Continuous Information Systems as a Finite Information Logic for Continuous Domains

Benjamín René Callejas Bedregal
Universidade Federal do Rio Grande do Norte - UFRN
Departamento de Informática e Matemática Aplicada - DIMAp
Laboratório de Lógica e Inteligência Computacional - LabLIC
Campus Universitário - Lagoa Nova
CEP: 59.072-970, Natal-RN, Brazil.
Fone: (55)(84)215-3814 Fax: (55)(84)215-3781
`bedregal@dimap.ufrn.br`

Abstract

Continuous information systems, introduced by Hoofman in [Hoo93], are alternative descriptions for continuous domains, where the idea of information becomes explicit, in the sense that each element of a domain is seen as a collection of information that it "satisfies".

In this work, we will introduce a finite information logic for continuous domains, as a complementary logic to the domain logic of Abramsky [Abr91, BA95]. Here, the logical aspects of a continuous domain are analysed instead of the whole class at all. In order to do it using the continuous information system representation of continuous domains and we relate both structures via a satisfaction relation. From its relation we define several notions, such as: Information content, models, theory, logical consequence, information content order, etc.

Keywords: Domain theory, continuous information systems, logical systems, models, consequence relations.

1 Introduction

Domain theory studies a class of appropriate spaces to solve recursive equations in order to obtain semantic functions to give denotational semantics of programming languages [Sco82, Gun85]. The (algebraic) domain logic was introduced by Abramsky [Abr91] in order to study the logical aspects (a proof system) of domains as it is used in denotational semantics for programming languages. However, algebraic domains does not support a natural and a consistent topologically interpretation of the real interval data type. Nevertheless, continuous domains have a desirable interpretation for this data type.

Analogously to the work of Scott [Sco82] to represent concretely algebraic domains, Hoofman in [Hoo93] introduced the alternative description of continuous information systems to continuous domains. Roughly speaking, a continuous information system is a triple composed by a set of tokens, a consistency predicate and an entailment relation.

The tokens are finite units of information, in the sense that they are information of computations derived from observing a program execution at a finite time with a finite amount of work. Thus, the logical language associated to continuous information systems is the finite observations logic of Vickers [Vic89]. The main difference between Scott information system and continuous information systems is that, of the logical point of view, the entailment relation $\vdash$ in a Scott information system is a consequence relation [Gab94], i.e. is reflexive, monotonic and transitive. Whenever, in a continuous information system the entailment relation not satisfy the property of reflexivity [Hoo93].

The continuous information system representation to turn out the idea of information, in the sense that each element of a domain is seen as a collection of information that it "satisfies". Although the domains and information systems are categorically equivalent, information systems allow us to capture the logical aspects of domains, in the sense that the properties of domains can be derived from the entailment between propositions about the properties of computations.

In [BA95] the Abramsky approach to domain logic [Abr91] was extended and types are interpreted as continuous information systems instead of algebraic domains. The Abramsky domain logic proposed denotational semantics as a logic of program and, hence, studies the logical relations among domains. The finite information logic for continuous domains, introduced here, studies a complementary aspect of Abramsky domain logic, in the sense that the logical aspects of a continuous domain are analysed instead of the whole class of continuous domains at all. Thus, we will use the continuous information systems representation for continuous domains and relate both structures by using a satisfaction relation which will give us several notions, such as, information content, models, theory, logical consequence, theorem, information content order, etc.

2 Continuous Domains

Since the seminal work of Scott and Strachey, several mathematical structures have been widely employed as models for denotational semantics of programming language. Continuous domains, which are, basically, a continuous lattice [Sco72] without, possibly, the top element is one of them. Continuous domain allows us to develop a theory of interval arithmetic and numerical analysis [Aci91]. This theory has the advantage of being constructive and computational, besides of unifying the theory of programming languages semantics and computational mathematics. In what follows is formally define this.

