A Normal Form which Preserve 1-Tautologies and 0-Contradictions in a Class of Residuum-based Propositional Fuzzy Logics

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***Abstract*— Each t-norm determine in the propositional fuzzy language a set of 1-tautologies, i.e. formulas which are 1 for each evaluation, and the set of 0-contradictions, i.e. formulas which are 0 for each evaluation.**

In this paper we will introduce a normal form which for any continuous t-norm preserves partially 1-tautology, i.e. if a formula is a 1-tautology then their normal form is also a 1-tautology but where the reverse not always is hold. This normal form preserve, for t-norm without zero divisors, 0-contradictions i.e. a formula is 0-contradiction iff their normal form also is 0-contradiction. Finally, we will showed that the normal form proposed could be used to implement an automated theorem proving for a class of residuum-based of propositional fuzzy logics.

I. Introduction

Schweizer and Sklar in [14] introduced the triangular norm notion in order to model the distance in probabilistic metric spaces.

Alsina, Trillas and Valverde in [1] studied t-norms as models for conjunction and disjunction connectives of fuzzy logics which generalize the several definitions for these connectives provides, among others, by Lotfi Zadeh in [19], Bellman and Zadeh in [3], [4] and Yager in [18] (which define a general class of interpretations). Each t-norm determine a different set of true formulas (1-tautologies) and false formulas (0-contradictions).

Normal forms in logics are usually used to automated theorem proving, geometric proof of the completeness, analytical and geometrical checking of tautologies, etc. [17]. The disjunctive and conjunctive normal form (DNF and CNF, respectively) for propositional classic logic are equivalent. Nevertheless, in fuzzy logic, DNF and CNF not necessarily are equals. In fact, Tursken in [15], interpreting conjunction as the t-norm and disjunction as the respective t-conorm, proved that for some families of t-norms $DNF(\alpha) \leq CNF(\alpha)$ for each value attributed to propositional symbols. When the t-norm is the Gödel (also known as minimal t-norm) both normal forms are equivalent [11]. A problem with CNF and DNF is that it is not describe arbitrary fuzzy logical operations [11]. For example, how define a CNF and DNF for $\alpha = p \to (q \to p)$ in such a way that when the connective $\to$ is interpreted for the residuum operators, the CNF (DNF) of α also be a 1-tautology for each t-norm and the CNF (DNF) of $\alpha \to \mathsf{f}$, where f is evaluated as 0, also be a 0-contradiction for each t-norm? this question at the present has not answered satisfactory. There exists several other versions for disjunctive and conjunctive normal form for fuzzy logics, such as some one provided by Tursken in [16], Kreinovich, Nguyen and Sprecher in [11], Perfilieva in [13], etc. But, all these versions have the same problem with the implication connective when modelled by residuum operators and therefore are not adequate for triangular propositional fuzzy logic [2], i.e. logics modelled by t-norms.

The most of automated theorem proving are based on the refutation principle. The fundamental of a normal form is that the resulted formula not necessarily will be equivalent to original formula, but is necessary that the contradiction (for refutation methods) of the normal form implies in the contradiction of the original formula and that the normal form of any contradiction also be a contradiction. Also is necessary that in the logic a formula is universal valid iff their negation is a contradiction. Analogously, if the method is not of refutation, then the normal form only must guarantee that the universal validness of the normal form implies in the universal validness of the original formula and vice-versa.

Nevertheless, another problem with the classical DNF and CNF is that these not preserves 1-tautologies neither 0-contradiction. For example, the formula

$$\alpha' = ((\alpha \to \beta) \to \gamma) \to (((\beta \to \alpha) \to \gamma) \to \gamma)$$

is a BL axiom and therefore is a 1-tautology for any t-norm. However, their classical conjunctive normal form (follows the algorithm in [5]) is the formula:

$$(\neg\alpha\vee\beta\vee\alpha\vee\gamma)\wedge(\neg\gamma\vee\neg\beta\vee\alpha\vee\gamma)\wedge(\neg\alpha\vee\beta\vee\neg\gamma\vee\gamma)\wedge(\neg\gamma\vee\gamma)$$

which is not a 1-tautology for the product t-norm neither for Gödel t-norm. Since $DNF \leq CNF$ then the DNF of α' also is not a 1-tautology for these t-norms. Analogously, the negation of α' is a 0-contradiction for all t-norm but their DNF, that is the formula

$$(\alpha\wedge\neg\beta\wedge\neg\alpha\wedge\neg\gamma)\vee(\gamma\wedge\beta\wedge\neg\alpha\wedge\neg\gamma)\vee(\alpha\wedge\neg\beta\wedge\gamma\wedge\neg\gamma)\vee(\gamma\wedge\neg\gamma),$$

is not a 0-contradiction for the Lukasiewicz t-norm.

