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Two's Company: "The Humbug of Many Logical Values"

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How was it possible that
the humbug of many logical values
persisted over the last fifty years?
—Roman Suszko, 1976.

Abstract. The Polish logician Roman Suszko has extensively pleaded in the 1970s for a restatement of the notion of many-valuedness. According to him, as he would often repeat, "there are but two logical values, true and false." As a matter of fact, a result by Wójcicki-Lindenbaum shows that any tarskian logic has a many-valued semantics, and results by Suszko-da Costa-Scott show that any many-valued semantics can be reduced to a two-valued one. So, why should one even consider using logics with more than two values? Because, we argue, one has to decide how to deal with bivalence and settle down the trade-off between logical 2-valuedness and truth-functionality, from a pragmatical standpoint.

This paper will illustrate the ups and downs of a two-valued reduction of logic. Suszko's reductive result is quite non-constructive. We will exhibit here a way of effectively constructing the two-valued semantics of any logic that has a truth-functional finite-valued semantics and a sufficiently expressive language. From there, as we will indicate, one can easily go on to provide those logics with adequate canonical systems of sequents or tableaux. The algorithmic methods developed here can be generalized so as to apply to many non-finitely valued logics as well —or at least to those that admit of computable quasi tabular two-valued semantics, the so-called dyadic semantics.

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1. Suszko's Thesis

"After 50 years we still face an illogical paradise of many truths and falsehoods". Thus spake Suszko in 1976, at the 22nd Conference on the History of Logic, in Cracow (cf. [25]). He knew all too well who was the first to blame for that state of affairs: "Łukasiewicz is the chief perpetrator of a magnificent conceptual deceipt lasting out in mathematical logic to the present day." Suszko was perfectly aware, of course, that there are logics that can only be characterized truth-functionally with the help of n-valued matrices, for $n > 2$. He also knew that there were logics, such as most logics proposed by Łukasiewicz or by Post, that were characterizable by finite-valued matrices, and he knew that there were logics, such as Łukasiewicz's $Ł_\omega$, intuitionistic logic, or all the usual modal systems, that could only be characterized by infinite-valued matrices. Suszko was even ready to concede, in his reconstruction of the Fregean distinction between 'sense' and 'reference' of sentences, that the talk about many truth-values, in a sense, could not be avoided, "unless one agrees that thought is about nothing, or, rather, stops talking with sentences" (cf. [23]).

Still, Suszko insisted that "obviously any multiplication of logical values is a mad idea" (cf. [25]). How come? The point at issue is, according to Suszko, a distinction between the *algebraic truth-values* of many-valued logics, that were supposed to play a merely referential role, while only two *logical truth-values* would really exist. The philosophical standpoint according to which "there are but two logical values, true and false" receives nowadays the label of *Suszko's Thesis* (cf. [16, 18, 26]).

Suszko illustrated his proposition by showing how Łukasiewicz's 3-valued logic $Ł_3$ could be given a 2-valued (obviously non-truth-functional) semantics (cf. [24]). He did not explain though how he obtained the latter semantics, or how the procedure could be effectively applied to other logics. The present paper shows how that can be done for a large class of finite-valued logics. It also illustrates some uses for that 2-valued reduction in the mechanization of proof procedures. Our initial related explorations in the field appeared in our reports [8, 6]. A detailed appraisal and an extended investigation of both the technical and the philosophical issues involved in Suszko's Thesis can be found in our [7].

The plan of the present paper is as follows. Section 2 explains the general reductive results that make tarskian logics n-valued and 2-valued. Section 3 presents the technology for separating truth-values, which is the cornerstone of our reductive procedure. Section 4 introduces gentzenian semantics as an appropriate format for presenting bivaluation axioms, and proposes dyadic semantics so as to define the class of computable 2-valued semantics. Section 5 obtains, in an effective way, 2-valued semantics for many-valued logics, applying the algorithm from the main Theorem 5.2. Several detailed examples are given in Section 6. The question of obtaining 'bivalent' tableaux for such logics is treated in Section 7. Finally, Section 8 briefly summarizes the obtained results and calls for a continuation of the present investigations.

2. Reductive results

Let $\mathcal{S}$ denote a non-empty set of *formulas* and let $\mathcal{V}$ denote a non-empty set of *truth-values*. Any $\Gamma \subseteq \mathcal{S}$ will be called a *theory*. Assume $\mathcal{V} = \mathcal{D} \cup \mathcal{U}$ for suitable disjoint sets $\mathcal{D}$ and $\mathcal{U}$ of *designated* and *undesignated* values. Any mapping $\S_k^{\mathcal{V}} : \mathcal{S} \to \mathcal{V}_k$ is called a (n-*valued*) *valuation*, where n is $|\mathcal{V}_k|$ (the cardinality of $\mathcal{V}_k = \mathcal{D}_k \cup \mathcal{U}_k$); if both $\mathcal{D}_k$ and $\mathcal{U}_k$ are singletons, $\S_k^{\mathcal{V}}$ is called a *bivaluation*. Any collection sem of valuations is called a (n-*valued*) *semantics*, where n is the cardinality of the largest $\mathcal{V}_k$ such that $\S_k^{\mathcal{V}} \in \mathsf{sem}$. A *model* of a formula φ is any valuation $\S_k^{\mathcal{V}}$ such that $\S_k^{\mathcal{V}}(\varphi) \in \mathcal{D}_k$. A canonical notion of *entailment* given by a *consequence relation* $\vDash_{\mathsf{sem}} \subseteq \mathsf{Pow}(\mathcal{S}) \times \mathcal{S}$ associated to the semantics sem can be defined by saying that a formula $\varphi \in \mathcal{S}$ follows from a set of formulas $\Gamma \subseteq \mathcal{S}$ whenever all models of all formulas of Γ are also models of φ, that is,

$$\Gamma \vDash_{\mathsf{sem}} \varphi \text{ iff } \S_k^{\mathcal{V}}(\varphi) \in \mathcal{D}_k \text{ whenever } \S_k^{\mathcal{V}}(\Gamma) \subseteq \mathcal{D}_k, \text{ for every } \S_k^{\mathcal{V}} \in \mathsf{sem}. \quad \text{(DER)}$$

That much for a semantic (many-valued) account of consequence. Now, for an abstract account of consequence, consider the following set of properties:

(CR1) $\Gamma, \varphi, \Delta \Vdash \varphi$; (inclusion)
(CR2) If $\Delta \Vdash \varphi$, then $\Gamma, \Delta \Vdash \varphi$; (dilution)
(CR3) $(\forall \beta \in \Delta)(\Gamma \Vdash \beta$ and $\Delta \Vdash \alpha)$ implies $\Gamma \Vdash \alpha$. (cut for sets)

A *logic* $\mathcal{L}$ will in this section be defined simply as a set of formulas together with a consequence relation defined over it. Logics respecting axioms (CR1)–(CR3) are called *tarskian*. Notice, in particular, that when sem is a singleton, one also defines a tarskian logic. Furthermore, an arbitrary intersection of tarskian logics also defines a tarskian logic. Given some logic $\mathcal{L} = \langle \mathcal{S}, \Vdash \rangle$, a theory $\Gamma \subseteq \mathcal{S}$ will be called *closed* in case it contains all of its consequences; the closure $\overline{\Gamma}$ of a theory Γ may be obtained by setting $\varphi \in \overline{\Gamma}$ iff $\Gamma \Vdash \varphi$. A *Lindenbaum matrix* for a theory Γ is defined by taking $\mathcal{V} = \mathcal{S}$, $\mathcal{D} = \overline{\Gamma}$ and $\mathsf{sem}[\Gamma] = \{\mathbf{id}_{\mathcal{S}}\}$ (the identity mapping on the set of formulas).

It is easy to check that every n-valued logic is tarskian. It can be shown that the converse is also true:

Theorem 2.1. (Wójcicki's Reduction)
Every tarskian logic $\mathcal{L} = \langle \mathcal{S}, \Vdash \rangle$ is n-valued, for some $n \leq |\mathcal{S}|$.