Let $\mathbf{D} = \langle D, \leq \rangle$ be a partially ordered set (poset). A set $\Delta \subseteq D$ is called **directed** if each finite subset has an upper bound, or equivalently, $\forall a, b \in \Delta \;\; \exists c \in \Delta$ such that $a \leq c$ and $b \leq c$. A poset $\mathbf{D}$ is **complete** (for short cpo) if each directed set Δ has a supremum or least upper bound (denoted by $\bigsqcup \Delta$) and has a bottom element. We say that a is **way below** b (denoted by $a \ll b$) if for every directed set Δ such that $b \leq \bigsqcup \Delta$ then $a \leq x$ for some $x \in \Delta$.

Let $\mathrel{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}} x$ be the set $\{y \in D \mid y \ll x\}$. A cpo $\mathbf{D}$ is called **continuous** if, for all $x \in D$, the set $\mathrel{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}} x$ is directed and $x = \bigsqcup \mathrel{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}} x$. A continuous cpo $\mathbf{D}$ where each consistent set (a set has an upper bound in $\mathbf{D}$) has a supremum in D is called **continuous domain**.

An element x of a continuous domain $\mathbf{D} = \langle D, \leq \rangle$ is said to be **total** if $x \leq y$ implies $x = y$. We denote the set of total elements of a domain $\mathbf{D}$ by $Tot(\mathbf{D})$.

Example 2.1 *Let $\mathbb{I}(\mathbb{R}) = \{[r,s] \mid r,s \in \mathbb{R} \text{ and } r \leq s\} \cup \{[-\infty, +\infty]\}$ be the set of closed real intervals with real numbers as end points or simply real intervals. Define on $\mathbb{I}(\mathbb{R})$ the follow partial order:*

$$[r,s] \sqsubseteq [t,u] \text{ iff } [r,s] = [-\infty, +\infty] \text{ or } (r \leq_{\mathbb{R}} t \text{ and } u \leq_{\mathbb{R}} s).$$

The pair $\Re = \langle \mathbb{I}(\mathbb{R}), \sqsubseteq \rangle$ is a continuous domain. The way below relation associated to $\Re$ is defined by

$$[r,s] \ll [t,u] \text{ iff } r <_{\mathbb{R}} t \text{ and } u <_{\mathbb{R}} s.$$

A basis for it continuous domain is the set $\mathbb{I}(\mathbb{Q}) = \{[p,q] \mid p,q \in \mathbb{Q} \text{ and } p <_{\mathbb{Q}} q\}$ of rational intervals with rational numbers as end points. The total elements of $\Re$ are the degenerate intervals, i.e. intervals with the same end point and therefore can be consider as real numbers. Thus,

$$Tot(\Re) = \{[r,r] \mid r \in \mathbb{R}\} \cong \mathbb{R}.$$

The morphisms between continuous domains, called continuous functions, are the monotonic functions w.r.t. the order associated to domains preserving supremum of directed sets. The continuous domains $\mathbf{D} = \langle D, \sqsubseteq_D \rangle$ and $\mathbf{E} = \langle E, \sqsubseteq_E \rangle$ are given isomorphic orders if there exists a bijective continuous function $f : D \longrightarrow E$.

3 Continuous Information Systems

Continuous information systems give a more concrete and logic representation for continuous domains. It can be viewed as prescription or program saying how to build a domain. It has the following formal definition.

Definition 3.1 [*Hoo93*] *A* **continuous information system**, *CIS in short, is a triple $\mathcal{I} = \langle I, Con, \vdash \rangle$, where I is a non empty countable* **information set**, *Con is a non empty subset of $\mathcal{P}^{fin}(I)$ (finite parts of I), called* **consistency predicate**, *and $\vdash$ is a subset of $Con \times I$, named* **entailment relation** *such that*

1. If $X \in Con$ and $Y \subseteq X$ then $Y \in Con$
2. If $a \in I$ then, $\{a\} \in Con$
3. If $X \vdash a$ then, $X \cup \{a\} \in Con$
4. If $Y \subseteq X$, $X \in Con$ and $Y \vdash a$ then, $X \vdash a$
5. If $X \vdash Y$ and $Y \vdash a$ then, $X \vdash a$
6. If $X \vdash a$ then $\exists Y \in Con$ such that $X \vdash Y$ and $Y \vdash a$

The notation $X \vdash Y$ is a shorthand for $\forall a \in Y$, $X \vdash a$.