In this paper we will introduce a normal form for residuum-based propositional fuzzy logics [10] also known by triangular propositional logics [2]. The languages for these logics are based on two connectives: strong conjunction and implication, which are interpreted as a t-norm and their residuum, respectively. The other connectives can be defined from this in a canonical way. We will prove that our normal form preserves 0-contradictions for any t-norm without zero divisors, and preserves partially 1-tautologies for any t-norm, in the sense that if a formula is 1-tautology for some t-norm then their normal form also is, but the reverse not necessarily is hold, up to least the t-norm be the Gödel. Finally, we will showed that the normal form proposed could be used to implement an automated theorem proving for a class of residuum-based propositional fuzzy logics.

II. T-NORMS AND RESIDUUM

In this section we will give the notions of t-norms and their residuum operator, as well as, several properties of them. Nevertheless, because are well known results, the proves will be omitted.

A mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is a **triangular norm**, t-norm in short, if it satisfy the follow properties:

1) Symmetry: for each $x, y \in [0,1]$, $T(x,y) = T(y,x)$,
2) Associativity: for each $x,y,z \in [0,1]$, $T(x,T(y,z)) = T(T(x,y),z)$,
3) Monotonicity: for each $x_1,y_1,x_2,y_2 \in [0,1]$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ then $T(x_1,y_1) \leq T(x_2,y_2)$ and
4) One identity: for each $x \in [0,1]$, $T(x,1) = x$.

Some basic t-norms are:

1) Gödel or minimum: $T_G(x,y) = \min\{x,y\}$
2) Lukasiewicz: $T_L(x,y) = \max\{x+y-1,0\}$
3) Product: $T_P(x,y) = xy$
4) Weak:

$$T_W(x,y) = \begin{cases} \min\{x,y\} & \text{, if } \max\{x,y\} = 1 \\ 0 & \text{, otherwise} \end{cases}$$

5) Hamacher: For each $\gamma \geq 0$ define

$$T_{H,\gamma}(x,y) = \frac{xy}{\gamma + (1-\gamma)(x+y-xy)}$$

6) Dubois and Prade: For each $\alpha \in (0,1)$ define

$$T_{DP,\alpha}(x,y) = \frac{xy}{\max\{x,y,\alpha\}}$$

We can establish a partial order on t-norms as follow:

$$T_1 \leq T_2 \text{ iff } \forall x,y \in [0,1],\;\; T_1(x,y) \leq T_2(x,y)$$

Proposition 2.1: Let T be a t-norm. Then

$$T_W \leq T \leq T_G$$

Corollary 2.1: Let T be a t-norm. Then $T(x,y) \leq y$ for each $x,y \in [0,1]$.

Proposition 2.2: $T(x,y) = 1$ iff $x = y = 1$

Lemma 2.1: For each $x,y \in [0,1]$, if $x \leq T(y,z)$ then $x \leq y$ and $x \leq z$.

Proposition 2.3: Let T be a t-norm. Then $T(0,y) = 0$ for each $y \in [0,1]$.

An element $x \neq 0$ is called **zero divisor** if there exists $y \neq 0$, such that $T(x,y) = 0$. For example, $T_W(0.5, 0.5) = 0$ and therefore 0.5 is a zero divisor, in the hold for T_W each $x \in (0,1)$ is a zero divisor. So,

Proposition 2.4: Let T be a t-norm without zero divisors $T(x,y) = 0$ iff $x = 0$ or $y = 0$.

A t-norm T is called **Archimedean** if for each $x \in (0,1)$, $T(x,x) < x$. A continuous Archimedean t-norm which has at least one zero divisor is called **nilpotent** and is called **strict** otherwise. It implies in that a t-norm T is strict iff for each $x,y,z \in [0,1]$ such that $x < y$ and $0 < z$, $T(x,z) < T(y,z)$ [10], [12].