Proof. For each theory Γ of $\mathcal{L}$, notice that the corresponding Lindenbaum matrix defines a sound semantics for that logic, that is, $\Gamma \Vdash \varphi$ implies $\Gamma \vDash_{\mathsf{sem}[\Gamma]} \varphi$. To obtain completeness, one can now consider the intersection of all Lindenbaum matrices and check that $\Vdash \; = \; \cap_{\Gamma \subseteq \mathcal{S}} \vDash_{\mathsf{sem}[\Gamma]}$. □

This result shows that the above semantic and the abstract accounts of consequence define exactly the same class of logics. While we know that classical propositional logic can be characterized in fact by a collection of 2-valued matrices, and several other tarskian logics can be similarly characterized by other collections of finite-valued matrices, Wójcicki's Reduction shows that any tarskian logic has, in general, an infinite-valued characteristic matrix. Apart from the 'many truths and falsehoods' allowed by many-valued semantics, it should be observed that such semantics retain, in a sense, a shadow of bivalence, as reflected in the distinction between designated and undesignated values. Capitalizing on that distinction, one can show in fact that every tarskian logic also has an adequate 2-valued semantics:

Theorem 2.2. (Suszko's Reduction)
Every tarskian n-valued logic can also be characterized as 2-valued.

Proof. For any n-valuation § of a given semantics $\mathsf{sem}(n)$, and every consequence relation based on $\mathcal{V}_n$ and $\mathcal{D}_n$, define $\mathcal{V}_2 = \{T, F\}$ and $\mathcal{D}_2 = \{T\}$ and set the characteristic total function $b_\S : \mathcal{S} \to \mathcal{V}_2$ to be such that $b_\S(\varphi) = T$ iff $\S(\varphi) \in \mathcal{D}$. Now, collect all such bivaluations $b_\S$'s into a new semantics $\mathsf{sem}(2)$, and notice that $\Gamma \vDash_{\mathsf{sem}(2)} \varphi$ iff $\Gamma \vDash_{\mathsf{sem}(n)} \varphi$. □

The above results deserve a few brief comments. First of all, the standard formulations of Wójcicki's Reduction (cf. [27]) and of Suszko's Reduction (cf. [18]) usually presuppose more about the set of formulas (more specifically, they assume that it is a free algebra) and about the consequence relation (among other things, they assume that it is structural, i.e., that it allows for uniform substitutions). As we have seen, however, such assumptions are unnecessary for the more general formulation of the reductive results. From the next section on, however, we will incorporate those assumptions in our logics. Second, reductive theorems similar in spirit to Suszko's Reduction have in fact been independently proposed in the 70s by other authors, such as Newton da Costa and Dana Scott (a summary of important results from the theory of bivaluations that sprang from those approaches can be found in [4]). Third, it might seem paradoxical that the same logic is characterized by an n-valued semantics, for some sufficiently large n, and also by a 2-valued semantics. As we will see, though, the tension is resolved when we notice that the whole issue involves a trade-off between 'algebraic' truth-functionality and 'logical' bivalence. From the point of view of Suszko's Thesis, explained in the last section, these results can only lend some plausibility to the idea that "there are but two logical values, true and false": At the very least, we now know that the assertion makes perfect sense once we are talking about tarskian logics. Last, but not least, it should be noticed that the above reductive results are quite non-constructive. In case the logic *has* a finite-valued truth-functional semantics, Wójcicki's Reduction tells you nothing, in general, about how it can be obtained. Furthermore, Suszko's Reduction does not give you any hint, in general, on how a 2-valued semantics could be determined by anything like a finite recursive set of clauses, even for the case of logics with finite-valued truth-functional semantics.

In the present paper we obtain an effective method that assigns a useful 2-valued semantics to every finite-valued truth-functional logic provided that the 'algebraic values' of the semantics can be individualized by means of the linguistic resources of the logic.

3. Separating truth-values

Let's begin by adding some standard structure to the sets of formulas of our logics. Let $ats = \{p_1, p_2, \ldots\}$ be a denumerable set of *atomic sentences*, and let $\Sigma = \{\Sigma_n\}_{n\in\mathbb{N}}$ be a propositional signature, where each Σ_n is a set of *connectives* of arity n. Let $cct = \bigcup_{n\in\mathbb{N}} \Sigma_n$ be the whole set of connectives. The set of formulas $\mathcal{S}$ is then defined as the algebra freely generated by ats over Σ. Let's also add here some structure to the set of truth-values of our logics. Unless explicitly stated otherwise, from now on $\mathcal{L}$ will stand for a propositional finite-valued logic. Additionally, $\mathbb{V}$ will be a fixed Σ-algebra defining a truth-functional semantics for $\mathcal{L}$ over a finite non-empty set of truth-values $\mathcal{V} = \mathcal{D} \cup \mathcal{U}$. Assume that $\mathcal{D} = \{d_1, \ldots, d_i\}$ and $\mathcal{U} = \{u_1, \ldots, u_j\}$ are the sets of designated and undesignated truth-values, respectively, with $\mathcal{D} \cap \mathcal{U} = \emptyset$. Assume also that the valuations composing the semantics of *genuinely n-valued* logics (logics having n-valued characterizing matrices, but no m-valued such matrices, for $m < n$) are given by the homomorphisms $\S : \mathcal{S} \to \mathbb{V}$. A *uniform substitution* is an endomorphism $\varepsilon : \mathcal{S} \to \mathcal{S}$. Let us denote by $\varphi(p_1, \ldots, p_n)$ a formula φ whose set of atomic sentences appear among $p_1, \ldots, p_n$. From now on, we write $\varphi(p_1/\alpha_1, \ldots, p_n/\alpha_n)$ instead of $\varepsilon(\varphi(p_1, \ldots, p_n))$ whenever $\varepsilon(p_k) = \alpha_k$. Given a genuinely n-valued logic $\mathcal{L}$ whose semantics is determined by $\langle \mathcal{V}, cct, \mathcal{D}\rangle$, we shall denote by $\mathcal{L}^{\mathbf{c}}$ any functionally complete genuinely n-valued (conservative) extension of it (extending, if necessary, the signature Σ), that is, a logic $\mathcal{L}^{\mathbf{c}}$ with the same number of (un)designated values as $\mathcal{L}$, but which can define all n-valued matrices —had they not been already definable from the start.

Def. 3.1. A set of *interpretation maps* $[.] : \mathcal{V}^n \to \mathcal{V}$ over $\mathcal{S}$, for each $n \in \mathbb{N}^+$, is defined as follows, given $\vec{v} = (v_1, \ldots, v_n) \in \mathcal{V}^n$:

(i) $[p_k](\vec{v}) = v_k$, if $1 \leq k \leq n$;
(ii) $[\otimes(\varphi_1, \ldots, \varphi_m)](\vec{v}) = \otimes([\varphi_1](\vec{v}), \ldots, [\varphi_m](\vec{v}))$, where $\otimes$ is an m-ary connective and $\otimes$ is identified with the corresponding operator in the algebra $\mathbb{V}$.

Remark 3.2. Given formulas $\varphi(p)$ and α of $\mathcal{L}$, and a homomorphism $\S : \mathcal{S} \to \mathbb{V}$, then we have:

$$[\varphi](\S(\alpha)) = \S(\varphi(p/\alpha)). \qquad (*)$$

Def. 3.3. Let $v_1, v_2 \in \mathcal{V}$. We say that v_1 and v_2 are *separated*, and we write $v_1 \sharp v_2$, in case v_1 and v_2 belong to different classes of truth-values, that is, in case either $v_1 \in \mathcal{D}$ and $v_2 \in \mathcal{U}$, or $v_1 \in \mathcal{U}$ and $v_2 \in \mathcal{D}$. Given some genuinely n-valued logic $\mathcal{L}$, there is always some formula $\varphi(p)$ of $\mathcal{L}^{\mathbf{c}}$ which *separates* v_1 and v_2, that is, such that $[\varphi](v_1)\sharp[\varphi](v_2)$ (or else one of these two values would be redundant, and the logic would thus not be genuinely n-valued). Equivalently, one can say that $\varphi(p)$ separates v_1 and v_2 if the truth-values obtained in the truth-table for φ when p takes the values v_1 and v_2 are separated. We say that v_1 and v_2 are *effectively separated* by a logic $\mathcal{L}$ in case there is some separating formula $\varphi(p)$ to be found among the original set of formulas of $\mathcal{L}$. In that case we will also say that the values v_1 and v_2 of $\mathcal{L}$ are *effectively separable*.

