ω-continuos cpo's, ω-algebraic cpo's, SFP domains and retract of SFP domains as information systems

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Abstract

In this paper we define a representation for ω-continuous cpo's, ω-algebraic cpo's, SFP domains and retracts of SFP domains, with the intention of keeping as close as possible to the spirit of Scott's original construction of information systems and, in this form, to provide a more concrete and logical account of theses domains. In this sense, we introduce the notion of pre-information system and Plotkin information system, which are simple generalizations of Scott Information Systems [Sco82b], and we show their equivalence with ω-algebraic cpo and SFP domains. Analogously, we show that pre-continuous information systems representing adequately ω-continuous cpo's and a subclass of them constructed from Plotkin Information systems correspond to retract of SFP domains.

Keywords: Domain Theory, partial orders, SFP domains, retracts, information systems

1 Introduction

Domain theory study a class of appropriate spaces to solve recursive equations in order to obtain semantic functions to give denotational semantics of programming languages [Sco82b, Plo83]. Scott introduced the notion of *information system*, initially with the intention to make domain theory accessible to a wider audience [Sco82b]. In this representation of Scott domains the idea of information is made explicit, in the sense that properties of domains can be derived from assumptions about the entailment between propositions expressing properties of computations. Thus, this representation allows us to obtain a logical of partial information of semantic domains which include a notion of finite consistency and a consequence relation.

In order to treat, adequately, continuous data type (for example with the real numbers) is necessary work with continuous domains. Some attempts to represent continuous domains were made. For example, in [Smy77] are defined the R-structures to represent continuous domains, via Dedekind completion. In [Vic93] is used a structure, denomined as *infosys*, which consists of a set of token of information joint with an entailment relation which is transitive and interpolative, but not necessarily reflexive. Infosys generalize the R-structures and can be used to represent continuous poset. However, none of the structures, are completely faithfull with the spirit of Scott information systems (infosys and R-structures do not have a consistency predicates).

In [Hoo93] was introduced the continuous information systems, which extend in a natural way the information system, and are appropriate to represent continuous domain. Continuous information systems have a non necessarily reflexive consequence relation which also satisfy the interpolation (reverse cut) rule.

The notion of *powerdomain* was introduced by Plotkin in [Plo76] in order to extend Scott's framework to consider simpler nondeterministic features of programming languages. Unfortunately, Scott domains are not closed under the Plotkin powerdomain constructor. A more general framework is the SFP domains (Sequence of Finite complete Partial orders), introduced by Plotkin to solve these dificulties.

It has been an open question how SFP domain can be represented as information system in a better way following Scott lines [Zha94]. A couple of attempts were made to find a natural extension of Scott representation. However, none of these seems to provide a fully satisfactory treatment of SFP domains.

In this work, we define the Plotkin information systems and show that this type of information systems represent SFP domains as close as possible to the spirit of Scott's original construction, providing, therefore, a more concrete and logical account of SFP domains. In order to make it possible we introduce an information system representation for ω-continuous cpo's and ω-algebraic cpo's. Finally, we define an information system structure to give a non internal representation to retract of SFP domains.

2 SFP domains

We will assume that the readers are familiar with the basic concepts of domain theory and category theory. Some references are [Sco82b, Plo83, Gun85, Law87] and [AL91], respectively.

Let $\mathbf{D}$ and $\mathbf{E}$[1] be cpo's. A continuous function $f : E \to D$ is a **retraction** if there is a continuous function $g : D \to E$ such that $f \circ g = Id_D$, in this case D is a **retract** of E. If g satisfies the further conditions that $g \circ f \leq Id_E$, i.e. $g \circ f(x) \leq_E x$, for each $x \in E$, then f is called a **projection**.

Let $\mathcal{D} = \{\mathbf{D_1}, \mathbf{D_2}, \cdots, \mathbf{D_n}, \cdots\}$ be a family of cpo's. $\mathcal{D}$ is a **directed sequence** if for each m and n, with $m \leq n$, there exists a projection $f_{m,n} : D_m \to D_n$ such that $f_{m,m} = Id_m$ and $f_{m,n} \circ f_{k,m} = f_{k,n}$ when $0 \leq k \leq m \leq n$.