From each t-norm T we can obtain a model for the implication through of the residuum operator $\Rightarrow$: $[0,1] \times [0,1] \longrightarrow [0,1]$ defined by:

$$x \Rightarrow y = \sup\{z \mid T(x,z) \leq y\}$$

For example the residuum derived from T_L, T_G and T_W are defined, respectively, by

$$x \Rightarrow_L y = \begin{cases} 1 & \text{, if } x \leq y \\ 1-x+y & \text{, otherwise} \end{cases}$$

$$x \Rightarrow_G y = \begin{cases} 1 & \text{, if } x \leq y \\ y & \text{, otherwise} \end{cases}$$

$$x \Rightarrow_W y = \begin{cases} 1 & \text{, if } x \neq 1 \\ y & \text{, if } x = 1 \end{cases}$$

Proposition 2.5: Let T be a t-norm and $\Rightarrow$ their residuum. If $x \leq y$ then $x \Rightarrow y = 1$

If the t-norm is continuous [8], [2] in the usual sense then

$$x \Rightarrow y = \max\{z \mid T(x,z) \leq y\}$$

Proposition 2.6: Let T be a continuous t-norm and $\Rightarrow$ their residuum. Then

$$x \Rightarrow y = 1 \text{ iff } x \leq y$$

Notice that the continuity of T is fundamental in the above proposition. For example, T_W is not continuous and $0.5 \Rightarrow_W 0.3 = 1$.

Proposition 2.7: Let T_1 and T_2 t-norms. If $T_1 \leq T_2$ then $\Rightarrow_2 \leq \Rightarrow_1$, where as usual $\Rightarrow_2 \leq \Rightarrow_1$ if for each $x,y \in [0,1]$, $x \Rightarrow_2 y \leq x \Rightarrow_1 y$.

Proposition 2.8: Let T be a t-norm. If $x \Rightarrow y = 0$ then $y = 0$ and $x \neq 0$.

Proposition 2.9: Let T be a t-norm without zero divisors. If $y = 0$ and $x \neq 0$ then $x \Rightarrow y = 0$

III. 1-Tautologies and 0-Contradictions

First we will defined a formal language for propositional fuzzy logic. Let $\Sigma_P = \{p_i \mid i \in I\}$ be a set of propositional symbols, where I is a set of index. Define L_P as being the lesser set contain Σ_P, the constant f and such that if $\alpha, \beta \in L_P$ then $(\alpha\&\beta), (\alpha \rightarrow \beta) \in L_P$. As usual, are used rules for eliminate some parentheses follow priority criterium. Also, other connectives can be seen as abbreviation of formulas in L_P. As usual in fuzzy logic we have the follows one:

$$\alpha \wedge \beta \stackrel{\text{def}}{=} \alpha\&(\alpha \rightarrow \beta)$$
$$\alpha \vee \beta \stackrel{\text{def}}{=} ((\alpha \rightarrow \beta) \rightarrow \beta) \wedge ((\beta \rightarrow \alpha) \rightarrow \alpha)$$
$$\neg\alpha \stackrel{\text{def}}{=} \alpha \rightarrow \mathsf{f}$$
$$\alpha \leftrightarrow \beta \stackrel{\text{def}}{=} (\alpha \rightarrow \beta)\&(\beta \rightarrow \alpha)$$

The connective $\&$ is called **strong conjunction** while $\wedge$ is called min-conjunction [7].

An evaluation of propositional symbols is any function $e : \Sigma_P \longrightarrow [0,1]$. Given a t-norm T an their residuum $\Rightarrow$ we can extend an evaluation e for a function $e_T : L_P \longrightarrow [0,1]$ as follow:

$$e_T(p) = e(p) \text{ for each } p \in \Sigma_P, \tag{1}$$

$$e_T(\mathsf{f}) = 0, \tag{2}$$

$$e_T(\alpha\&\beta) = T(e_T(\alpha), e_T(\beta)), \text{ and} \tag{3}$$

$$e_T(\alpha \rightarrow \beta) = e_T(\alpha) \Rightarrow e_T(\beta). \tag{4}$$

A formula $\alpha \in L_P$ is a 1-tautology w.r.t. the t-norm T, or in short is a 1_T-tautology, if for each evaluation e, $e_T(\alpha) = 1$. For example, the formula $\alpha \rightarrow \alpha$, trivially is a 1_T-tautology for any t-norm T. But there exists formulas which are 1-tautology w.r.t. some t-norm but not for some other t-norm. For example, the formula $\neg\neg\alpha \rightarrow \alpha$ is a 1_{T_L}-tautology but is not a 1_{T_G}-tautology. Analogously, the formula $(\alpha \rightarrow \beta) \rightarrow \alpha$ is a 1_{T_G}-tautology but is not a 1_{T_L}-tautology. Thus, the order between the t-norms not implies in an inclusion of their sets of 1-tautologies.

Proposition 3.1: Let $\alpha_1, \ldots, \alpha_n \in L_P$ with $n \geq 1$ and T be a t-norm. Then $\alpha_1\&\ldots\&\alpha_n$ is a 1_T-tautology iff for each $i = 1, \ldots, n$ α_i is a 1_T-tautology.

Proof: Straightforward from proposition 2.2 and equation 3. ■

Analogously to 1-tautology, a formula $\alpha \in L_P$ is called a **0-contradiction** w.r.t. a t-norm T, 0_T**-contradiction** in short, if for each evaluation e, $e_T(\alpha) = 0$.

Proposition 3.2: Let T be a t-norm and $\alpha \in L_P$. If α is a 1_T-tautology then $\neg\alpha$ is a 0_T-contradiction.

Proof: If α is a 1_T-tautology then $e_T(\alpha) = 1$ for each evaluation e. So,

$$\begin{aligned} e_T(\neg\alpha) &= e_T(\alpha \rightarrow \mathsf{f}) \\ &= e_T(\alpha) \Rightarrow e_T(\mathsf{f}) \\ &= 1 \Rightarrow 0 \\ &= \sup\{z \mid T(1,z) = 0\} \\ &= \sup\{0\} \text{ by one identity} \\ &= 0 \end{aligned}$$

Therefore, $\neg\alpha$ is a 0_T-contradiction. ■

Proposition 3.3: Let T be a continuous t-norm and $\alpha \in L_P$. α is a 0_T-contradiction iff $\neg\alpha$ is a 1_T-tautology.

Proof: If α is a 0_T-contradiction then $e_T(\alpha) = 0$ for each evaluation e. So,

$$\begin{aligned} e_T(\neg\alpha) &= e_T(\alpha \rightarrow \mathsf{f}) \\ &= e_T(\alpha) \Rightarrow e_T(\mathsf{f}) \\ &= 0 \Rightarrow 0 \\ &= 1 \end{aligned}$$

On the other hand, if $\neg\alpha$ is a 1_T-tautology then $e_T(\alpha) \Rightarrow 0 = 1$. So, because T is continuous, by proposition 2.6, $e_T(\alpha) = 0$. Therefore, α is a 0_T-tautology. ■

Proposition 3.4: Let T be a continuous without zero divisors t-norm and $\alpha \in L_P$. If α is a 0_T-contradiction the $\neg\neg\alpha$ is a 0_T-contradiction.

Proof: If α is a 0_T-contradiction then by proposition 3.3, $\neg\alpha$ is a 1_T-tautology and by proposition 3.2, $\neg\neg\alpha$ is 0_T-contradiction.

If $\neg\neg\alpha$ is a 0_T-contradiction then $(e_T(\alpha) \Rightarrow 0) \Rightarrow 0 = 0$ for each evaluation e. Suppose that α is not a 0_T-contradiction, then for some evaluation e', $e'_T(\alpha) \neq 0$. But, because T is continuous, by proposition 2.6, $e'_T(\alpha) \Rightarrow 0 \neq 0$. Since T has not zero divisors, if $T(e'_T(\alpha), z) = 0$ then $z = 0$. So, $\max\{z \mid T(e'_T(\alpha), z) = 0\} = \max\{0\} = 0$ and therefore $e'_T(\alpha) \Rightarrow 0 = 0$ which is a contradiction. Therefore α is a 0_T-contradiction. ■

Proposition 3.5: Let T be a continuous t-norm and $\alpha \in L_P$. If α is a 1_T-tautology then $\neg\neg\alpha$ is a 1_T-tautology.