Example 3.2 *We can to see each rational interval $[p,q]$ as a partial or finite information of some real number r belong to it. Thus, finite set of information (rational intervals) are consistent if all the intervals, inform on a same real number, i.e.*

$$X \in Con_{\mathbb{I}(\mathbb{Q})} \text{ iff } X \subseteq^{fin} \mathbb{I}(\mathbb{Q}) \text{ and } Max(left(X)) < Min(right(X)),$$

where $left(X) = \{p \mid [p,q] \in X$ for some $q \in \mathbb{Q}\}$, $right(X) = \{q \mid [p,q] \in X$ for some $p \in \mathbb{Q}\}$, $Max(S)$ is the greater element of S w.r.t. the usual order, and $Min(S)$ is the lesser element of S.

A entailment relation can be defined as follow:

$$X \vdash_{\mathbb{I}(\mathbb{Q})} [p,q] \text{ iff } p <_{\mathbb{Q}} Max(left(X)) \text{ and } Min(right(X)) < q.$$

The structure $\mathcal{I}_{\Re} = \langle \mathbb{I}(\mathbb{Q}), Con_{\mathbb{I}(\mathbb{Q})}, \vdash_{\mathbb{I}(\mathbb{Q})} \rangle$ is a continuous information system.

A CIS informs about elements of a domain. These elements are identified with the set of information, in the system, which are satisfied by them.

Definition 3.3 [*Hoo93*] *Let $\mathcal{I} = \langle I, Con, \vdash \rangle$ be a CIS. The set $x \subseteq I$ is an* **element** *of $\mathcal{I}$ if the the following conditions are satisfied:*

1. *If $X \subseteq^{fin} x$, then $X \in Con$;*[1].
2. *If $X \subseteq x$ and $X \vdash a$, then $a \in x$;*
3. *If $a \in x$, then $\exists Y \subseteq^{fin} x$ such that $Y \vdash a$.*

Let $|\mathcal{I}|$ be the set of elements of a CIS $\mathcal{I} = \langle I, Con, \vdash \rangle$. The poset $\langle |\mathcal{I}|, \subseteq \rangle$ is called **domain of elements** of the CIS $\mathcal{I}$.

Lemma 3.4 [*Hoo93*] *For all CIS $\mathcal{I} = \langle I, Con, \vdash \rangle$, if $X \in Con$ then $\overline{X} = \{a \in I \mid X \vdash a\} \in |\mathcal{I}|$.*

Theorem 3.5 [*Hoo93*] *For all CIS $\mathcal{I} = \langle I, Con, \vdash \rangle$, $\langle |\mathcal{I}|, \subseteq \rangle$ is a continuous domain, with $\overline{Con} = \{\overline{X} \mid X \in Con\}$ as a countable basis*

Example 3.6 *The domain of elements of the continuous information system $\mathcal{I}_{\Re}$ is an isomorphic order to the continuous domain $\Re$. The isomorphisms is given by the function $f : \mathbb{I}(\mathbb{R}) \longrightarrow |\mathcal{I}_{\Re}|$ defined by:*

$$f([r,s]) = \{[p,q] \in \mathbb{I}(\mathbb{Q}) \mid [p,q] \ll [r,s]\}.$$

There are technical advantages to working with CIS rather than directly with continuous domains. First CIS uses the set theory languages and second the properties of domains can be derived rather than postulated.