Example 3.4. Clearly, if $v_1 \sharp v_2$ then p separates v_1 and v_2. Therefore, every pair of separated truth-values is always effectively separable. As another example, note that $\varphi(p) = \neg p$ separates 0 and $\frac{1}{2}$ in Łukasiewicz's logic Ł$_3$ (see the formulation of its matrices at Example 3.9), given that $[\neg p](0) = \neg 0 = 1$, $[\neg p](\frac{1}{2}) = \neg\frac{1}{2} = \frac{1}{2}$, and $1\sharp\frac{1}{2}$. The separability of the truth-values of a logic $\mathcal{L}$ clearly depends on the original expressibility of this logic, i.e., the range of matrices that it can define by way of interpretations of its formulas. The truth-values of a functionally complete logic, for instance, are all obviously separable. Consider, in contrast, a logic whose semantics is given by $\langle\{0, \frac{1}{2}, 1\}, \{\otimes\}, \{1\}\rangle$, where $v_1 \otimes v_2 = v_1$ if $v_1 = v_2$, otherwise $v_1 \otimes v_2 = 1$. The values 0 and $\frac{1}{2}$ of this logic are obviously not separable.

Assumption 3.5. (Separability)
From this point on we will assume that, for any finite-valued logic we consider, every pair $\langle v_1, v_2\rangle \in \mathcal{D}^2 \cup \mathcal{U}^2$ such that $v_1 \neq v_2$ is effectively separable.

It follows from the last assumption that it is possible to individualize every truth-value in terms of its membership to $\mathcal{D}$ (to be represented here by the 'logical' value T) or to $\mathcal{U}$ (to be represented by the 'logical' value F). As it will be shown, together with this assumption about the expressibility of our logics, the residual bivalence embodied in the distinction between designated and undesignated values will permit us to effectively reformulate our original n-valued semantics using at most two truth-values.

Remark 3.6. Consider the mapping $t : \mathcal{V} \to \{T, F\}$ such that $t(v) = T$ iff $v \in \mathcal{D}$, for some logic $\mathcal{L}$. Note that:

$$\varphi \text{ separates } v_1 \text{ and } v_2 \text{ iff } t([\varphi](v_1)) \neq t([\varphi](v_2)). \qquad (**)$$

Now, suppose that φ_{mn} separates d_m and d_n (for $1 \leq m < n \leq i$), and ψ_{mn} separates u_m and u_n (for $1 \leq m < n \leq j$). Given a variable x and $d \in \mathcal{D}$, consider the equation:

$$t([\varphi_{mn}](x)) = q^d_{mn}$$

where $q^d_{mn} = t([\varphi_{mn}](d))$. Observe that $q^d_{mn} \in \{T, F\}$ and $q^{d_m}_{mn} \neq q^{d_n}_{mn}$, using $(**)$. Thus, if $\vec{\varphi}_d(x)$ is the sequence $(t([\varphi_{mn}](x)) = q^d_{mn})_{1\leq m<n\leq i}$, the distinguished truth-value d can then be characterized through the sequence of equations $Q_d(x) : (t(x) = T, \vec{\varphi}_d(x))$, where commas represent conjunctions. That is,

$$x = d \text{ iff } t(x) = T \ \wedge \bigwedge_{1\leq m<n\leq i} t([\varphi_{mn}](x)) = q^d_{mn}$$

characterizes d in terms of membership to $\mathcal{D}$ or to $\mathcal{U}$ (or, equivalently, in terms of T/F), as desired. Analogously, if r^u_{mn} is $t([\psi_{mn}](u))$ for $1 \leq m < n \leq j$ and $u \in \mathcal{U}$, then the sequence of equations $R_u(x) : (t(x) = F, \vec{\psi}_u(x))$ characterizes u in terms of T/F, where $\vec{\psi}_u(x) = (t([\psi_{mn}](x)) = r^u_{mn})_{1\leq m<n\leq j}$. That is,

$$x = u \text{ iff } t(x) = F \ \wedge \bigwedge_{1\leq m<n\leq j} t([\psi_{mn}](x)) = r^u_{mn}$$

characterizes u in terms of T/F using t.

Remark 3.7. If $\mathcal{D} = \{d\}$ then we simply write $x = d$ iff $t(x) = T$. Analogously, if $\mathcal{U} = \{u\}$ then we simply write $x = u$ iff $t(x) = F$.

Remark 3.8. For any given logic $\mathcal{L}$, the composition $b = t \circ \S$ gives us exactly Suszko's 2-valued reduction, viz. a 2-valued (usually non-truth-functional) semantic presentation of $\mathcal{L}$. Given a logic that respects our Separability Assumption 3.5, we will see in the next section how this 2-valued semantics can be mechanically written down in terms of 'dyadic semantics'. A later section will show how such semantics can provide us with classic-like tableaux for those same logics.

Example 3.9. Consider the n-valued logics of Łukasiewicz, $n > 2$, which can be formulated by way of:

$$\text{Ł}_n = \langle\{0, \tfrac{1}{n-1}, \ldots, \tfrac{n-2}{n-1}, 1\}, \{\neg, \Rightarrow, \vee, \wedge\}, \{1\}\rangle.$$

The above operations over the truth-values can be defined as follows:

$$\neg v_1 := 1 - v_1; \qquad (v_1 \Rightarrow v_2) := \mathsf{Min}(1, 1 - v_1 + v_2);$$
$$(v_1 \vee v_2) := \mathsf{Max}(v_1, v_2); \quad (v_1 \wedge v_2) := \mathsf{Min}(v_1, v_2).$$

Consider now the particular case of Ł_5. Then we can take, for instance:

$$\psi_{0\frac{1}{4}} = \psi_{0\frac{2}{4}} = \psi_{0\frac{3}{4}} = \neg p; \quad \psi_{\frac{1}{4}\frac{2}{4}} = \psi_{\frac{1}{4}\frac{3}{4}} = (\neg p \Rightarrow p); \quad \psi_{\frac{2}{4}\frac{3}{4}} = (p \Rightarrow \neg p).$$

To save on notation, take $\triangle(p) = \psi_{\frac{1}{4}\frac{2}{4}}$ and $\nabla(p) = \psi_{\frac{2}{4}\frac{3}{4}}$, and consider next the table:

v	$\neg v$	$\triangle(v)$	$\nabla(v)$
0	1	0	1
$\frac{1}{4}$	$\frac{3}{4}$	$\frac{2}{4}$	1
$\frac{2}{4}$	$\frac{2}{4}$	1	1
$\frac{3}{4}$	$\frac{1}{4}$	1	$\frac{2}{4}$

Note that (the reduced version of) each $\vec{\psi}_k(x)$ is as follows:

$$\vec{\psi}_0(x) = \langle t(\neg x) = T,\, t(\triangle(x)) = F,\, t(\nabla(x)) = T\rangle,$$
$$\vec{\psi}_{\frac{1}{4}}(x) = \langle t(\neg x) = F,\, t(\triangle(x)) = F,\, t(\nabla(x)) = T\rangle,$$
$$\vec{\psi}_{\frac{2}{4}}(x) = \langle t(\neg x) = F,\, t(\triangle(x)) = T,\, t(\nabla(x)) = T\rangle,$$
$$\vec{\psi}_{\frac{3}{4}}(x) = \langle t(\neg x) = F,\, t(\triangle(x)) = T,\, t(\nabla(x)) = F\rangle.$$

We obtain thus the following characterizations of the truth-values:

$$x = 0 \text{ iff } t(x) = F \wedge t(\neg x) = T \wedge t(\triangle(x)) = F \wedge t(\nabla(x)) = T,$$
$$x = \tfrac{1}{4} \text{ iff } t(x) = F \wedge t(\neg x) = F \wedge t(\triangle(x)) = F \wedge t(\nabla(x)) = T,$$
$$x = \tfrac{2}{4} \text{ iff } t(x) = F \wedge t(\neg x) = F \wedge t(\triangle(x)) = T \wedge t(\nabla(x)) = T,$$
$$x = \tfrac{3}{4} \text{ iff } t(x) = F \wedge t(\neg x) = F \wedge t(\triangle(x)) = T \wedge t(\nabla(x)) = F.$$

Obviously, the sole distinguished truth-value 1 is characterized simply by:

$$x = 1 \;\text{ iff }\; t(x) = T.$$

A similar procedure can be applied to all the remaining finite-valued logics of Łukasiewicz, making use for instance of the well-known Rosser-Turquette (definable) functions so as to produce the appropriate effective separations of truth-values.

4. Dyadic semantics

Suszko's Reduction is quite general, and it applies to any tarskian logic, be it truth-functional or not. In the next section we will exhibit our algorithmic reductive method for automatically obtaining 2-valued formulations of any sufficiently expressive finite-valued logic. To that purpose, it will be convenient to make use of an appropriate equational language, made explicit in the following.