Proof: If α is a 1_T-tautology then, by proposition 3.2, $\neg\alpha$ is a 0_T-contradiction. So, by proposition 3.3, $\neg\neg\alpha$ is a 1_T-tautology. ■

Proposition 3.6: Let T be a continuous t-norm such that $\neg\neg\alpha \rightarrow \alpha$ is a 1_T-tautology for each $\alpha \in L_P$. If $\neg\beta$ is a 0_T-contradiction then β is a 1_T-tautology.

Proof: By above observation, if $\neg\beta$ is a 0_T-contradiction then $\neg\neg\beta$ is a 1_T-tautology. By hypothesis, $\neg\neg\beta \rightarrow \beta$ is a 1_T-tautology. Since T is continuous, then by proposition 2.6, $e_T(\neg\neg\beta) \leq e_T(\beta)$ for each evaluation e. So, $e_T(\beta) = 1$ for each evaluation e and therefore β is a 1_T-tautology. ■

IV. Conjunctive Implicative Normal Form

We will establish a normal form to propositional fuzzy language L_P. $\alpha \in L_P$ is an **implicative formula** if $\&$ no occurs in α, i.e. all propositional symbol p and f are implicative formulas and if α_1 and α_2 are implicative formulas then $\alpha_1 \Rightarrow \alpha_2$ also is an implicative formula. A formula α is in the **conjunctive implicative normal form**, (CINF in short) if it is a strong conjunction of implicative formulas, i.e. each implicative formula is CINF and if α_1 and α_2 are CINF then $\alpha_1\&\alpha_2$ also is CINF.

The CINF of a formula $\alpha' \in L_P$, denoted by $CINF(\alpha')$, is obtained applying exhaustively the follows substitutions:

1) right-& substitution: if a subformula $\alpha \to (\beta\&\gamma))$ occur in α' then substitute $\alpha \to (\beta\&\gamma)$ by $(\alpha \to \beta)\&(\alpha \to \gamma)$ in α'
2) left-& substitution: if a subformula $(\alpha\&\beta) \to \gamma$ occur in α' then substitute $(\alpha\&\beta) \to \gamma$ by $\alpha \to (\beta \to \gamma)$ in α'

Remark 4.1: When a normal form is used in a method of refutation to prove that some formula α is a 1-tautology, it is necessary that:

1) If the normal form of $\neg\alpha$ is a 0-contradiction then $\neg\alpha$ also is a 0-contradiction, and
2) If $\neg\alpha$ is a 0-contradiction then α is a 1-tautology.

Thus, if we can prove using some method that the normal form of the negation of a formula is a 0-contradiction we would have certainty that the original formula is a 1-tautology. Still, it is not sufficient, because it is not guarantee that all 1-tautology could be proved in this way, i.e. can be exist a 1-tautology α such that the normal form of their negation is not a 0-contradiction. Therefore, in order to turn reliable the method, also is necessary guarantee that

3) If α is a 1-tautology then $\neg\alpha$ is a 0-contradiction and
4) If $\neg\alpha$ is a 0-contradiction then the normal form of $\neg\alpha$ also is a 0-contradiction.

In the case of logics based on t-norm interpretation, we must consider the language L_P and the notions of 1_T-tautology, 0_T-contradictions and the conjunctive implicative normal form.

The items 2) and 3), above, not depend upon the normal form and was proved in propositions 3.6 and 3.3, respectively. In the next of this section we will provide some conditions which must satisfy the t-norms in order to satisfy the items 1 and 4, which depend of the normal form.

Proposition 4.1: Let $\alpha, \beta, \gamma \in L_P$, T a continuous t-norm and e an evaluation. Then

1) If $e_T(\alpha \to (\beta\&\gamma)) = 1$ then $e_T((\alpha \to \beta)\&(\alpha \to \gamma)) = 1$.
2) If $e_T((\alpha\&\beta) \to \gamma) = 1$ then $e_T(\alpha \to (\beta \to \gamma)) = 1$.

