

# Actions of Automorphisms on Some Classes of Fuzzy Bi-implications

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**Abstract.** In a previous paper we have studied two classes of fuzzy bi-implications based on t-norms and r-implications, and shown that they are increasingly weaker subclasses of the Fodor-Roubens bi-implication. Now we prove that each of these three classes of bi-implications are closed under automorphisms.

**Keywords:** fuzzy bi-implications, automorphisms, fuzzy operators

## 1 Introduction

In [9] we studied the relation between the more well-known definition proposed by Fodor and Roubens and other appealing definitions, old or new, of fuzzy operators that extend the interpretation of the classical bi-implication.

On the other hand, automorphisms, i.e., isomorphisms between the same lattices, with the composition form a group [22, 3]. Automorphisms had played an interesting role in fuzzy connectives, because when a class of fuzzy connectives is closed under automorphisms the action of the group of automorphisms establishes an equivalence relation between the connectives and therefore determine a partition among these connectives. These partitions, in some case have characterized important subclasses of fuzzy connective. For example, the class of strict t-norms is the equivalence class of the product t-norm [16], the class of nilpotent t-norms agree with the equivalence class of the Łukasiewicz t-norm [16], the class of strong negations is the same that the equivalence class of the standard negation [26], and the class of implications which are both strong and residual is the equivalence class of the Łukasiewicz implication [1]. So it is reasonable to study the action of automorphisms on fuzzy bi-implications. In this paper we prove that each of the three classes of fuzzy bi-implications studied in [9] are closed under automorphisms.

## 2 Fuzzy extension of conjunction and implication

**Definition 21** A triangular norm (*in short* t-norm) is a binary operator  $T$  on the unit interval  $[0, 1]$  that: on  $\{0, 1\}$  behaves as classical conjunction, is

commutative, is associative, is increasing on both arguments, and has 1 as neutral element.

The notions of continuity are the usual ones. In particular:

**Definition 22** A  $t$ -norm  $T$  is left-continuous if for all non-decreasing sequences  $(x_n)_{n \in \mathbb{N}}$  we have that  $\lim_{n \rightarrow \infty} T(x_n, y) = T(\lim_{n \rightarrow \infty} x_n, y)$ .

**Definition 23** A fuzzy implication is a binary operator  $I$  on the unit interval  $[0, 1]$  that: on  $\{0, 1\}$  behaves as classical implication, is decreasing on the first argument and is increasing on the second argument.

**Definition 24** The residuum of a left-continuous  $t$ -norm  $T$  is the operation  $I$  such that  $I(x, y) \geq z$  iff  $T(z, x) \leq y$ .

**Theorem 21 ([2])** The residuum of a left-continuous  $t$ -norm is unique.

A particularly interesting class of fuzzy implications is the one based on residua:

**Definition 25** A binary operator  $I$  on  $[0, 1]$  is called an  $r$ -implication if there is a  $t$ -norm  $T$  such that:

$$I(x, y) = \sup\{z \in [0, 1] \mid T(z, x) \leq y\} \tag{1}$$

In such case we may say also that  $I$  is an  $r$ -implication based on  $T$ , and denote it by  $I^T$ . We say that  $I^T$  is of type  $\mathbb{LC}$  in case  $T$  is left-continuous. In the later situation we also say that  $(T, I^T)$  form an adjoint pair, or that  $I^T$  is the adjoint companion of  $T$ .

### 2.1 Automorphisms and their action on the fuzzy connectives

**Definition 26** A function  $\rho: [0, 1] \rightarrow [0, 1]$  is an automorphism if it is bijective and increasing, i.e., if  $x \leq y$  then  $\rho(x) \leq \rho(y)$  [12, 22, 24].

**Theorem 22 ([8, 24])** A function  $\rho: [0, 1] \rightarrow [0, 1]$  is an automorphism iff it is continuous in the usual sense, strictly increasing and preserves bounds, i.e.,  $\rho(0) = 0$  and  $\rho(1) = 1$ .

Since the inverse of an automorphism is also an automorphism and automorphisms are closed under composition, then the set of automorphisms on  $[0, 1]$ ,  $Aut([0, 1])$ , with the composition operator forms a group [22]. Thus, as usual in algebra, see for example [14], we can consider the action of the group  $\langle Aut([0, 1]), \circ \rangle$  on a set of functions from  $[0, 1]^n$  into  $[0, 1]$ .