Def. 4.1. A *gentzenian semantics* for a logic $\mathcal{L}$ is an adequate (sound and complete) set of 2-valued valuations $b : \mathcal{S} \to \{T, F\}$ given by conditional clauses $(\Phi \to \Psi)$ where both Φ and Ψ are (meta)formulas of the form $\top$ (top), $\bot$ (bottom) or:

$$b(\varphi_1^1) = w_1^1, \ldots, b(\varphi_1^{n_1}) = w_1^{n_1} \mid \ldots \mid b(\varphi_m^1) = w_m^1, \ldots, b(\varphi_m^{n_m}) = w_m^{n_m}. \qquad (G)$$

Here, $w_i^j \in \{T, F\}$, each φ_i^j is a formula of $\mathcal{L}$, commas "," represent conjunctions, and bars "|" represent disjunctions. The (meta)logic governing these clauses is FOL, First-Order Classical Logic (further on, $\to$ will be used to represent the implication connective from this metalogic). We may alternatively write a clause of the form (G) as $\bigvee_{1\leq k\leq m} \bigwedge_{1\leq s\leq n_m} b(\varphi_k^s) = w_k^s$.

A dyadic semantics will consist in a specialization of a gentzenian semantics, in a deliberate intent to capture the computable class of such semantics, as follows. It should be noticed, at any rate, that not all decidable 2-valued semantics will come with a built-in gentzenian presentation. Moreover, as shown in Example 4.6, there are many logics that are characterizable by gentzenian or even by dyadic semantics, yet not by any genuinely finite-valued semantics.

Remark 4.2. (i) Given an algebra of formulas $\mathcal{S}$, an appropriate *measure of complexity* of these formulas may be defined as the output of some schematic mapping $\ell : \mathcal{S} \to \mathbb{N}$, with the restriction that $\ell(p_k) = 0$, for each $p_k \in ats$. As a particular case, the *canonical* measure of complexity of $\varphi = \otimes(\varphi_1, \ldots, \varphi_m)$ has the additional restriction that $\ell(\varphi) = 1 + \ell(\varphi_1) + \ldots + \ell(\varphi_m)$, for each $\otimes \in cct$.

(ii) Let $var : \mathcal{S} \to \mathsf{Pow}(ats)$ be a mapping that associates to each formula its set of atomic subformulas. Given an algebra of formulas $\mathcal{S}$, denote by $\mathcal{S}[n]$, for $n \geq 1$, the set $\mathcal{S}[n] = \{\varphi \in \mathcal{S} : var(\varphi) = \{p_1, \ldots, p_n\}\}$. There are surely non-empty (and possibly finite) families of formulas $(\psi_i)_{i\in I}$, for some $I = \{1, 2, \ldots\} \subseteq \mathbb{N}^+$, and there are $1 \leq n_i \leq \aleph_0$, for each $i \in I$, with $\psi_i \in \mathcal{S}[n_i]$, which cover the whole set of formulas up to some substitution, that is, such that $\mathcal{S} = \bigcup_{i\in I}\{\varepsilon(\psi_i) : \varepsilon \text{ is a substitution}\}$. A minimal example of such a covering family is given by $\{\otimes(p_1, \ldots, p_n) : \otimes \in \Sigma_n \text{ and } n \in \mathbb{N}\}$.

Def. 4.3. A logic $\mathcal{L}$ is said to be *quasi tabular* in case:

(i) There is some measure of complexity ℓ and there is some covering family of formulas $\{\psi_i\}_{i\in I}$, with $\psi_i \in \mathcal{S}[n_i]$, for some (possibly finite) set $I = \{1, 2, \ldots\} \subseteq \mathbb{N}^+$ such that for each ψ_i there is a finite sequence $\langle\phi_s^i\rangle_{s=1,\ldots,k_i}$ of formulas such that $var(\phi_s^i) \subseteq \{p_1, \ldots, p_{n_i}\}$, and $\ell(\phi_s^i) < \ell(\psi_i)$, for $1 \leq s \leq k_i$.
(ii) There is an adequate $|\mathcal{V}|$-valued set of valuations $\S : \mathcal{S} \to \mathcal{V}$ for $\mathcal{L}$, for some finite set of truth-values $\mathcal{V}$, such that for each $i \in I$ there is some recursive function $\lceil . \rceil_i : \mathcal{V}^{k_i} \to \mathcal{V}$ according to which, if $\phi = \varepsilon(\psi_i)$ for some substitution ε, then $\S(\phi) \bowtie_i \lceil \S(\varepsilon(\phi_1^i)), \ldots, \S(\varepsilon(\phi_{k_i}^i)) \rceil_i$ for every $\S$, where $\bowtie_i$ is one of the following partial ordering relations defined on $\mathcal{V}$: $=$, $\leq$, or $\geq$.

The reader will have remarked that the above definition of quasi-tabularity extends, in a sense, the usual Fregean notion of semantic compositionality.

Def. 4.4. A quasi tabular logic is called *tabular* in case ℓ can be taken to be the canonical measure of complexity and, accordingly, for each $i \in I$, one can take $\langle\phi_s^i\rangle_{s=1,\ldots,k_i}$ as the immediate subformulas of ψ_i. In that case, also, the covering set $\{\psi_i\}_{i\in I}$ can be taken to be the minimal one (check Remark 4.2(ii)), and each $\bowtie_i$ can be limited to the equality symbol $=$.

Tabular logics define exactly the class of *truth-functional logics*, given that the former logics are always genuinely n-valued, for some $1 \leq n \leq |\mathcal{V}|$.

Def. 4.5. A quasi tabular logic $\mathcal{L}$ is said to have a *dyadic semantics* in case the set $\mathcal{V}$ of Def. 4.3(ii) is $\{T, F\}$, and additionally $\mathcal{L}$ can be endowed with an adequate gentzenian semantics.

The class of quasi tabular logics is quite wide: Genuinely finite-valued logics are but a very special case of them, and the former class in fact coincides with the class of logics which can be given a so-called 'society semantics with complex base' (cf. [17]). It even includes logics that cannot be characterized as genuinely finite-valued, as the following example shows:

Example 4.6. Consider the paraconsistent logic $\mathcal{C}_1$ (cf. [14]). It is well known that this logic has no genuinely finite-valued characterizing semantics, though it *can* be decided by way of 'quasi matrices' (cf. [15]). In fact, a dyadic semantics for $\mathcal{C}_1$ is promptly available (cf. [9]). To that effect, recall that α° abbreviates $\neg(\alpha \wedge \neg\alpha)$ in $\mathcal{C}_1$, and consider the following bivaluational axioms (where $\sqcap$, $\sqcup$, $-$ are the usual lattice operators):

(4.6.1) $b(\neg\alpha) \geq -b(\alpha)$;
(4.6.2) $b(\neg\neg\alpha) \leq b(\alpha)$;
(4.6.3) $b(\alpha \wedge \beta) = b(\alpha) \sqcap b(\beta)$;
(4.6.4) $b(\alpha \vee \beta) = b(\alpha) \sqcup b(\beta)$;
(4.6.5) $b(\alpha \Rightarrow \beta) = -b(\alpha) \sqcup b(\beta)$;
(4.6.6) $b(\alpha^\circ) = -b(\alpha) \sqcup -b(\neg\alpha)$;
(4.6.7) $b((\alpha \otimes \beta)^\circ) \geq (-b(\alpha) \sqcup -b(\neg\alpha)) \sqcap (-b(\beta) \sqcup -b(\neg\beta))$, for $\otimes \in \{\wedge, \vee, \Rightarrow\}$.

As it will be clear further on, in case it is possible to obtain a tableau decision procedure from a gentzenian semantics $\mathcal{B}$ for a logic $\mathcal{L}$ then $\mathcal{B}$ is a dyadic semantics for $\mathcal{L}$.