Proof: If $e_T(\alpha \to (\beta\&\gamma)) = 1$ then $e_T(\alpha) \Rightarrow T(e_T(\beta), e_T(\gamma)) = 1$. So, by proposition 2.6, $e_T(\alpha) \leq T(e_T(\beta), e_T(\gamma))$. By lemma 2.1 $e_T(\alpha) \leq e_T(\beta)$ and $e_T(\alpha) \leq e_T(\gamma)$. Therefore, by proposition 2.5, $e_T(\alpha) \Rightarrow e_T(\beta) = 1$ and $e_T(\alpha) \Rightarrow e_T(\gamma) = 1$. By equation 4, $e_T(\alpha \to \beta) = 1$ and $e_T(\alpha \to \gamma) = 1$. By proposition 2.2 $T(e_T(\alpha \to \beta), e_T(\alpha \to \gamma)) = 1$ and therefore, by equation 3, $e_T((\alpha \to \beta)\&(\alpha \to \gamma)) = 1$

If $e_T((\alpha\&\beta) \to \gamma) = 1$ then, by proposition 2.5 and equations 3 and 4, $T(e_T(\alpha), e_T(\beta)) \leq e_T(\gamma)$. Suppose that $e_T(\alpha \to (\beta \to \gamma)) \neq 1$ then $e_T(\alpha) > e_T(\beta) \Rightarrow e_T(\gamma)$ and therefore $e_T(\alpha) > \sup\{z \mid T(e_T(\beta), z) \leq e_T(\gamma)\}$. So, by monotonicity of T, $T(e_T(\beta), e_T(\alpha)) > e_T(\gamma)$ which is a contradiction. ■

Remark 4.2: If T is a non-Gödel continuous t-norm then for some $x, y \in [0,1]$, $T(x,y) < T_G(x,y)$. Let $z = \min\{x,y\}$, then $z \Rightarrow_T x = 1$ and $z \Rightarrow_T y = 1$ but, by proposition 2.6, $z \Rightarrow_T T(x,y) < 1$. So, for any evaluation e not always $e_T((\alpha \to \beta)\&(\alpha \to \gamma)) = 1$ implies in $e_T(\alpha \to (\beta\&\gamma)) = 1$. Thus, the reverse of property 1 in proposition 4.1 only is satisfied when the t-norm is the Gödel one.

Proposition 4.2: Let $\alpha, \beta, \gamma \in L_P$ and e an evaluation. Then

1) If $e_{T_G}((\alpha \to \beta)\&(\alpha \to \gamma)) = 1$ then $e_{T_G}(\alpha \to (\beta\&\gamma)) = 1$
2) If $e_{T_G}(\alpha \to (\beta \to \gamma)) = 1$ then $e_{T_G}((\alpha\&\beta) \to \gamma) = 1$

Proof:

1) If $e_{T_G}((\alpha \to \beta)\&(\alpha \to \gamma)) = 1$ then $e_{T_G}(\alpha \to \beta) = 1$ and $e_{T_G}(\alpha \to \gamma) = 1$. So, $e_{T_G}(\alpha) \Rightarrow_{T_G} e_{T_G}(\beta) = 1$ and $e_{T_G}(\alpha) \Rightarrow_{T_G} e_{T_G}(\gamma) = 1$. By proposition 2.6, $e_{T_G}(\alpha) \leq e_{T_G}(\beta)$ and $e_{T_G}(\alpha) \leq e_G(\gamma)$. Thus, $e_{T_G}(\alpha) \leq \min\{e_{T_G}(\beta), e_{T_G}(\gamma)\}$ that is $e_{T_G}(\alpha) \leq T_G(e_{T_G}(\beta), e_{T_G}(\gamma))$ and by proposition 2.6 $e_{T_G}(\alpha) \Rightarrow_G T_G(e_{T_G}(\beta), e_{T_G}(\gamma)) = 1$. By equation 3 and 4, $e_{T_G}(\alpha \to (\beta\&\gamma)) = 1$.
2) If $e_{T_G}(\alpha \to (\beta \to \gamma)) = 1$ then $e_{T_G}(\alpha) \Rightarrow_{T_G} (e_{T_G}(\beta) \Rightarrow_{T_G} e_{T_G}(\gamma)) = 1$. If $e_{T_G}(\beta) \leq e_{T_G}(\gamma)$ then $\min\{e_{T_G}(\alpha), e_{T_G}(\beta)\} \leq e_{T_G}(\gamma)$ and therefore $e_{T_G}((\alpha\&\beta) \to \gamma) = 1$. If $e_{T_G}(\gamma) < e_{T_G}(\beta)$ then $e_{T_G}(\beta) \Rightarrow_{T_G} e_{T_G}(\gamma) = e_{T_G}(\gamma)$. So, $e_{T_G}(\alpha) \Rightarrow_{T_G} (e_{T_G}(\beta) \Rightarrow_{T_G} e_{T_G}(\gamma)) = e_{T_G}(\alpha) \Rightarrow_{T_G} e_{T_G}(\gamma)$ and therefore $e_{T_G}(\alpha) \Rightarrow_{T_G} e_{T_G}(\gamma) = 1$. By proposition 2.6, $e_{T_G}(\alpha) \leq e_{T_G}(\gamma)$. Hence, $\min\{e_{T_G}(\alpha), e_{T_G}(\beta)\} \leq e_{T_G}(\gamma)$ and therefore $e_{T_G}((\alpha\&\beta) \to \gamma) = 1$.