**Definition 27** The action of an automorphism  $\rho$  on a function  $f: [0, 1]^n \rightarrow [0, 1]$  is the function  $f^\rho: [0, 1]^n \rightarrow [0, 1]$  defined by

$$f^\rho(x_1, \dots, x_n) = \rho^{-1}(f(\rho(x_1), \dots, \rho(x_n))) \tag{2}$$

In this case  $f^\rho$  is called a conjugate of  $f$ .

A set  $\mathcal{F}$  of  $n$ -ary functions on  $[0, 1]$  is closed under automorphisms if for each  $f \in \mathcal{F}$  and  $\rho \in \text{Aut}([0, 1])$  we have that  $f^\rho \in \mathcal{F}$ . Clearly, if  $g$  is a conjugate of a function  $f$ , then also  $f$  is a conjugate of  $g$ , in fact if  $g = f^\rho$ , since  $(f^\rho)^{-\rho} = f$  then  $f = g^{\rho^{-1}}$ . In addition, if  $f$  is conjugate of  $g$  and  $g$  is conjugate of  $h$  then  $f$  is a conjugate of  $h$  and clearly each function is conjugate of itself. Therefore, the relation of conjugate on a set  $\mathcal{F}$  closed under automorphisms is an equivalence relation which allows us to partition  $\mathcal{F}$ . In particular, is well known that the sets of t-norms, t-conorms, fuzzy negations and implications are each closed under automorphisms (see for example [2, 8, 16]). In the following we will prove that the subclasses of left-continuous t-norms and of the r-implications are closed under automorphisms.

**Proposition 21** *Let  $T$  be a t-norm and  $\rho$  be an automorphism.  $T$  is left-continuous iff  $T^\rho$  is a left-continuous t-norm.*

*Proof.* ( $\Rightarrow$ ) Let  $(x_n)_{n \in \mathbb{N}}$  be a non-decreasing sequence. Then, because  $\rho$  is increasing,  $(\rho(x_n))_{n \in \mathbb{N}}$  also is a non-decreasing sequence. Thus, because  $\rho$  is continuous and  $T$  is left-continuous, we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} T^\rho(x_n, y) &= \lim_{n \rightarrow \infty} \rho^{-1}(T(\rho(x_n), \rho(y))) && \text{by Eq. (2)} \\ &= \rho^{-1}\left(\lim_{n \rightarrow \infty} T(\rho(x_n), \rho(y))\right) && \text{because } \rho^{-1} \text{ is continuous} \\ & && \text{it preserves limits} \\ &= \rho^{-1}(T(\lim_{n \rightarrow \infty} \rho(x_n), \rho(y))) && \text{because } T \text{ is left-continuous} \\ &= \rho^{-1}(T(\rho(\lim_{n \rightarrow \infty} x_n), \rho(y))) && \text{because } \rho \text{ is continuous} \\ & && \text{it preserves limits} \\ &= T^\rho(\lim_{n \rightarrow \infty} x_n, y) && \text{by Eq. (2)} \end{aligned}$$

( $\Leftarrow$ ) Follows straightforward from the ( $\Rightarrow$ ) side and the fact that  $(T^\rho)^{\rho^{-1}} = T$ .

**Proposition 22** *Let  $T$  be a t-norm and  $\rho$  be an automorphism. Then  $(I^T)^\rho = I^{(T^\rho)}$ .*

*Proof.*

$$\begin{aligned} (I^T)^\rho(x, y) &= \rho^{-1}(I^T(\rho(x), \rho(y))) && \text{by Eq. (2)} \\ &= \rho^{-1}(\sup\{z \in [0, 1] \mid T(\rho(x), z) \leq \rho(y)\}) && \text{by Eq. (1)} \\ &= \rho^{-1}(\sup\{z \in [0, 1] \mid \rho^{-1}(T(\rho(x), z)) \leq \rho^{-1}(\rho(y))\}) && \rho^{-1} \text{ is increasing} \\ &= \rho^{-1}(\sup\{z \in [0, 1] \mid \rho^{-1}(T(\rho(x), \rho(\rho^{-1}(z)))) \leq y\}) && \rho^{-1} \text{ is inverse of } \rho \\ &= \sup\{\rho^{-1}(z) \in [0, 1] \mid \rho^{-1}(T(\rho(x), \rho(\rho^{-1}(z)))) \leq y\} && \rho^{-1} \text{ is continuous} \\ & && \text{it preserves sups} \\ &= \sup\{\rho^{-1}(z) \in [0, 1] \mid T^\rho(x, \rho^{-1}(z)) \leq y\} && \text{by Eq. (1)} \\ &= I^{(T^\rho)}(x, y) && \text{by Eq. (2)} \end{aligned}$$

**Corollary 21** *Let  $I : [0, 1]^2 \rightarrow [0, 1]$  and  $\rho$  be an automorphism.  $I$  is an r-implication of type  $\mathbb{LC}$  iff  $I^\rho$  is an r-implication of type  $\mathbb{LC}$ .*

*Proof.* Straightforward from Propositions 21 and 22.