Let $\mathcal{D} = \{\mathbf{D_1}, \mathbf{D_2}, \cdots, \mathbf{D_n}, \cdots\}$ be a directed sequence of finite cpo's. A cpo $\mathbf{D}$ is a **SFP domain** if $\mathbf{D} \cong \lim_{\to} \mathcal{D}$. A practical characterization of SFP domains is obtained in terms of minimal upper bounds. Let $\mathbf{D}$ be a cpo. An element $u \in D$ is a **minimal upper bound** of a subset $X \subseteq D$ if it is an upper bound of X which is not strictly greater than any other upper bound of X. We write $MUB(X)$ for the set of minimal upper bounds of X. $MUB(X)$ is said **complete** if whenever u is an upper bound of X, $v \leq_D u$ for some $v \in MUB(X)$.

PROPOSITION 2.1
[Plo76] Let $\mathbf{D}$ be an algebraic cpo and $\mathbf{D^0}$ their set of compact elements. If $X \subseteq^{fin} \mathbf{D^0}$[2] and $MUB(X)$ is complete then, $MUB(X) \subseteq \mathbf{D^0}$.

For a subset X of D, let $\mathcal{U}^0(X) = X$ and $\mathcal{U}^{i+1}(X) = \{u \in D \mid u \in MUB(S)$ for some $S \subseteq^{fin} \mathcal{U}(X)\}$, for $i \geq 0$. It is convenient to write $\mathcal{U}^*(X)$ instead of $\bigcup_{i\geq 0} \mathcal{U}^i(X)$,

[1] In the follow we write $\mathbf{D}$ to indicate the cpo $\mathbf{D} = \langle D, \leq_D, \perp_D \rangle$. Analogously for the other cpo's $\mathbf{E}$, $\mathbf{D_1}$, etc.
[2] The notation $X \subseteq^{fin} Y$ means that X is a finite subset of Y.

that is $\mathcal{U}^*(X)$ is the least set containing X and closed under MUB.

THEOREM 2.2
[Plo76] $\mathbf{D}$ is a SFP domain if, and only if, the following statement holds: 1. $\mathbf{D}$ is algebraic, 2. $\forall X \subseteq^{fin} \mathbf{D}^0$, $MUB(X)$ is complete, and 3. $\mathcal{U}^*(X)$ is finite.

PROPOSITION 2.3
[Plo76] Let $\mathbf{D}$ be an algebraic cpo such that $\forall S \subseteq^{fin} \mathbf{D}^0$, $MUB(S)$ is complete. If $X \subseteq^{fin} \mathbf{D}^0$ then, $\mathcal{U}^*(X) \subseteq \mathbf{D}^0$.

The category **SDom**, of Scott domains with continuous functions, is a full sub-category (proper) of the category **SFP**, which has SFP domains as objects and continuous functions as morphisms, and the category **SFP** is a full sub-category of the category **Alg**, that have ω-algebraic cpo's as objects and continuous functions as morphisms. In fact, **SFP** is the largest full subcategory of **Alg** closed under the function space constructor [Plo76] and this is confirmed by Smyth theorem, that proved the following conjecture of Plotkin [Smy83].

THEOREM 2.4
[Smy83] Let $\mathbf{D}$ be an ω-algebraic cpo. If $\mathbf{D} \to \mathbf{D}$ is an ω-algebraic cpo then, $\mathbf{D}$ and $\mathbf{D} \to \mathbf{D}$ are SFP domains.

COROLLARY 2.5
[Smy83] Let $\mathcal{C}$ be a cartesian closed category. If $\mathcal{C} \subseteq \mathbf{Alg}$ then, $\mathcal{C} \subseteq \mathbf{SFP}$.

3 Representing ω-continuous cpo's and ω-algebraic cpo's

Algebraic cpo's are obtained as completion by ideals of poset's [Law87] and continuous cpo's are obtained as completion by Dedekind cuts of poset's [Bed96]. The elements of this poset's are tokens of information on the elements of the algebraic and continuous cpo's, respectively. Under this inspiration D. Scott introduced the notion of Information System, an alternative approach to Scott domains (algebraic) [Sco82b]. This notion was, posteriously, extended to continuous information systems by [Hoo93] in order to characterize continuous domains (consistently complete ω-continuous cpo's). We extend the original definition of information system and continuous information system in order to obtain as domain of elements an algebraic cpo instead of a Scott domain and a continuous cpo instead of a continuous domain.