[1] $X \subseteq^{fin} Y$ is an abbreviation for "$X \subseteq Y$ and X is finite"

4 Language of Finite Information

An information is said to be finite, if on the hypothesis of its correctness, we can verify it in a finite amount time and work. For example, the information "there exists a blue crow", seemingly false, to verify its truth, all we need is to observe the suppose crow and determinate its blueness, which will be made in a finite amount of time. But, the information "all the crows are black", seemingly true, can not be confirmed in an finite amount of time, because we need to observe all the crows (in the past, present and future). In our examples, because always is possible to determinate in a finite time and with a finite amount of work when a real number belong or not to a rational interval, we can consider each rational interval is a finite information of some real number. The logical operators that preserve the finiteness of the information are the conjunction (finite) and disjunction (perhaps infinite). We, also consider two canonical information: the always true information (T) and the always false information (F).

Definition 4.1 *Let $\mathcal{I} = \langle I, Con, \vdash\rangle$ be a continuous information system. The finite information language of $\mathcal{I}$, denoted by $\mathcal{L}(I)$, is defined recursively as follows:*

1. **The atomic information:**
   (a) *if $a \in I$, then $a \in \mathcal{L}(I)$;*
   (b) T, $\mathsf{F} \in \mathcal{L}(I)$;
2. *if $\alpha, \beta \in \mathcal{L}(I)$, then $\alpha \vee \beta \in \mathcal{L}(I)$ and $\alpha \wedge \beta \in \mathcal{L}(I)$.*
3. *if $\Gamma \subseteq \mathcal{L}(I)$, then $\bigvee_{\alpha\in\Gamma} \alpha \in \mathcal{L}(I)$.*

5 Logical System of a Continuous Domain

Let $\mathbf{D} = \langle D, \leq\rangle$ be a continuous domain. A CIS $\mathcal{I} = \langle I, Con, \vdash\rangle$ is said a **representation of D** if $\mathbf{D}$ and $\langle \mid \mathcal{I} \mid, \subseteq\rangle$ are isomorphic orders. The sentences in $\mathcal{L}(I)$ can be see as information on the elements of $\mathbf{D}$ and conversely, we can to see the elements of $\mathbf{D}$ as models of the information in $\mathcal{L}(I)$. Thus, we can introduce the relation "$\models$" between the elements of a continuous domain and the information of their CIS representation. The relation $\models \subseteq D \times \mathcal{L}(I)$ is defined recursively as follows:

1. $x \models \mathsf{T}$
2. $x \models a$ iff $a \in h(x)$ to each $a \in I$
3. $x \models \alpha \wedge \beta$ iff $x \models \alpha$ and $x \models \beta$.
4. $x \models \alpha \vee \beta$ iff $x \models \alpha$ or $x \models \beta$
5. $x \models \bigvee \Gamma$ iff $x \models \alpha$ for some $\alpha \in \Gamma$

The system $\langle \mathbf{D}, \models, \mathcal{I}\rangle$ is called **Logical System of D**, in short **LS**.

Example 5.1 *Since the continuous domain* $\Re$ *and* $|\mathcal{I}_{\Re}|$ *are isomorphic orders, it have associated a relation* $\models \subseteq \mathbb{I}(\mathbb{R}) \times \mathcal{L}(\mathbb{I}(\mathbb{Q}))$ *which is defined as above. Thus, in the case of the atomic elements of* $\mathcal{L}(\mathbb{I}(\mathbb{Q}))$*, this relations has the follow behaviour*

$$[r,s] \models [p,q] \text{ iff } [p,q] \ll [r,s].$$

Thus, the system $\langle \Re, \models, \mathcal{I}_{\Re} \rangle$ *is a logical system for* $\mathbb{I}(\mathbb{R})$.

According to Shannon [SW49], information reduces the uncertainty, which is relative with the contents of the information. However, Shannon information theory, is determined by a probabilistic measure [SW49]. While for us "reduction of uncertainty" is done by the reduction of the spectrum of elements. Thus, the **contents of an information** in a **LS** is associated with the set of their non models [Car90], i.e.

$$Cont(\alpha) = D - \{x \in D \mid x \models \alpha\}$$

Thus, for example, the contents of an information $[p,q] \in \mathbb{I}(\mathbb{Q})$ is the set

$$Cont([p,q]) = \mathbb{I}(\mathbb{R}) - \{[r,s] \in \mathbb{I}(\mathbb{R}) \mid [r,s] \models [p,q]\} = \{[r,s] \in \mathbb{I}(\mathbb{R}) \mid s \leq p \text{ or } q \leq r\}$$

The following proposition establishes a calculus of information contents to **LS**.