5. From finite matrices to dyadic valuations

Let $\otimes$ be some connective of $\mathcal{L}$; for the sake of simplicity, suppose that $\otimes$ is binary. If an entry of the truth-table for $\otimes$ states that $\otimes(v_1, v_2) = v$ then we can express this situation as follows:

$$\text{If } x = v_1 \text{ and } y = v_2, \text{ then } \otimes(x, y) = v.$$

Now, recall from Remark 3.6 the mapping $t : \mathcal{V} \to \{T, F\}$ such that $t(v) = T$ iff $v \in \mathcal{D}$. If the previous situation is expressed in terms of T/F using this mapping, we will get, respectively, systems of equations $E_{v_1}(x)$, $E_{v_2}(y)$ and $E_v(\otimes(x, y))$, and consequently the following statement in terms of T/F:

$$\text{if } E_{v_1}(x) \text{ and } E_{v_2}(y) \text{ then } E_v(\otimes(x, y)).$$

In the formal metalanguage of a gentzenian semantics (Def. 4.1), this statement is of the form:

$$\begin{array}{rl} & t([\beta_1](x)) = w_1, \ldots, t([\beta_m](x)) = w_m, \\ & t([\gamma_1](y)) = w'_1, \ldots, t([\gamma_{m'}](y)) = w'_{m'} \\ \to & t([\delta_1](\otimes(x, y))) = w''_1, \ldots, t([\delta_{m''}](\otimes(x, y))) = w''_{m''}, \end{array} \qquad (***)$$

where $w_n, w'_{k'}, w''_{s''} \in \{T, F\}$ for $1 \leq n \leq m$, $1 \leq k' \leq m'$ and $1 \leq s'' \leq m''$.

Now, suppose that v is $\S(\alpha)$ for some formula α. Then, using $(*)$ (check Remarks 3.2 and 3.8) we obtain:

$$t([\varphi](v)) = t([\varphi](\S(\alpha))) = t(\S(\varphi(p/\alpha))) = b(\varphi(p/\alpha))$$

for every formula $\varphi(p)$. Using this in $(***)$ we obtain an axiom for $\mathcal{B}$ of the form:

$$\begin{array}{rl} & b(\beta_1(p/\alpha)) = w_1, \ \ldots, \ b(\beta_m(p/\alpha)) = w_m, \\ & b(\gamma_1(p/\beta)) = w'_1, \ \ldots, \ b(\gamma_{m'}(p/\beta)) = w'_{m'} \\ \to & b(\delta_1(p/\otimes(\alpha, \beta))) = w''_1, \ \ldots, \ b(\delta_{m''}(p/\otimes(\alpha, \beta))) = w''_{m''}, \end{array}$$

for $w_n, w'_{k'}, w''_{s''} \in \{T, F\}$ etc. Obviously, we can repeat this process for each entry of each connective $\otimes$ of $\mathcal{L}$. For 0-ary connectives there is no input at the left-hand side; in such case, you should write conditional clauses of the form $(\top \to \Psi)$.

Example 5.1. In $Ł_5$ we have, for instance, the following entry in the truth-table for $\wedge$: If $v_1 = \frac{2}{4}$ and $v_2 = 1$ then $v_1 \wedge v_2 = \frac{2}{4}$. Or, in other words: If $\S(\alpha) = \frac{2}{4}$ and $\S(\beta) = 1$ then $\S(\alpha \wedge \beta) = \frac{2}{4}$, for any formulas α and β, and any homomorphism $\S$. From Example 3.9 we obtain, using t and $b = t \circ \S$:

$$\begin{array}{l} b(\alpha) = F, \ b(\neg\alpha) = F, \ b(\triangle(\alpha)) = T, \ b(\triangledown(\alpha)) = T, \ b(\beta) = T \\ \qquad \to \ b(\alpha \wedge \beta) = F, \ b(\neg(\alpha \wedge \beta)) = F, \ b(\triangle(\alpha \wedge \beta)) = T, b(\triangledown(\alpha \wedge \beta)) = T. \end{array}$$

So, each entry of the truth-table for each connective $\otimes$ of $\mathcal{L}$ determines an axiom for a gentzenian valuation $b : \mathcal{S} \to \{T, F\}$. We obtain thus, through the above method, a kind of unique (partial) 'dyadic print' of the original truth-functional logic.

Theorem 5.2. *Given a logic $\mathcal{L}$, let $\mathcal{B}$ be the set of gentzenian valuations $b : \mathcal{S} \to \{T, F\}$ satisfying the axioms obtained from the truth-tables of $\mathcal{L}$ using the above method, plus the following axioms:*

(C1): $\top \;\to\; b(\alpha) = T \mid b(\alpha) = F$;
(C2): $b(\alpha) = T,\ b(\alpha) = F \;\to\; \bot$;
(C3): $b(\alpha) = T \;\to\; \bigvee_{d\in\mathcal{D}} \bigwedge_{1\le m<n\le i} b(\varphi_{mn}(p/\alpha)) = q^d_{mn}$;
(C4): $b(\alpha) = F \;\to\; \bigvee_{u\in\mathcal{U}} \bigwedge_{1\le m<n\le j} b(\psi_{mn}(p/\alpha)) = r^u_{mn}$,

for every $\alpha \in \mathcal{S}$ (here, q^d_{mn} and r^u_{mn} are as in Remark 3.6*). Then $b \in \mathcal{B}$ iff $b = t \circ \S$ for some homomorphism $\S : \mathcal{S} \to \mathbb{V}$.*

Proof. Given $b \in \mathcal{B}$, define a homomorphism $\S : \mathcal{S} \to \mathbb{V}$ such that:

(i) $\S(\alpha) = d$ iff $b(\alpha) = T$ and $b(\varphi_{mn}(p/\alpha)) = q^d_{mn}$ for every $1 \le m < n \le i$;
(ii) $\S(\alpha) = u$ iff $b(\alpha) = F$ and $b(\psi_{mn}(p/\alpha)) = r^u_{mn}$ for every $1 \le m < n \le j$,

where α ranges over the atomic sentences $ats \in \mathcal{S}$. Note that $\mathcal{S}$ is well-defined as a total functional assignment because $b \in \mathcal{B}$ satisfies conditions (**C1**)–(**C2**) above. Since b satisfies all the axioms obtained from all the entries of the truth-tables of $\mathcal{L}$, it is straightforward to prove, by induction on the complexity of the formula $\alpha \in \mathcal{S}$, that (i) and (ii) hold when α ranges over all the formulas in $\mathcal{S}$. (Indeed, note that, in the light of conditions (**C2**)–(**C4**), given $b \in \mathcal{B}$ and $b(\alpha) = T$ we can conclude that there exists a unique $d \in \mathcal{D}$ such that $\bigwedge_{1\le m<n\le i} b(\varphi_{mn}(p/\alpha)) = q^d_{mn}$; similarly, given $b(\alpha) = F$ we can conclude that there exists a unique $u \in \mathcal{U}$ such that $\bigwedge_{1\le m<n\le j} b(\psi_{mn}(p/\alpha)) = r^u_{mn}$.) From this we obtain that $\S(\varphi) \in \mathcal{D}$ iff $b(\varphi) = T$, therefore $b = t \circ \S$ as desired. The converse (if $b = t \circ \S$ for some homomorphism $\S$, then $b \in \mathcal{B}$) is immediate. □

Thus, a new 2-valued adequate semantics based on but two 'logical values' can now be seen to realize Suszko's Thesis, through the above constructive method.

Corollary 5.3. (i) *For every bivaluation $b : \mathcal{S} \to \{T, F\}$ in $\mathcal{B}$ there exists a homomorphism $\S_b : \mathcal{S} \to \mathbb{V}$ such that:*

$$\S_b(\alpha) \in \mathcal{D} \;\text{ iff }\; b(\alpha) = T, \text{ for any } \alpha \in \mathcal{S}. \tag{1}$$

(ii) *For every $\S : \mathcal{S} \to \mathbb{V}$ there exists a $b_\S \in \mathcal{B}$ such that:*

$$b_\S(\alpha) = T \;\text{ iff }\; \S(\alpha) \in \mathcal{D}, \text{ for any } \alpha \in \mathcal{S}. \tag{2}$$

We now have two notions of semantic entailment for $\mathcal{L}$. The first one, $\models$, uses the truth-tables given by $\mathbb{V}$ and its corresponding homomorphic valuations, whereas the second one, $\models_{\mathcal{B}}$, uses the related gentzenian semantics $\mathcal{B}$. But both notions are in a sense 'talking about the same thing':

Theorem 5.4. *The set $\mathcal{B}$ of gentzenian valuations for $\mathcal{L}$ is adequate, that is, for any $\Gamma \cup \{\varphi\} \subseteq \mathcal{S}$:*

$$\Gamma \models \varphi \quad \textit{iff} \quad \Gamma \models_{\mathcal{B}} \varphi.$$

Proof. Suppose that $\Gamma \models \varphi$, and let $b \in \mathcal{B}$ be such that $b(\Gamma) \subseteq \{T\}$, if possible. By Corollary 5.3(i) there exists a homomorphism $\S_b$ such that $\S_b(\Gamma) \subseteq \mathcal{D}$. By hypothesis we get $\S_b(\varphi) \in \mathcal{D}$, whence $b(\varphi) = T$ by (1). This shows that $\Gamma \models_{\mathcal{B}} \varphi$. The converse is proven in an analogous way, using Corollary 5.3(ii). □

6. Some Illustrations

In this section we will give examples of gentzenian semantics for several genuinely finite-valued paraconsistent logics, obtained through applications of the reductive algorithm proposed in the last section. Instead of writing extensive lists of bivaluational axioms, one for each entry of each truth-table, plus some complementing axioms, we shall be using First-Order Classical Logic, FOL, in what follows, in order to manipulate and simplify the clauses written in our equational metalanguage. Moreover, we will often seek to reformulate things so as to make them more convenient for a tableaux-oriented approach, as in the next section.