DEFINITION 3.1
A **pre-continuous information system**, pre-cis in short, is a triple $\mathcal{I} = \langle I, Con, \vdash \rangle$, where I is a countable information set, Con is a subset of $\mathcal{P}^{fin}(I)$ (finite part of I), named consistency predicate, and $\vdash$ is a subset of $Con \times I$, named entailment relation, such that for each $X, Y \in Con$ and $a, b \in I$

1. $\emptyset \in Con$
2. $\{a\} \in Con$
3. If $X \vdash a$, then $X \cup \{a\} \in Con$
4. If $X \vdash a$ and $\{a\} \vdash b$, then $X \vdash b$
5. If $Y \subseteq X$ and $Y \vdash a$, then $X \vdash a$
6. If $X \vdash a$, then $X \vdash Z$ and $Z \vdash a$, for some $Z \in Con$.

The notation $X \vdash Z$ is a shorthand for $\forall a \in Z$, $X \vdash a$.

DEFINITION 3.2
Let $\mathcal{I} = \langle I, Con, \vdash \rangle$ be a pre-cis. $\mathcal{I}$ is said a **Pre-information system**, **pre-is** in short, if also satisfy: $5'$. If $a \in X$ and $X \in Con$ then, $X \vdash a$

The properties 5 and 6 of definition 3.1 can be obtained from 4 and from $5'$ of definition 3.2.

A pre-cis informs on the elements of a domain which can be identified with the set of information in the system which are satisfied by the element. Hence, such set must be consistent and must contain all information which can be deduced from it.

DEFINITION 3.3
The **elements** of a pre-cis $\mathcal{I} = \langle I, Con, \vdash \rangle$ are subsets x of I which satisfies:

1. pre-finitely consistent: If $X \subseteq^{fin} x$ then, $\exists Y \in Con$ such that $X \subseteq Y \subseteq x$,
2. $\vdash$-closed: If $X \subseteq x$ and $X \vdash a$ then, $a \in x$,
3. derivability of informations: If $a \in x$ then, $\exists X \subseteq^{fin} x$ such that $X \vdash a$.

The set of all elements of a pre-cis $\mathcal{I} = \langle I, Con, \vdash \rangle$ is denoted by $|\mathcal{I}|$. The poset $\langle |\mathcal{I}|, \subseteq \rangle$ is called domain of elements of $\mathcal{I}$.

PROPOSITION 3.4
[Bed96] Let $\mathcal{I} = \langle I, Con, \vdash \rangle$ be a pre-cis. If $X \in Con$ then $\overline{X} = \{a \in I \mid X \vdash a\} \in |\mathcal{I}|$.

PROPOSITION 3.5
[Bed99] Let $\mathcal{I} = \langle I, Con, \vdash \rangle$ be a pre-cis and $y \in |\mathcal{I}|$. Then,

1. $B(y) = \{\overline{X} \mid X \in Con \text{ and } X \subseteq y\}$ is a directed set with y as supremum.
2. The relation "way below" of the cpo $\langle |\mathcal{I}|, \subseteq \rangle$ is characterized by

$$x \ll y \text{ if, and only if, } \exists Y \subseteq^{fin} y, \text{ such that } \forall a \in x\ Y \vdash a$$

3. $B(y) \subseteq \{\overline{X} \mid X \in Con \text{ and } \overline{X} \ll y\}$.

LEMMA 3.6
[Bed99] If $\mathcal{I} = \langle I, Con, \vdash \rangle$ is a pre-cis then $\langle |\mathcal{I}|, \subseteq \rangle$ is a cpo.

THEOREM 3.7
[Bed99] Let $\mathcal{I} = \langle I, Con, \vdash \rangle$ be a pre-cis. Then, $\langle |\mathcal{I}|, \subseteq \rangle$ is a continuous cpo with $\overline{Con} = \{\overline{X} \mid X \in Con\}$ as a countable basis.

THEOREM 3.8
[Bed99] If $\mathcal{I} = \langle I, Con, \vdash \rangle$ is a pre-is then, $\langle |\mathcal{I}|, \subseteq \rangle$ is an ω-algebraic cpo with $\overline{Con}$ as the set of compact elements.