Proposition 5.2 *For all* **LS** $\langle \mathbf{D}, \models, \mathcal{I} \rangle$*, if* $\alpha, \beta \in \mathcal{L}(I)$ *and* $\Gamma \subseteq \mathcal{L}(I)$*, then, the following are valid:*

1. $Cont(\alpha \wedge \beta) = Cont(\alpha) \cup Cont(\beta)$
2. $Cont(\alpha \vee \beta) = Cont(\alpha) \cap Cont(\beta)$
3. $Cont(\bigvee \Gamma) = \bigcap_{\alpha \in \Gamma} Cont(\alpha)$
4. $Cont(\mathsf{T}) = \emptyset$
5. $Cont(\mathsf{F}) = D$

The property 4 says that a true information for all element has an empty content in the system, in other words these information is *irrelevant*. Analogously, the property 5 states that a false information has a greatest content, or in other words is *absurd*.

The dual notion of information content is the notion of **models of an information**. It is the set of objects in the domain which satisfies the information, and can be seen as the function $Mod : \mathcal{L}(I) \longrightarrow \mathcal{P}(D)$ defined by

$$Mod(\alpha) = \{x \in D \mid x \models \alpha\}.$$

Note that, $Mod(\alpha) \cup Cont(\alpha) = D$ and $Mod(\alpha) \cap Cont(\alpha) = \emptyset$.

We can extend the consistence predicate of a CIS to consider $\mathcal{L}(I)$ instead of I. A subset $\Gamma \subseteq^{fin} \mathcal{L}(I)$ is **consistent** if $Mod(\Gamma) \neq \emptyset$, (i.e. if there exists an element in D satisfying all the information in Γ). Notice, that this notion of consistency is coherent with the relation Con in $\mathcal{I}$. For example, if $X \in Con$ then, by lemma 3.4, $\overline{X} \in |\mathcal{I}|$, and therefore $h^{-1}(\overline{X}) \models X$, where h^{-1} is the inverse of the isomorphism (bijection) $h : D \longrightarrow |\mathcal{I}|$. So, $Mod(X) \neq \emptyset$ and consequently, X is consistent.

The inverse notion of models determines all the information in the system about the objects of a domain, it is called **theory of the object** in the **LS**, which can be see as their specification in the **LS**. Thus, the theory of the objects in a **LS**, is the function $Theo : D \longrightarrow \mathcal{L}(I)$ defined by

$$Theo(x) = \{\alpha \in \mathcal{L}(I) \ \mid \ x \models \alpha\}.$$

Thus, for example, the theory of a real interval $[r, s]$ in the logical system of our examples contain the set of rational intervals which inform to it, i.e.

$$\{[p, q] \in \mathbb{I}(\mathbb{Q})) \ \mid \ [p, q] \ll [r, s]\} \subseteq Theo([r, s]).$$

The following proposition states that two diferent objects of a domain satisfies diferent sets of informations in a **LS**.

Proposition 5.3 [*BL94*] *Let $\langle \mathbf{D}, \models, \mathcal{I}\rangle$ be a* **LS** *and let $x, y \in D$. If $x \neq y$ then $Theo(x) \neq Theo(y)$. If $x \leq y$ then $Theo(x) \subseteq Theo(y)$.*

This functions can be extended to a set of informations (Γ) or objects (X), as follows:

$$Cont(\Gamma) = \bigcup_{\alpha\in\Gamma} Cont(\alpha) \qquad Mod(\Gamma) = \bigcap_{\alpha\in\Gamma} Mod(\alpha) \qquad Theo(X) = \bigcap_{x\in X} Theo(x).$$