Example 6.1. The paraconsistent logic $\mathbf{P}^1_3 = \langle\{0, \frac{1}{2}, 1\}, \{\neg, \Rightarrow\}, \{\frac{1}{2}, 1\}\rangle$, was introduced by Sette in [22] (where it was called P^1), having as truth-tables:

	0	$\frac{1}{2}$	1
$\neg$	1	1	0

$\Rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	1	1
1	0	1	1

Note that $\neg p$ separates $\frac{1}{2}$ and 1. Indeed:

$$[\neg p](1) = 0, \; [\neg p](\tfrac{1}{2}) = 1,$$

and $0 \sharp 1$. Thus:

$$\begin{array}{lll} x = 0 & \text{iff} & t(x) = F; \\ x = \frac{1}{2} & \text{iff} & t(x) = T, \, t(\neg x) = T; \\ x = 1 & \text{iff} & t(x) = T, \, t(\neg x) = F. \end{array}$$

Applying our reductive algorithm to the truth-tables of $\neg$ and $\Rightarrow$ we may, after some simplification, obtain the following axioms for b:

(i) $b(\alpha) = F \;\rightarrow\; b(\neg\alpha) = T, \; b(\neg\neg\alpha) = F$;
(ii) $b(\alpha) = T, \; b(\neg\alpha) = T \;\rightarrow\; b(\neg\alpha) = T, \; b(\neg\neg\alpha) = F$;
(iii) $b(\alpha) = T, \; b(\neg\alpha) = F \;\rightarrow\; b(\neg\alpha) = F$;
(iv) $b(\alpha) = F \mid b(\beta) = T \;\rightarrow\; b(\alpha \Rightarrow \beta) = T, \; b(\neg(\alpha \Rightarrow \beta)) = F$;
(v) $b(\alpha) = T, \; b(\beta) = F \;\rightarrow\; b(\alpha \Rightarrow \beta) = F$.

In this case, axiom **(C3)** corresponds to $b(\alpha) = T \;\to\; b(\neg\alpha) = T \mid b(\neg\alpha) = F$, which can be derived from **(C1)**. Axiom **(C4)** corresponds to $b(\alpha) = F \;\to\; b(\neg\alpha) = T$, which is derivable from the above clause (i). Using FOL we may rewrite clauses (i)–(v) equivalently as:

(6.1.1) $b(\neg\alpha) = F \;\to\; b(\alpha) = T$;
(6.1.2) $b(\neg\neg\alpha) = T \;\to\; b(\neg\alpha) = F$;
(6.1.3) $b(\alpha \Rightarrow \beta) = T \;\to\; b(\alpha) = F \mid b(\beta) = T$;
(6.1.4) $b(\alpha \Rightarrow \beta) = F \;\to\; b(\alpha) = T,\ b(\beta) = F$;
(6.1.5) $b(\neg(\alpha \Rightarrow \beta)) = T \;\to\; b(\alpha) = T,\ b(\beta) = F$.

Note that (6.1.3)–(6.1.5) axiomatize a sort of 'classic-like' implication. Axioms (6.1.1)–(6.1.5) plus **(C1)**–**(C2)** characterize a dyadic semantics for $\mathbf{P}^1_3$.

Example 6.2. The paraconsistent logic $\mathbf{P}^1_4 = \langle\{0, \frac{1}{3}, \frac{2}{3}, 1\}, \{\neg, \Rightarrow\}, \{\frac{1}{3}, \frac{2}{3}, 1\}\rangle$, was introduced in [11] and [19], and studied under the name P^2 in [17]. The truth-tables of its connectives are as follows:

	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\neg$	1	$\frac{2}{3}$	1	0

$\Rightarrow$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
0	1	1	1	1
$\frac{1}{3}$	0	1	1	1
$\frac{2}{3}$	0	1	1	1
1	0	1	1	1

It is easy to see that $\neg p$ separates 1 and $\frac{1}{3}$, as well as 1 and $\frac{2}{3}$. On the other hand, $\neg\neg p$ separates $\frac{1}{3}$ and $\frac{2}{3}$. From this we get:

$x = 0$ iff $t(x) = F$;
$x = \frac{1}{3}$ iff $t(x) = T,\ t(\neg x) = T,\ t(\neg\neg x) = T$;
$x = \frac{2}{3}$ iff $t(x) = T,\ t(\neg x) = T,\ t(\neg\neg x) = F$;
$x = 1$ iff $t(x) = T,\ t(\neg x) = F,\ t(\neg\neg x) = T$.

From the truth-table for $\neg$ we obtain, after applying FOL:

(6.2.1) $b(\neg\alpha) = F \;\to\; b(\alpha) = T$;
(6.2.2) $b(\neg\neg\alpha) = T \;\to\; b(\alpha) = T$;
(6.2.3) $b(\neg\neg\neg\alpha) = T \;\to\; b(\neg\neg\alpha) = F$.

Once again, axiom **(C3)** is derivable from **(C1)**, and axiom **(C4)** is derivable from the clauses above. The implication $\Rightarrow$ is again 'classic-like', in the same sense as in the last example. Therefore, axioms (6.2.1)–(6.2.3), (6.1.3)–(6.1.5) and **(C1)**–**(C2)** characterize together a dyadic semantics for $\mathbf{P}^1_4$. Similar procedures can be applied to each paraconsistent logic of the hierarchy $\mathbf{P}^1_{n+2}(= P^n,$ from [17]), for $n \in \mathbb{N}^+$.

Example 6.3. Having already used negation in the two above examples in order to separate truth-values, let us now make it differently. Consider the paraconsistent propositional logic **LFI1** $= \langle\{0, \frac{1}{2}, 1\}, \{\neg, \bullet, \Rightarrow, \wedge, \vee\}, \{\frac{1}{2}, 1\}\rangle$, studied in detail in [13], whose matrices are:

	0	$\frac{1}{2}$	1
$\neg$	1	$\frac{1}{2}$	0
$\bullet$	0	1	0

$\Rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	$\frac{1}{2}$	1
1	0	$\frac{1}{2}$	1

plus conjunction $\wedge$ and disjunction $\vee$ defined as in Łukasiewicz's logics (see Example 3.9). Clearly, $\bullet p$ separates 1 and $\frac{1}{2}$. Thus:

$x = 0$ iff $t(x) = F$;
$x = \frac{1}{2}$ iff $t(x) = T$, $t(\bullet x) = T$;
$x = 1$ iff $t(x) = T$, $t(\bullet x) = F$.

From the truth-table for $\neg$, and using FOL, we obtain:

(6.3.1) $b(\neg\alpha) = T \;\to\; b(\alpha) = F \mid b(\bullet\alpha) = T$;
(6.3.2) $b(\neg\alpha) = F \;\to\; b(\alpha) = T,\; b(\bullet\alpha) = F$.

Axiom **(C3)** is again derivable from **(C1)**; axiom **(C4)** is derivable from (6.3.2). Now, these are the axioms for $\bullet$:

(6.3.3) $b(\bullet\alpha) = T \;\to\; b(\alpha) = T$;
(6.3.4) $b(\bullet\bullet\alpha) = T \;\to\; b(\bullet\alpha) = F$;
(6.3.5) $b(\bullet\neg\alpha) = T \;\to\; b(\bullet\alpha) = T$;
(6.3.6) $b(\bullet\neg\alpha) = F \;\to\; b(\neg\alpha) = F \mid b(\alpha) = F$.