DEFINITION 3.9
Let $\mathcal{I} = \langle I, Con, \vdash \rangle$ be a pre-cis. Define $R(\mathcal{I}) = \langle I, Con, \mid\!\sim \rangle$, where

$$X \mid\!\sim a \text{ if, and only if, } X \cup \{a\} \in Con \text{ and for each } b \in I, \text{ if } \{a\} \vdash b \text{ then, } X \vdash b.$$

The follow proposition consider the basic relations between the entailment relations $\vdash$ and $\mid\!\sim$ of a pre-cis. Notice, the similarity of this properties with the usual properties of the "way-below" order in a continuous cpo [Law87].

PROPOSITION 3.10
Let $\mathcal{I} = \langle I, Con, \vdash \rangle$ be a pre-cis. Then

1. If $X \in Con$ and $a \in X$ then $X \mid\!\sim a$

2. If $X \vdash a$ then $X \mid\!\sim a$
3. If $X \vdash a$ and $X \vdash b$ then $\exists Y \in Con$ such that $Y \mid\!\sim a$, $Y \mid\!\sim b$ and $X \vdash Y$
4. If $X \mid\!\sim Y$ and $Y \vdash a$ then $X \vdash a$
5. If $X \vdash a$ then $\exists Y \in Con$ such that $X \vdash Y$ and $Y \vdash a$

PROOF. 1. Let $X \in Con$ and $a \in X$. If $\{a\} \vdash b$ then, by definition 3.1.4, $X \vdash b$. So, $X \mid\!\sim a$.

2. If $X \vdash a$ and $\{a\} \vdash b$ then, by definition 3.1.4, $X \vdash b$. So, by definition 3.9, $X \mid\!\sim a$.
3. If $X \vdash a$ and $X \vdash b$, then $\{a, b\} \subseteq^{fin} \overline{X}$. So, by pre-finitely property, there exist $Y \in Con$ such that $\{a, b\} \subseteq Y \subseteq \overline{X}$. Therefore, by property 1 of this proposition, $Y \mid\!\sim a$ and $Y \mid\!\sim b$ and, by definition of $\overline{X}$, $X \vdash Y$.
4. If $X \mid\!\sim Y$, $Y \vdash a$ then, by definition of $\mid\!\sim$, $X \vdash a$.
5. Direct from definition 3.1.6. ■

PROPOSITION 3.11
Let $\mathcal{I} = \langle I, Con, \vdash\rangle$ be a pre-cis. Then $R(\mathcal{I})$ is a pre-is and $\langle|\mathcal{I}|, \subseteq\rangle$ is a retract of $\langle|R(\mathcal{I})|, \subseteq\rangle$.

PROOF. Since $\mathcal{I}$ is a pre-cis a $R(\mathcal{I})$ has the same consitency relation that $\mathcal{I}$, the properties 1, 2 and 3 of the definition 3.2 are straighforward. Thus, all we have to prove are the properties 4 and $5'$ of definition 3.2.

4. Let $X, Y \in Con$ and $a \in I$ such that $X \mid\!\sim Y$ and $Y \mid\!\sim a$. By definition of $\mid\!\sim$, if $\{a\} \vdash b$, then $Y \vdash b$. Since, by proposition 3.10.1, $\{b\} \mid\!\sim b$, then, by proposition 3.10.4, $X \vdash b$. So, by definition 3.9, $X \mid\!\sim a$.

$5'$. Direct from proposition 3.10.1.

In order to prove that $\langle|\mathcal{I}|, \subseteq\rangle$ is a retract of $\langle|R(\mathcal{I})|, \subseteq\rangle$ we need define a pair embeeding-projection $g : |\mathcal{I}| \longrightarrow |R(\mathcal{I})|$ and $f : |R(\mathcal{I})| \longrightarrow |\mathcal{I}|$, respectively. Define $g(x) = \{a \in I \mid X \mid\!\sim a \text{ for some } X \subseteq^{fin} x\}$ and $f(x) = \{a \in x \mid X \vdash a \text{ for some } X \subseteq^{fin} x\}$. Clearly, $x \subseteq g(x)$ and $f(x) \subseteq x$. Thus, $f(g(x)) = x$ e $g(f(x)) \subseteq x$. So, $\langle|\mathcal{I}|, \subseteq\rangle$ is a retract of $\langle|R(\mathcal{I})|, \subseteq\rangle$. ■

On the other hand, we can construct, in a canonical way, pre-cis's and pre-is's from ω-continuos cpo's and ω-algebraic cpo's, respectively.