■

Corollary 4.1: Let $\alpha, \beta, \gamma \in L_P$ and e be an evaluation. Then

1) $e_{T_G}((\alpha \to \beta)\&(\alpha \to \gamma)) = 1$ iff $e_{T_G}(\alpha \to (\beta\&\gamma)) = 1$
2) $e_{T_G}(\alpha \to (\beta \to \gamma)) = 1$ iff $e_{T_G}((\alpha\&\beta) \to \gamma) = 1$

Proof: Straightforward of propositions 4.2 and 4.1. ■

Proposition 4.3: Let $\alpha \in L_P$. α is a 1_{T_G}-tautology iff $CINF(\alpha)$ is a 1_{T_G}-tautology.

Proof: Straightforward of definition of CINF and corollary 4.1. ■

Theorem 4.1: Let $\alpha \in L_P$ and T be a continuous t-norm. If α is a 1_T-tautology then $CINF(\alpha)$ is a 1_T-tautology.

Proof: Straightforward from proposition 4.1. ■

Remark 4.3: Let T be a non-Gödel continuous t-norm. By remark 4.2 there exists $\alpha \in L_P$ such that $CINF(\alpha)$ is 1_T-tautology but α is not. That is the reverse of this theorem not always is valid.

Lemma 4.1: Let $\alpha, \beta, \gamma \in L_P$, T be a t-norm without zero divisors and e an evaluation. Then

1) $e_T(\alpha \to (\beta\&\gamma)) = 0$ iff $e_T((\alpha \to \beta)\&(\alpha \to \gamma)) = 0$.
2) $e_T((\alpha\&\beta) \to \gamma) = 0$ iff $e_T(\alpha \to (\beta \to \gamma)) = 0$.

Proof: If $e_T(\alpha \to (\beta\&\gamma)) = 0$ then, by equations 4 and 3, $e_T(\alpha) \Rightarrow T(e_T(\beta), e_T(\gamma)) = 0$. So, by proposition 2.8, $e_T(\alpha) \neq 0$ and $T(e_T(\beta), e_T(\gamma)) = 0$. By proposition 2.4, $e_T(\beta) = 0$ or $e_T(\gamma)) = 0$. So, by proposition 2.9, $e_T(\alpha) \Rightarrow e_T(\beta) = 0$ or $e_T(\alpha) \Rightarrow e_T(\gamma) = 0$. Therefore, by proposition 2.3 and symmetry of T, $T(e_T(\alpha) \Rightarrow$

$e_T(\beta), e_T(\alpha) \Rightarrow e_T(\gamma)) = 0$. So, by equations 4 and 3, $e_T((\alpha \to \beta)\&(\alpha \to \gamma)) = 0$.