From the truth-tables for the binary connectives, and using FOL, we obtain:

(6.3.7) $b(\alpha \wedge \beta) = T \;\to\; b(\alpha) = T,\; b(\beta) = T$;
(6.3.8) $b(\alpha \wedge \beta) = F \;\to\; b(\alpha) = F \mid b(\beta) = F$;
(6.3.9) $b(\alpha \vee \beta) = T \;\to\; b(\alpha) = T \mid b(\beta) = T$;
(6.3.10) $b(\alpha \vee \beta) = F \;\to\; b(\alpha) = F,\; b(\beta) = F$;
(6.3.11) $b(\alpha \Rightarrow \beta) = T \;\to\; b(\alpha) = F \mid b(\beta) = T$;
(6.3.12) $b(\alpha \Rightarrow \beta) = F \;\to\; b(\alpha) = T, b(\beta) = F$.

To those we may add, furthermore:

(6.3.13) $b(\bullet(\alpha \wedge \beta)) = T$
$\to\; b(\alpha) = T,\; b(\bullet\beta) = T \mid b(\beta) = T,\; b(\bullet\alpha) = T$;
(6.3.14) $b(\bullet(\alpha \wedge \beta)) = F$
$\to\; b(\alpha) = F \mid b(\beta) = F \mid b(\alpha) = T,\; b(\bullet\alpha) = F,\; b(\beta) = T,\; b(\bullet\beta) = F$;
(6.3.15) $b(\bullet(\alpha \vee \beta)) = T$
$\to\; b(\alpha) = F,\; b(\bullet\beta) = T \mid b(\beta) = F,\; b(\bullet\alpha) = T \mid b(\bullet\alpha) = T,\; b(\bullet\beta) = T$;
(6.3.16) $b(\bullet(\alpha \vee \beta)) = F$
$\to\; b(\alpha) = F,\; b(\beta) = F \mid b(\alpha) = T,\; b(\bullet\alpha) = F \mid b(\beta) = T,\; b(\bullet\beta) = F$;
(6.3.17) $b(\bullet(\alpha \Rightarrow \beta)) = T \;\to\; b(\alpha) = T,\; b(\bullet\beta) = T$;
(6.3.18) $b(\bullet(\alpha \Rightarrow \beta)) = F \;\to\; b(\alpha) = F \mid b(\bullet\beta) = F$.

So, if the above axioms are taken together with **(C1)**–**(C2)**, then we obtain a natural dyadic semantics for **LFI1**. Two slightly different (non-gentzenian) bivaluation semantics for **LFI1** were explored in [13].

Example 6.4. Belnap's paraconsistent and paracomplete 4-valued logic (cf. [2]), $B_4 = \langle\{0, \frac{1}{3}, \frac{2}{3}, 1\}, \{\neg, \wedge, \vee\}, \{\frac{2}{3}, 1\}\rangle$, can be presented by way of the following matrices:

	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\neg$	0	$\frac{2}{3}$	$\frac{1}{3}$	1

$\wedge$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
0	0	0	0	0
$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$
$\frac{2}{3}$	0	0	$\frac{2}{3}$	$\frac{2}{3}$
1	0	$\frac{1}{3}$	$\frac{2}{3}$	1

$\vee$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
0	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	1
$\frac{2}{3}$	$\frac{2}{3}$	1	$\frac{2}{3}$	1
1	1	1	1	1

Clearly, $\neg p$ separates 1 and $\frac{2}{3}$ and also separates $\frac{1}{3}$ and 1. Thus:

$x = 0$ iff $t(x) = F$, $t(\neg x) = F$;
$x = \frac{1}{3}$ iff $t(x) = F$, $t(\neg x) = T$;
$x = \frac{2}{3}$ iff $t(x) = T$, $t(\neg x) = F$;
$x = 1$ iff $t(x) = T$, $t(\neg x) = T$.

Now, from the truth-table for $\neg$, and using FOL, we obtain:

(6.4.1) $b(\neg\neg\alpha) = T \ \rightarrow\ b(\alpha) = T$;
(6.4.2) $b(\neg\neg\alpha) = F \ \rightarrow\ b(\alpha) = F$.

Both axioms **(C3)** and **(C4)** are now derivable from **(C1)**. From the truth-tables of conjunction and disjunction, using FOL, we obtain the positive clauses (6.3.7)–(6.3.10) again, but also:

(6.4.3) $b(\neg(\alpha \wedge \beta)) = T \ \rightarrow\ b(\alpha) = F,\ b(\neg\alpha) = T,\ b(\beta) = F,\ b(\neg\beta) = T \mid b(\alpha) = F,\ b(\neg\alpha) = T,\ b(\beta) = T,\ b(\neg\beta) = T \mid b(\alpha) = T,\ b(\neg\alpha) = T,\ b(\beta) = F,\ b(\neg\beta) = T \mid b(\alpha) = T,\ b(\neg\alpha) = T\ b(\beta) = T,\ b(\neg\beta) = T$;

(6.4.4) $b(\neg(\alpha \wedge \beta)) = F \ \rightarrow\ b(\alpha) = F,\ b(\neg\alpha) = F \mid b(\alpha) = T,\ b(\neg\alpha) = F \mid b(\beta) = F,\ b(\neg\beta) = F \mid b(\beta) = T,\ b(\neg\beta) = F$;

(6.4.5) $b(\neg(\alpha \vee \beta)) = T \ \rightarrow\ b(\alpha) = F,\ b(\neg\alpha) = T \mid b(\alpha) = T,\ b(\neg\alpha) = T \mid b(\beta) = F,\ b(\neg\beta) = T \mid b(\beta) = T,\ b(\neg\beta) = T$;

(6.4.6) $b(\neg(\alpha \vee \beta)) = F \ \rightarrow\ b(\alpha) = F,\ b(\neg\alpha) = F,\ b(\beta) = F,\ b(\neg\beta) = F \mid b(\alpha) = F,\ b(\neg\alpha) = F,\ b(\beta) = T,\ b(\neg\beta) = F \mid b(\alpha) = T,\ b(\neg\alpha) = F,\ b(\beta) = F,\ b(\neg\beta) = F \mid b(\alpha) = T,\ b(\neg\alpha) = F\ b(\beta) = T,\ b(\neg\beta) = F$.

A dyadic semantics for B_4 is given by the above axioms, plus **(C1)**–**(C2)**.

7. Application: tableaux for logics with dyadic semantics

In the examples from the last section we found axioms for the set $\mathcal{B}$ of bivaluation mappings b (defining a gentzenian semantics for a genuinely finite-valued logic $\mathcal{L}$) expressed as conditional clauses of the form:

$$b(\alpha) = w$$
$$\to b(\alpha_1^1) = w_1^1, \ldots, b(\alpha_1^{n_1}) = w_1^{n_1} \mid \ldots \mid b(\alpha_m^1) = w_m^1, \ldots, b(\alpha_m^{n_m}) = w_m^{n_m},$$

where $w, w_k^s \in \{T, F\}$ and α_k^s has smaller complexity, under some appropriate measure (recall Remark 4.2 and Def. 4.3), than α. Each clause as above generates a tableau rule for $\mathcal{L}$ as follows: Translate $b(\beta) = T$ as the signed formula $T(\beta)$, and $b(\beta) = F$ as the signed formula $F(\beta)$. Then, a conditional clause such as the one above induces the following tableau-rule:

$$\begin{array}{ccc} & w(\alpha) & \\ \diagup & \cdots & \diagdown \\ w_1^1(\alpha_1^1) & & w_m^1(\alpha_m^1) \\ \vdots & & \vdots \\ w_1^{n_1}(\alpha_1^{n_1}) & & w_m^{n_m}(\alpha_m^{n_m}) \end{array}$$

where $w, w_k^s \in \{T, F\}$. In that case, it is routine to prove that the set of tableau rules for $\mathcal{L}$ obtained from the clauses for $\mathcal{B}$ characterizes a sound and complete tableau system for $\mathcal{L}$ (check [7] for details). We are supposing that there exists a basic common rule known as *branching rule*, as follows:

$$\frac{\cdots}{T(\varphi) \mid F(\varphi)}$$

This rule is generated by clause (C1) of Theorem 5.2. In certain cases it may be possible to dispense with such rule, but taking into consideration that tableau rules are not mandatory but permissive there is little loss of generality in keeping such rule. The branching rule is not analytic, but can be bounded in certain cases so as to guarantee the termination of the decidable tableau procedure. Moreover, the variables occurring in the formula φ must in general be contained in the finite collection of variables occurring in the tableau branch.

