \mathbb{I}(Y)$ be de£ned by

$$f([(a, b), (c, d)]) = ([a, c], [b, d]).$$

It is easy to show that f is a well de£ned uniformly continuous isomorphism. ■

6.3. The Natural Transformation

Let $\mathbf{X}$ be a quasi-metric space. De£ne $m(X) : \mathbb{I}(X) \longrightarrow X \times X^{op}$ by

$$m(X)([x, y]) = (x, y).$$

Lemma 9 $m(X) : \mathbb{I}(X) \longrightarrow X \times X^{op}$ *is uniformly continuous.*

Proof: Let $\epsilon > 0$. If $\mathbb{I}(q)([r, s], [t, u]) \leq \epsilon$ (considering $\delta = \epsilon$) then or $\mathbb{I}(q)([r, s], [t, u]) = 0$ or $max\{q(r, t), q(u, s)\} \leq \epsilon$.

If $\mathbb{I}(q)([r, s], [t, u]) = 0$ then $q(t, r) = 0$ and $q(s, u) = 0$ (i.e. $q^{op}(u, s) = 0$). So, $q \times q^{op}((t, u), (r, s)) = 0$. Thus, $q \times q^{op}(m(X)([t, u]), m(X)([r, s])) = 0$. Therefore, $q \times q^{op}(m(X)([t, u]), m(X)([r, s])) \leq \epsilon$.

If $max\{q(r, t), q(u, s)\} \leq \epsilon$ then $q(r, t) \leq \epsilon$ and $q(u, s) \leq \epsilon$ (i.e. $q^{op}(s, u) \leq \epsilon$). So, $q \times q^{op}((r, s), (t, u)) \leq \epsilon$. Therefore, $q \times q^{op}(m(X)([t, u]), m(X)([r, s])) \leq \epsilon$. ■

Lemma 10 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be quasi-metric spaces. Let* $f : X \longrightarrow Y$ *be a uniformly continuous map. Then the following diagram*

$$\begin{tikzcd}
\mathbb{I}(X) \arrow[r, "m(X)"] \arrow[d, "\mathbb{I}(f)"'] & X \times X^{op} \arrow[d, "f \times f^{op}"] \\
\mathbb{I}(Y) \arrow[r, "m(Y)"] & Y \times Y^{op}
\end{tikzcd}$$

commutes.

Proof: Let $[x, y] \in \mathbb{I}(X)$. Then

$$\begin{aligned} f \times f^{op}(m(X)([x,y])) &= f \times f^{op}(x,y) \\ &= (f(x), f^{op}(y)) \\ &= m(Y)([f(x), f^{op}(y)]) \\ &= m(Y)(\mathbb{I}(f)([x,y])). \end{aligned}$$

Hence, $(f \times f^{op}) \circ m(X) = m(Y) \circ \mathbb{I}(f)$.

Therefore the above diagram commutes. ■

The above lemmas guarantees that m is a natural transformation from functor $\mathbb{I}$ to functor $Prod_{op}$.

6.4. The Functor *F*

Let $(\mathbf{X}, q)$ be a quasi-metric space. De£ne the poset $F(X) = (X, \leq_X)$ where

$$x \leq_X y \Leftrightarrow q(y, x) = 0.$$

Clearly, $\leq_X$ is a partial order on X.

Lemma 11 *If $f : X \longrightarrow Y$ is a uniformly continuous map then f is monotonic with respect to the above partial order.*

Proof: Straightforward from proposition 6. ■

De£ne $F(f) = f$. The above lemma guarantees that $F(f)$ is a morphism from the poset $F(X)$ to the poset $F(Y)$. Thus $F : \mathbf{QMS} \longrightarrow \mathbf{POSET}$ is a covariant functor.

Proposition 8 $(\mathbf{QMS}, {}^{op}, \mathbb{I}, m)$ *is an Aciόly-Scott interval category.*

Proof: Properties 1), 2), 3) and 4) of de£nition 3 follows from the above discussions. Property 5) follows straightforward from the de£nition of F, m, ^{op} and $\mathbb{I}$. We will only show that $F(\mathbb{I}(X)) = I(F(X))$ for all quasi-metric space $\mathbf{X}$.

Notice that

$$\begin{aligned} [r,s] \leq_{\mathbb{I}(X)} [t,u] &\Leftrightarrow \mathbb{I}(q)([t,u],[r,s]) = 0 \\ &\Leftrightarrow q(r,t) = 0 \text{ and } q(u,s) = 0 \\ &\Leftrightarrow t \leq_{F(X)} r \text{ and } u \leq_{F(X)} s \\ &\Leftrightarrow t \leq_{F(X)} r \text{ and } s \leq_{F(X)}^{op} u \\ &\Leftrightarrow [r,s] \sqsubseteq [t,u] \text{ in } I(F(X)). \end{aligned}$$

■

7. Conclusions

We de£ned Aciόly-Scott interval categories in order to generalize usual results of the Moore interval theory to other kind of categories.

The main difference of this work with a previous one [Callejas-Bedregal and Bedregal, 2001a] which also generalize the Moore interval theory in a categorical sight, rest in the order considered to de£ne the interval constructor. The choice in this paper of the information order is better from a computational point of view, because it re¤ect the information or approximation nature of an interval as well as the error bound [Moore, 1979]. That is, an interval $[a, b] \sqsubseteq [c, d]$ means that the information or approximation given by the interval $[c, d]$ on an exact solution is better than the one provided by $[a, b]$ and therefore the length of the maximal error (error bound) in $[c, d]$ is lesser than the one provided by $[a, b]$[5].

We showed that some important categories are Aciòly-Scott interval categories, such as the category **QMS** of quasi-metric spaces with uniformly continuous functions, and several domain categories. This is an interesting result since **QMS** is not a subcategory of **POSET** and it is important in computer science as showed in [Smyth, 1992], [Aciòly and Bedregal, 1997], [J.S. Jung and Choi, 2000] and [Gregori and Romaguera, 2000]. It is clear that for a category be a non trivial Aciòly-Scott interval category it is necessary that it embeds (implicit or explicitly) a partial order (different of equal for some object) on their objects. But not necessarily all category with a partial order embedding is a non trivial Aciòly-Scott interval category.

As a secondary product we de£ne an interval constructor I on the **POSET** category which generalizes the interval constructor on the real set. This result is important in order to have a formal treatment of parametric interval data type. Therefore, we are given a theoretical foundations to develop programming languages with such parametric interval data type as primitive type.

Further work includes:

1. To de£ne interval categories in an intrinsic way;
2. To prove that some categories of continuous posets, such as the category of ω-continuous consistently complete cpo´s whose reverse is also an ω-continuous consistently complete cpo and continuous functions, are Aciòly-Scott interval categories;
3. To prove that the interval functor is well behaved under other domain constructors; and
4. To extend interval arithmetic for some Aciòly-Scott interval categories.

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[5]Although, for calculating the maximal errors of a real interval, we need arithmetic operations, turning this process not generic since not every Aciòly-Scott category has objects with intrinsic arithmetic operations, the intuitive idea is that when smaller is the interval smaller is the error.

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