If $e_T((\alpha \to \beta)\&(\alpha \to \gamma)) = 0$ then, by equations 3 and 4, $T(e_T(\alpha) \Rightarrow e_T(\beta), e_T(\alpha) \Rightarrow e_T(\gamma)) = 0$. So, because T has not zero divisor, $e_T(\alpha) \Rightarrow e_T(\beta) = 0$ or $e_T(\alpha) \Rightarrow e_T(\gamma) = 0$. Thus, by proposition 2.8, $e_T(\alpha) \neq 0$ and $e_T(\beta) = 0$ or $e_T(\gamma)) = 0$. Therefore, by proposition 2.4, $T(e_T(\beta), e_T(\gamma)) = 0$. So, by equations 3 and 4 and proposition 2.9, $e_T(\alpha \to (\beta\&\gamma)) = 0$.

If $e_T((\alpha\&\beta) \to \gamma) = 0$ then, by equations 4 and 3, $T(e_T(\alpha), e_T(\beta)) \Rightarrow e_T(\gamma) = 0$. So, by proposition 2.8, $T(e_T(\alpha), e_T(\beta)) \neq 0$ and $e_T(\gamma) = 0$. By proposition 2.4, $e_T(\alpha) \neq 0$ and $e_T(\beta)) \neq 0$. So, by proposition 2.9, $e_T(\beta) \Rightarrow e_T(\gamma) = 0$. Thus, again by proposition 2.9, $e_T(\alpha) \Rightarrow (e_T(\beta) \Rightarrow e_T(\gamma)) = 0$. Therefore, by equation 4, $e_T(\alpha \to (\beta \to \gamma)) = 0$.

If $e_T(\alpha \to (\beta \to \gamma)) = 0$ then, by equation 4, $e_T(\alpha) \Rightarrow (e_T(\beta) \Rightarrow e_T(\gamma)) = 0$. So, by proposition 2.8, $e_T(\alpha) \neq 0$, $e_T(\beta) \neq 0$ and $e_T(\gamma) = 0$. Thus, by proposition 2.4, $T(e_T(\alpha), e_T(\beta)) \neq 0$. So, by proposition 2.9, $T(e_T(\alpha), e_T(\beta)) \Rightarrow e_T(\gamma) = 0$. Therefore, by equations 3 and 4, $e_T((\alpha\&\beta) \to \gamma) = 0$. ■

Theorem 4.2: Let T be a t-norm without zero divisors. Then $\alpha \in L_P$ is a 0_T-contradiction iff $CINF(\alpha)$ is a 0_T contradiction.

Proof: Straightforward of lemma 4.1 and definition of $CINF(\alpha)$. ■

A formula $\alpha \in L_P$ is **satisfiable** in a t-norm T if there exists an evaluation e such that $e_T(\alpha) \neq 0$.

Corollary 4.2: Let T be a t-norm without zero divisors. Then $\alpha \in L_P$ is satisfiable in T iff $CINF(\alpha)$ is satisfiable in T.

Proof: Straightforward of theorem 4.2 and definition of satisfiable. ■

V. Final Remarks

The conjunctive implicative normal form here introduced is simples to obtain. The resulted formula has, in the worst case, only 33% more connectives than original formula, while the FCNF Tursken normal form in [16], for example, in the worst case has fifteen times more connectives than the original formula. But the main characteristic of CINF is their potential to will be used in automated theorem proved methods. As said in remark 4.1, as usual, in order to use a normal form in an automated theorem proved method of a logic, it is necessary satisfy the 4 items listed in these remark. For the case of propositional logics based on t-norms, i.e. triangular logics as seen in [2], the satisfaction of these properties are proved in propositions 3.6 and 3.3 and theorem 4.2. Nevertheless, the restrictions imposed in these propositions and theorem are different and perhaps inconsistent, that is, there exists some continuous t-norm without zero divisors and where $((x \Rightarrow 0) \Rightarrow 0) \Rightarrow x = 1$ for any x? if there exists some one then our normal form could be useful to prove theorems in the logics associated to is class of t-norms. In this case, a further work will be develop a computational method to prove that a formula in the conjunctive implicative normal form is a 0-contradiction for any t-norm in this class. Still, if this class is empty then this formal norm cannot used for this proposed. The problem come upon interpretation of negation. In fact, if we consider strong fuzzy negations [6], [10] instead of the canonical negation obtained from a t-norm (more specifically from a residuum), this problem would be solved. However, this would implies in to include the negation in the language and consequently the normal form should to consider its, but it is will be matter for a next research.

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