DEFINITION 3.12
Let $\mathbf{D} = \langle D, \sqsubseteq, \perp\rangle$ be an ω-continuous cpo with B as a countable basis. Define $PCIS(\mathbf{D}, B) = \langle B, Con_D, \vdash_D\rangle$, where

1. $X \in Con_D$ if, and only if, $X \subseteq^{fin} B$ and $\bigsqcup X$ exist.
2. $X \vdash_D a$ if, and only if, $X \in Con_D$ and $a \ll \bigsqcup X$.

DEFINITION 3.13
Let $\mathbf{D} = \langle D, \sqsubseteq, \perp\rangle$ be an ω-algebraic cpo. Define $PIS(\mathbf{D}) = \langle \mathbf{D^0}, Con_D, \vdash_D\rangle$, where

1. $X \in Con_D$ if, and only if, $X \subseteq^{fin} \mathbf{D^0}$ and $\bigsqcup X$ exist.
2. $X \vdash_D a$ if, and only if, $X \in Con_D$ and $a \sqsubseteq \bigsqcup X$.

THEOREM 3.14
[Bed99] Let $\mathbf{D} = \langle D, \sqsubseteq, \perp \rangle$ be an ω-continuous (algebraic) cpo. Then, $PCIS(\mathbf{D}, B)$ ($PIS(\mathbf{D})$) is a pre-cis (pre-is) and $\mathbf{D}$ and $|PCIS(\mathbf{D}, B)|$ ($|PIS(\mathbf{D})|$) are isomorphic orders.

4 Representing SFP domains and their retracts

In order to represent SFP domains and retract of SFP domains we introduce two new kinds of pre-cis's .

DEFINITION 4.1
let $\mathcal{I} = \langle I, Con, \vdash \rangle$ be a pre-is. $\mathcal{I}$ is called a **Plotkin information system**, **Pis** in short, if $\forall Y \subseteq^{fin} I$, $\exists \underline{X} = \{X_1, \cdots, X_n\} \subseteq^{fin} Con$ such that

1. If $Y' \subseteq Y$ and $Y' \in Con$ then, $Y' \in \underline{X}$,
2. $\forall \underline{X'} \subseteq \underline{X}$, if $\forall X_j \in \underline{X'}$, $Z \vdash X_j$ then, $\exists X_i \in \underline{X}$ such that, $Z \vdash X_i$ and $X_i \vdash X_j$ for each $X_j \in \underline{X'}$.

The following theorems state that all Pis define canonically a SFP domain and vice-versa.

THEOREM 4.2
[Bed99] Let $\mathcal{I} = \langle I, Con, \vdash \rangle$ be a Pis. Then $|\mathcal{I}| = \langle |\mathcal{I}|, \subseteq \rangle$ is a SFP domain.

THEOREM 4.3
[Bed96] Let $\mathbf{D} = \langle D, \sqsubseteq, \perp \rangle$ be a SFP domain. Then $PIS(\mathbf{D})$ is a Pis and $\langle |PIS(D)|, \subseteq \rangle$ and $\mathbf{D}$ are isomorphic orders.

DEFINITION 4.4
Let $\mathcal{I} = \langle I, Con, \vdash \rangle$ be a pre-cis. Then $\mathcal{I}$ is called a **R-Plotkin information system**, R-Pis in short, if $\mathcal{I} = \langle I, Con, \mid\!\sim \rangle$, defined as in definition 3.9, is a Pis's.

THEOREM 4.5
$\mathcal{I} = \langle I, Con, \vdash \rangle$ is a R-Pis if, and only if, $\langle |\mathcal{I}|, \subseteq \rangle$ is a Retract of some SFP domain.

PROOF. (only the first part) By definition 4.4, if $\mathcal{I}$ is a R-Pis then, $R(\mathcal{I}) = \langle I, Con, \mid\!\sim \rangle$ constructs conform definition 3.9, is a Pis. Therefore, by theorem 4.2, $\langle |R(\mathcal{I})|, \subseteq \rangle$ is a SFP domain. We will show that $\langle |\mathcal{I}|, \subseteq \rangle$ is a retract of $\langle |R(\mathcal{I})|, \subseteq \rangle$.