The structural similarity between the tableau rules so obtained and the classical ones is not fortuitous. Applying the above idea to the gentzenian semantics obtained in the last section for a large class of many-valued logics, one can devise two-signed tableau systems for them. Many-signed tableau systems for many-valued logics, constructed with the help of their many truth-values used as labels may be obtained as in [10]. Here, though, we learn that we can forget about those 'algebraic truth-values' and work only with the 'logical values' T and F, just like in the classical case. While the former many-signed tableaux enjoy the so-called *subformula property*, according to which each formula α_k^s obtained from the application to α of a tableau rule as the one above is a subformula of the initial formula α, the latter related two-signed tableaux obtained through our method will often fail this property, reflecting the loss of the truth-functionality of the many-valued homomorphisms in transforming them into bivaluations. We will still have, though, a *shortening property* which is as advantageous for efficiency as the subformula property: Each formula α_k^s will be *less complex* (under some appropriate measure, recall Def. 4.3) than the initial formula α being analyzed by the tableau rules, the only exception being the above mentioned branching rule.

Example 7.1. The following set of rules characterizes a tableau system for the paraconsistent logic $\mathbf{P}^1_3$, according to clauses (6.1.1)–(6.1.5) of Example 6.1:

$$(7.1.1)\ \frac{F(\neg\alpha)}{T(\alpha)} \qquad (7.1.2)\ \frac{T(\neg\neg\alpha)}{F(\neg\alpha)}$$

$$(7.1.3)\ \frac{T(\alpha\Rightarrow\beta)}{F(\alpha)\mid T(\beta)} \qquad (7.1.4)\ \frac{F(\alpha\Rightarrow\beta)}{T(\alpha),\ F(\beta)} \qquad (7.1.5)\ \frac{T(\neg(\alpha\Rightarrow\beta))}{T(\alpha),\ F(\beta)}$$

Example 7.2. Following Example 6.2, an adequate set of tableau rules for the paraconsistent logic $\mathbf{P}^1_4$ is given by (7.1.3)–(7.1.5) plus:

$$(7.2.1)\ \frac{F(\neg\alpha)}{T(\alpha)} \qquad (7.2.2)\ \frac{T(\neg\neg\alpha)}{T(\alpha)} \qquad (7.2.3)\ \frac{T(\neg\neg\neg\alpha)}{F(\neg\neg\alpha)}$$

Example 7.3. Here is a tableau system for the paraconsistent logic **LFI1** (see Example 6.3), based on its dyadic semantics:

$$(7.3.1)\ \frac{T(\neg\alpha)}{F(\alpha)\mid T(\bullet\alpha)} \qquad (7.3.2)\ \frac{F(\neg\alpha)}{T(\alpha),F(\bullet\alpha)} \qquad (7.3.3)\ \frac{T(\bullet\alpha)}{T(\alpha)}$$

$$(7.3.4)\ \frac{T(\bullet\bullet\alpha)}{F(\bullet\alpha)} \qquad (7.3.5)\ \frac{T(\bullet\neg\alpha)}{T(\bullet\alpha)} \qquad (7.3.6)\ \frac{F(\bullet\neg\alpha)}{F(\neg\alpha)\mid F(\alpha)}$$

$$(7.3.7)\ \frac{T(\alpha\wedge\beta)}{T(\alpha),\ T(\beta)} \qquad (7.3.8)\ \frac{F(\alpha\wedge\beta)}{F(\alpha)\mid F(\beta)}$$

$$(7.3.9)\ \frac{T(\alpha\vee\beta)}{T(\alpha)\mid T(\beta)} \qquad (7.3.10)\ \frac{F(\alpha\vee\beta)}{F(\alpha),\ F(\beta)}$$

$$(7.3.11)\ \frac{T(\alpha\Rightarrow\beta)}{F(\alpha)\mid T(\beta)} \qquad (7.3.12)\ \frac{F(\alpha\Rightarrow\beta)}{T(\alpha),F(\beta)}$$

$$(7.3.13)\ \frac{T(\bullet(\alpha\wedge\beta))}{\begin{array}{c}T(\alpha),\ \mid T(\beta),\\ T(\bullet\beta)\mid T(\bullet\alpha)\end{array}} \qquad (7.3.14)\ \frac{F(\bullet(\alpha\wedge\beta))}{\begin{array}{l}F(\alpha)\mid F(\beta)\mid\ T(\alpha),T(\beta),\\ \qquad\qquad\quad \mid F(\bullet\alpha),F(\bullet\beta)\end{array}}$$

$$(7.3.15)\ \frac{T(\bullet(\alpha\vee\beta))}{\begin{array}{c}F(\alpha),\ \mid F(\beta),\ \mid T(\bullet\alpha),\\ T(\bullet\beta)\mid T(\bullet\alpha)\ \mid T(\bullet\beta)\end{array}} \qquad (7.3.16)\ \frac{F(\bullet(\alpha\vee\beta))}{\begin{array}{c}F(\alpha),\ \mid T(\alpha),\ \mid T(\beta),\\ F(\beta)\mid F(\bullet\alpha)\ \mid F(\bullet\beta)\end{array}}$$

$$(7.3.17)\ \frac{T(\bullet(\alpha\Rightarrow\beta))}{T(\alpha),T(\bullet\beta)} \qquad (7.3.18)\ \frac{F(\bullet(\alpha\Rightarrow\beta))}{F(\alpha)\mid F(\bullet\beta)}$$

Compare this tableau system for **LFI1** with the tableau system for this same logic presented in [12]. The latter is based on a non-gentzenian semantics. As a result, (decidable) tableaux without the shortening property (in fact, tableaux allowing for loops) were thereby obtained.

Example 7.4. A tableau system for Belnap's 4-valued logic (see Example 6.4), B_4, can be obtained by adding to (7.3.7)–(7.3.10) the following rules:

$$\text{(7.4.1)}\ \frac{T(\neg\neg\alpha)}{T(\alpha)} \qquad \text{(7.4.2)}\ \frac{F(\neg\neg\alpha)}{F(\alpha)}$$

$$\text{(7.4.3)}\ \frac{T(\neg(\alpha\wedge\beta))}{\begin{array}{c|c|c|c} F(\alpha), T(\neg\alpha), & F(\alpha), T(\neg\alpha), & T(\alpha), T(\neg\alpha), & T(\alpha), T(\neg\alpha), \\ F(\beta), T(\neg\beta) & T(\beta), T(\neg\beta) & F(\beta), T(\neg\beta) & T(\beta), T(\neg\beta) \end{array}}$$

$$\text{(7.4.4)}\ \frac{F(\neg(\alpha\wedge\beta))}{F(\alpha), F(\neg\alpha) \mid T(\alpha), F(\neg\alpha) \mid F(\beta), F(\neg\beta) \mid T(\beta), F(\neg\beta)}$$

$$\text{(7.4.5)}\ \frac{T(\neg(\alpha\vee\beta))}{F(\alpha), T(\neg\alpha) \mid T(\alpha), T(\neg\alpha) \mid F(\beta), T(\neg\beta) \mid T(\beta), T(\neg\beta)}$$

$$\text{(7.4.6)}\ \frac{F(\neg(\alpha\vee\beta))}{\begin{array}{c|c|c|c} F(\alpha), F(\neg\alpha), & F(\alpha), F(\neg\alpha), & T(\alpha), F(\neg\alpha), & T(\alpha), F(\neg\alpha), \\ F(\beta), F(\neg\beta) & T(\beta), F(\neg\beta) & F(\beta), F(\neg\beta) & T(\beta), F(\neg\beta) \end{array}}$$

As done in [5], similar algorithmic procedures can be devised so as to provide adequate sequent systems to all the 2-valued semantics hereby constructed.

8. Conclusions

While Suszko's *Thesis* is a philosophical stance concerning the scope of Universal Logic as a general theory of logical structures (cf. [3]), Suszko's *Reduction* is presented in this paper as a general non-constructive result about the comprehensive class of tarskian logics.

We have exhibited here a method for the effective implementation of Suszko's Reduction by transforming any finite-valued truth-functional semantics whose truth-values can be individualized in the sense of Assumption 3.5 into homologous 2-valued semantics. The specific form of the gentzenian axioms we obtain permits us then to automatically define a (decidable) tableau system for each logic subjected to that 2-valued reduction. The same methods can be applied to many other well-known logics such as Łukasiewicz's $Ł_n$, Kleene's K_3, Gödel's G_3 etc. Our reductive method builds bulk in the reductive results from [20, 21] and [1].

It is an open problem to extend our 2-valued reductive procedure so as to cover other classes of logics such as modal or infinite-valued logics.

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