Proposition 5.4 [*Par83*] *Let $\langle \mathbf{D}, \models, \mathcal{I}\rangle$ be a* **LS***. Mod and $Theo$ are a Galois collection. I.e. if $\Gamma, \Delta \subseteq \mathcal{L}(I)$ and $X, Y \subseteq D$, then*

1. *If $\Gamma \subseteq \Delta$ then $Mod(\Delta) \subseteq Mod(\Gamma)$*
2. *If $X \subseteq Y$ then $Theo(Y) \subseteq Theo(X)$*
3. *$\Gamma \subseteq Theo(Mod(\Gamma))$ and $X \subseteq Mod(Theo(X))$.*

Let $\langle \mathbf{D}, \models, \mathcal{I}\rangle$ be a **LS** and $\Gamma \subseteq \mathcal{L}(I)$. Γ is **complete** iff Γ has an unique model, i.e. $Mod(\Gamma) = \{x\}$ for some $x \in D$. Γ is **complete greatest** iff it becomes inconsistent on the addition of any new information to Γ, i.e. to each $\alpha \in \mathcal{L}(I)$, if $\alpha \notin \Gamma$ then $Mod(\Gamma \cup \{\alpha\}) = D$. Γ is **complete minimally** iff Γ is complete and there does not exist any proper subset $\Delta \subset \Gamma$ such that $Mod(\Delta) = Mod(\Gamma) = \{x\}$. So, Γ is the information set which are sufficient and necessary to specify x.

Proposition 5.5 *Let $S = \langle \mathbf{D}, \models, \mathcal{I}\rangle$ be a* **LS** *and let $h : D \longrightarrow \mid \mathcal{I} \mid$ be the isomorphism between the continuous domain and the domain of elements of I. Then, for each $x \in Tot(\mathbf{D})$ the set $h(x)$ is complete.*

The idea to characterise the notion of complete information is to emphasize that continuous information systems allows us to specify the total element of continuous domains in terms of finite information.

An important concept, which arises when we see a CIS as a logical theory, is that of logical consequence for information.

Definition 5.6 *Let $\langle \mathbf{D}, \models, \mathcal{I}\rangle$ be a* **LS**. *The function $CnI : \mathcal{L}(I) \longrightarrow \mathcal{P}(\mathcal{L}(I))$ defined by $CnI(\alpha) = Theo(Mod(\alpha))$, determines the information which can be deduced from α, i.e. their* **logical consequences**.

Proposition 5.7 *Let $\langle \mathbf{D}, \models, \mathcal{I}\rangle$ be a* **LS** *and $\Gamma, \Delta \subseteq \mathcal{L}(I)$. Then, $CnI(\Gamma \cup \Delta) = CnI(\Gamma) \cup CnI(\Delta)$.*[2]

The notion of logical consequence extends the entailment relation in a CIS $\mathcal{I} = \langle I, Con, \vdash\rangle$ to $\mid\!\sim$, which consider $\mathcal{L}(I)$ instead of I. Thus, given $\Gamma \subseteq \mathcal{L}(I)$ and $\alpha \in \mathcal{L}(I)$, $\Gamma \mid\!\sim \alpha$ if $\alpha \in CnI(\Gamma)$. If $\alpha \in CnI(\emptyset)$ (if $\emptyset \mid\!\sim \alpha$), then α is a **theorem**, denoted by $\mid\!\sim \alpha$.