Let $f : |R(\mathcal{I})| \longrightarrow |\mathcal{I}|$ and $g : |\mathcal{I}| \longrightarrow |\mathcal{R}(\mathcal{I})|$ defined by

$$f(x) = \{a \in I \mid X \mid\!\sim a \text{ for some } X \subseteq^{fin} x\} \text{ and } g(x) = x$$

Let $x \in |\mathcal{I}|$. If $a \in x$ then, by proposition 3.10, $\{a\} \mid\!\sim a$. So, $a \in f(x)$ and therefore $a \in f(g(x))$, i.e. $a \in f \circ g(x)$. Hence, $f \circ g(x) = x$, i.e. $f \circ g = id_{|\mathcal{I}|}$. ■

5 Approximable mappings

Pre-cis's are equiped with morphisms, called *approximable mappings*, which correspond to continuous functions between the associated domains of elements.

DEFINITION 5.1
Let $\mathcal{I}_1 = \langle I_1, Con_1, \vdash_1 \rangle$ and $\mathcal{I}_2 = \langle I_2, Con_2, \vdash_2 \rangle$ be pre-cis's. An **approximable mapping** $\rho : \mathcal{I}_1 \to \mathcal{I}_2$ is a relation $\rho \subseteq Con_1 \times I_2$ satisfying:

1. If $X\rho a_1 \wedge \cdots \wedge X\rho a_n$ then $\{a_1, \cdots, a_n\} \in Con_2$
2. If $X\rho a_1 \wedge \cdots \wedge X\rho a_n \wedge \{a_1, \cdots, a_n\} \vdash_2 a$ then, $X\rho a$
3. If $X \vdash_1 Y$ and $Y\rho a$ then, $X\rho a$.
4. If $X\rho a$ then exists $Y \in Con_1$, such that $X \vdash_1 Y$ and $Y\rho a$.

Pis's with approximable mappings is a category, denoted by **PISys**, in which approximable mappings are composed by the usual composition of relations and the identity morphism on a Pis $\mathcal{I} = \langle I, Con, \vdash\rangle$ is the relation $\vdash$.

We have seen how a Pis $\mathcal{I}$ represents a SFP domain $|\mathcal{I}|$. In a similar way, an approximable mapping ρ represents a continuous function $|\rho|$.

PROPOSITION 5.2
[Bed96] Let $\mathcal{I}_1 = \langle I_1, Con_1, \vdash_1\rangle$ and $\mathcal{I}_2 = \langle I_2, Con_2, \vdash_2\rangle$ be Pis's. If $\rho : \mathcal{I}_1 \to \mathcal{I}_2$ is an approximable mapping then, $|\rho| : |\mathcal{I}_1| \to |\mathcal{I}_2|$, defined by

$$|\rho|(x) = \{a \in I_2 \mid X\rho a \text{ for some } X \subseteq^{fin} x\}$$

is a continuous function between the SFP domains $\langle |\mathcal{I}_1|, \subseteq\rangle$ and $\langle |\mathcal{I}_2|, \subseteq\rangle$.

Thus, the operation $|\_|$ is a functor from the category **PISys** to the category **SFP**. On the other hand, to each SFP domain **D** corresponds an associated Pis $PIS(\mathbf{D})$. In a similar way a continuous function f represents an aproximable mapping PIS_f.

PROPOSITION 5.3
[Bed96] Let **D** and **E** be SFP domains. If $f : D \to E$ is a continuous function then the relation $(PIS_f) \subseteq Con_D \times \mathbf{E^0}$, defined by

$$X(PIS_f)a \text{ if, and only if, } a \sqsubseteq_E b \text{ for some } b \in MUB(X),$$

is an approximable mapping between the Pis's $PIS(\mathbf{D}) = \langle \mathbf{D^0}, Con_D, \vdash_D\rangle$ and $PIS(\mathbf{E}) = \langle \mathbf{E^0}, Con_E, \vdash_E\rangle$.

Thus, PIS is a functor from the category **SFP** to the category **PISys**. Observe that for any SFP domain **D** the function $\phi : D \to | PIS(\mathbf{D}) |$, given by $\phi(x) = \{y \in \mathbf{D^0} \mid y \sqsubseteq_D x\}$, is a continuous bijection, which determines therefore that **D** and $| PIS(\mathbf{D}) |$ are isomorphic orders. Thus, the functors $| \_ |$: **PISys** $\to$ **SFP** and PIS : **SFP** $\to$ **PISys** establish an equivalence between the categories **PISys** and **SFP** and therefore, from a category-theoretical point of view, they are essentially the same. Nevetheless, we have gained something. The rather abstract category of SFP domains with continuous functions is represented by the more concrete category of Pis's and approximable mappings. We can work concretely, with basic set theory, to produce constructions on Pis's.

Clearly we can prove an analougous proposition to proposition 5.2, but considering R-Pis's and retract of SFP domains, pre-cis's and ω-continuous cpo's, and pre-is's and ω-algebraic cpo's. Thus, we can say that the categories of R-Pis's with approximable mappings, denoted by **RPIS**, and the category of retracts of SFP domains with continuous functions, denoted by **RSFP**, are equivalent. Analogously, the categories **PCIS** (pre-cis's with approximable mappings) and **PIS** (pre-is's with approximable mappings) are equivalent to categories **CCPO** (ω-continuous cpo's with continuous functions) and **ACPO** (ω-algebraic cpo's with continuous functions), respectively.

6 Contructions on information systems structures

Tipically, some constructors are defined on semantic domains, and are used for semantic definitions. In this section, we will see only two of them.

6.1 Function space constructor

Because, there exists a straight correspondency between continuous functions and approximable mappings, a function space constructor on information system structures must produce an information system structure whose domain of elements will be the set of approximable mappings between the information systems structures.

The basic information of an approximable mapping, $\rho : \mathcal{I}_1 \longrightarrow \mathcal{I}_2$, are pairs (X, a), where $X \in Con_1$ and $a \in I_2$, such that $X \rho a$. Therefore, with an adequate notion of consistency and entailment relation, to this pairs, we will obtain an information systems structure where their elements are approximable mappings.

DEFINITION 6.1
Let $\mathcal{I}_1 = \langle I_1, Con_1, \vdash_1 \rangle$ and $\mathcal{I}_2 = \langle I_2, Con_2, \vdash_2 \rangle$ be pre-cis's. Define their **function space** by $\mathcal{I}_1 \to \mathcal{I}_2 = \langle Con_1 \times I_2, Con, \vdash \rangle$ where

- Let $\underline{X} = \{(X_1, a_1), \cdots, (X_n, a_n)\} \subseteq^{fin} Con_1 \times I_2$. $\underline{X} \in Con$ if, and only if, $\forall F \subseteq \{1, \cdots, n\}, (\bigcup_{i \in F} X_i \in Con_1 \Rightarrow \{a_i \mid i \in F\} \in Con_2)$
- $\underline{X} \vdash (X, a)$ if, and only if, $\{a_1 \mid \ X \vdash_1 X_i\} \vdash_2 a$

PROPOSITION 6.2
[Bed96] Let $\mathcal{I}_1 = \langle I_1, Con_1, \vdash_1 \rangle$ and $\mathcal{I}_2 = \langle I_2, Con_2, \vdash_2 \rangle$ be (R-)Pis's. Then

1. $\mathcal{I}_1 \to \mathcal{I}_2$ is a (R-)Plotkin information system,
2. $\rho : \mathcal{I}_1 \longrightarrow \mathcal{I}_2$ is an approximable mapping if, and only if, $\rho \in \mid \mathcal{I}_1 \to \mathcal{I}_2 \mid$, and
3. $\mid \mathcal{I}_1 \to \mathcal{I}_2 \mid$ is an isomorphic order of the function space $\mid \mathcal{I}_1 \mid \to \mid \mathcal{I}_2 \mid$.

6.2 Plotkin powerdomains constructor

In order to give informations on the final states of a nondeterministic program, we must give as information a set of basic information on the states, i.e. if I is the set of basic information on S then $\mathcal{P}_\emptyset^{fin}(I)$[3] is the set of basic information on $\mathcal{P}(S)$, since basic information must be finite in the sense of [Vic89]. Thus, we must define an unary constructor on pre-cis's which has as basic information the finite powerset of the original basic information and an adequately notion of consistency and entailment relations in order to interpret the diferents powerdomain constructors. In our case we will define the Plotkin powerdomain.

DEFINITION 6.3
Let $\mathcal{I} = \langle I, Con, \vdash \rangle$ be a pre-cis. The **Plotkin powerdomain** of $\mathcal{I}$ is defined by $\mathcal{I}^P = \langle \mathcal{P}_\emptyset^{fin}(I), Con^P, \vdash^P \rangle$ where

- $\underline{X} \in Con^P$ if, and only if, there exists $\bigsqcup \underline{X}$ in $\langle \mathcal{P}_\emptyset^{fin}(I), \preceq^P \rangle$
- Let $\underline{X} \in Con^P$ and $A \in \mathcal{P}_\emptyset^{fin}(I)$. $X \vdash^P A$ if, and only if, $A \prec^P \bigsqcup \underline{X}$.