Proposition 5.8 *Let $\langle \mathbf{D}, \models, \mathcal{I}\rangle$ be a* **LS**. *Then*

1. *if $X \vdash a$, then $X \mid\!\sim a$;*
2. *if $\alpha \in \Gamma$ then $\Gamma \mid\!\sim \alpha$;*
3. *if $\Gamma \mid\!\sim \alpha$, to each $\alpha \in \Delta$, and $\Delta \mid\!\sim \beta$ then $\Gamma \mid\!\sim \beta$;*
4. *if $\Gamma \subseteq \Delta$ and $\Gamma \mid\!\sim \alpha$ then $\Delta \mid\!\sim \alpha$;*
5. *if $\Gamma \mid\!\sim \alpha$ then $\Gamma \cup \{\alpha\}$ is consistent;*
6. *if Γ not is consistent then $\Gamma \mid\!\sim \alpha$ to each $\alpha \in \mathcal{L}(I)$.*

The properties 2, 3 and 4 of $\mid\!\sim$ are knows as reflexivity, transitivity and monotonicity, respectively. So, $\mid\!\sim$ is a **consequence relation** in $\mathcal{L}(I)$ [Gab94].

From the definition of information content we establish some interesting relations on information.

Definition 5.9 *Let $\langle \mathbf{D}, \models, \mathcal{I}\rangle$ be a* **LS** *and $\alpha, \beta \in \mathcal{L}(I)$. The information α is* **greatest** *than or* **more exact** *than the information β, denoted by $\beta \sqsubseteq \alpha$ iff $Cont(\beta) \subseteq Cont(\alpha)$. Analogously, α and β are* **equivalent information**, *denoted by $\alpha \equiv \beta$, iff $Cont(\alpha) = Cont(\beta)$.*

Example 5.10 *The follow information are equivalent w.r.t.* **LS** $\langle \Re, \models, \mathcal{I}_{\Re}\rangle$*:*

1. *if $p_1 = p$, $q_2 = q$, $q_1 < q$ and $p < p_2$ then $[p, q] \equiv [p_1, q_1] \vee [p_2, q_2]$*
2. *if $p_1 < p$, $q < q_2$, $q_1 = q$ and $p_2 = p$ then $[p, q] \equiv [p_1, q_1] \wedge [p_2, q_2]$*
3. $[-\infty, +\infty] \equiv \mathbf{T}$

The *information content order* $\sqsubseteq$ is an intensive order. On the other hand, the dual order, $\sqsupseteq$, models-based, is an extensional information content order in the sense of [Bun74].

[2] The expression $CnI(A)$ is an abreviation of $\bigcup_{\alpha \in A} CnI(\alpha)$.

Proposition 5.11 *Let $\langle \mathbf{D}, \models, \mathcal{I} \rangle$ be a* **LS**, *$\alpha, \beta \in \mathcal{L}(I)$ and $\Gamma \subseteq \mathcal{L}(I)$. Then*

1. $\alpha \vee \beta \equiv \beta \vee \alpha$ *and* $\alpha \wedge \beta \equiv \beta \wedge \alpha$
2. $\alpha \wedge \bigvee_{\beta \in \Gamma} \beta \equiv \bigvee_{\beta \in \Gamma} (\alpha \wedge \beta)$
3. $\alpha \equiv \beta$ *iff* $\alpha \sqsubseteq \beta$ *and* $\beta \sqsubseteq \alpha$
4. $\alpha \sqsubseteq \alpha \wedge \beta$
5. $\alpha \vee \beta \sqsubseteq \alpha$
6. $\mathsf{T} \sqsubseteq \alpha$ *and* $\alpha \sqsubseteq \mathsf{F}$
7. *if* $\alpha \mid\!\sim \beta$ *then* $\beta \sqsubseteq \alpha$

6 Conclusion

In this work, we used continuous information system to give a logical characterisation for continuous domains. Perhaps it is not a novelty, because that original motivation of Scott and Hoofman, latter, to introduce the Scott and the continuous information systems, respectively, was to present domains in a logical form. However, these approach was limited to deal with information system structures as a concrete representation of domains. Since it just explore the categorical relations between domains and information systems without both, a treatment of the deep implicit logical aspects of information system and a logical conection between domains and information systems. Thus, this work gives the relation between a continuous domain and their continuous information system representation, by using a satisfaction notion. This notion, reveals several logical concepts, such as: consequence relation, information content, models of an information, theory of an elements, theorems, information content order, equivalence between information, etc.

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