[3] $\mathcal{P}_\emptyset^{fin}(I)$ is the set of finite subsets of I, except the emptyset

where $\prec^P$ and $\preceq^P$ are pre-orders on $\mathcal{P}_{\emptyset}^{fin}(I)$ defined by

$A \prec^P B$ if, and only if, $\forall a \in A\ \exists b \in B,\ \{a\} \vdash b$ and $\forall b \in B\ \exists a \in A,\ \{a\} \vdash b$

$A \preceq^P B$ if, and only if, $\forall a \in A\ \exists b \in B,\ \{a\} \mid\!\sim b$ and $\forall b \in B\ \exists a \in A,\ \{a\} \mid\!\sim b$

Proposition 6.4
[Bed96] Let $\mathcal{I} = \langle I, Con, \vdash\rangle$ be a (R-)Pis. Then $\mathcal{I}^P$ is a (R-)Pis and $\mid \mathcal{I}^P \mid$ is an isomorphic order of the (retract of) SFP domain $\mid \mathcal{I}\mid^P$ (the Plotkin powerdomain of $\mid\mathcal{I}\mid$).

7 Conclusion

The main departure of our definitions of information system structures w.r.t. Scott's original spirit is that the consistency predicate need not be closed under subset. Since, the consistency predicate of tradicional Scott information systems has straightforward logical status in *domain logic*, we should give an other name to this predicate ("Prime", for example, will be an adequate name, because the open set determined by any set of Con is a prime element in the lattice of Scott open sets). However, in order to define an unifiqued class of information system structures, we opt to use the same name (Con) introduzed by Scott.

Several attempts were made to provide a fully satisfactory treatment of SFP domains. The first of this tentative is the Scott manuscript [Sco82a] in which he to give a representation only for 2/3 SFP domains. However, that treatment is not near to the spirit of information systems, because it allows the explicit use of logical operators and employs heavy machinery. [DG90] gives an axiomatization of the domains determined by non-deterministic information systems. Since entailment such as $\emptyset \vdash \emptyset$ are allowed, we have that a domain of elements do not need to be a cpo. [Abr91] uses pre-locale domains to represent SFP domains (and their function space and power domains), with the advantage that propositions of high order types can be construct more naturally, because its construction allows conjunctions. But its representation is not faithful with the spirit of Scott information systems, because it allow to use logical operators and employ a hard formalism. [ES91, ES92] uses an abstraction and enhancement of idea original Scott information system, represented SFP domains and retract of SFP domains in terms of their finite elements. Finally the work of [Zha94], modify the original structure of information systems, eliminating the consistency predicate (it can be recovered from the entailment relation) and instead of using entailment of the form $X \vdash a$, he works with the classical Gentzen style $X \vdash Y$, providing explicit disjunction of proposition in Y. This approximation is better than the previous one, however it is not completely faithful to the original structure of information system and it is much more complex than the given here.

We extend with success, in a simple way, the original notion of information systems and we prove that this extension is an adequate representation for SFP domains. In fact, in our representation, all Scott information system are Plotkin information systems, without any transformation of the structure as in the other tentatives.

In this work, we show that the categories **PSys** and **SFP** are equivalent. From the point of view of categorical construction, Pis's say no more than SFP domains, they are just a representation of the category of SFP domains. However, Pis's do have potentially more structure than SFP domains and quite likely the morphism of them have a greatest amount of details. This would be a strong point to close-up the conection between denotational semantics and the proof of properties of non-deterministic programs, and could explore the idea that the denotation of a programming constructor, for example Plotkin powerdomain or function spaces, can be seen as the set of the assertions on the constructors.

On the other hand, we also introduce the notions of pre-cis, pre-is and R-Pis and we show that their domain of elements are ω-continuous cpo's, ω-algebraic cpo's and retracts of SFP domains, respectively. It is clear, that it is possible to extend, for these categories, the equivalence results between **PSys** and **SFP**, in analogous way.

In order to represent the hierarchy of domains (L-domains, Scott domains, Coherent domains, ω-algebraic lattice as well as SFP and their retracts) in a uniform and coherent way with our notions of pre-continuous, we will define, in a future work, information systems structures to this categories of domains.

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