### 3 Fuzzy bi-implication and automorphisms

#### 3.1 Automorphisms on an axiomatized class of fuzzy bi-implications

**Definition 31** *The class of  $f$ -bi-implications contains all binary operators  $B$  on the unit interval  $[0, 1]$  respecting the following axioms:*

- (B1)  $B(x, y) = B(y, x)$  ( $B$ -commutativity)
- (B2)  $B(x, x) = 1$  ( $B$ -identity)
- (B3)  $B(0, 1) = 0$
- (B4) *If  $w \leq x \leq y \leq z$ , then  $B(w, z) \leq B(x, y)$*

*In view of (B1), (B2) and (B3), it is easy to see that any Fodor-Roubens fuzzy bi-implication is bound to agree with classical bi-implication on  $\{0, 1\}$ .*

The following are some examples of  $f$ -bi-implications:

**Example 31**

1.  $B_M(x, y) = \begin{cases} 1 & \text{if } x = y \\ \min(x, y) & \text{otherwise} \end{cases}$
2.  $B_P(x, y) = \begin{cases} 1 & \text{if } x = y \\ \frac{\min(x, y)}{\max(x, y)} & \text{otherwise} \end{cases}$
3.  $B_L(x, y) = 1 - |x - y|$
4.  $B_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 1 & \text{otherwise} \end{cases}$
5.  $B_B^{TI1}(x, y) = \begin{cases} 1 & \text{if } x = y \text{ or } \max(x, y) \neq 1 \\ 0 & \text{otherwise} \end{cases}$

**Proposition 31** *Let  $B : [0, 1]^2 \rightarrow [0, 1]$  and  $\rho$  be an automorphism.  $B$  satisfies (Bi) iff  $B^\rho$  satisfies (Bi), for  $i = 1, \dots, 4$ .*

*Proof.* ( $\Rightarrow$ ) If  $B$  satisfies (Bi) for  $i = 1, \dots, 3$  then, from Eq. (2) and the fact that  $\rho(1) = 1$  and  $\rho(0) = 0$ , trivially  $B^\rho$  satisfy (Bi). On the other hand, if  $w \leq x \leq y \leq z$ , then because  $\rho$  is increasing,  $\rho(w) \leq \rho(x) \leq \rho(y) \leq \rho(z)$  and so, since  $B$  satisfies (B4), we have that  $B(\rho(w), \rho(z)) \leq B(\rho(x), \rho(y))$ . Therefore, because  $\rho^{-1}$  is increasing,  $\rho^{-1}(B(\rho(w), \rho(z))) \leq \rho^{-1}(B(\rho(x), \rho(y)))$ , i.e.,  $B^\rho(w, z) \leq B^\rho(x, y)$  and so  $B^\rho$  satisfies (B4).

( $\Leftarrow$ ) Follows straightforward from the ( $\Rightarrow$ ) side and the fact that  $(B^\rho)^{\rho^{-1}} = B$ .

**Corollary 31** *Let  $B : [0, 1]^2 \rightarrow [0, 1]$  and  $\rho$  be an automorphism.  $B$  is an  $f$ -bi-implication iff  $B^\rho$  is also an  $f$ -bi-implication.*

*Proof.* Straightforward from previous proposition.

**Example 32** *Let  $\rho$  be the following automorphism:  $\rho(x) = x^2$  for each  $x \in [0, 1]$ , then*

1.  $B_M(x, y) = B_M^\rho(x, y)$
2.  $B_P(x, y) = B_P^\rho(x, y)$
3.  $B_L^\rho(x, y) = \sqrt{1 - |x^2 - y^2|}$
4.  $B_D(x, y) = B_D^\rho(x, y)$ .
5.  $B_B^{TI1}(x, y) = (B_B^{TI1})^\rho(x, y)$

**Definition 32** *An  $f$ -bi-implication  $B$  is said to satisfy:*

- the diagonal principle, if  $B(x, y) \neq 1$  whenever  $x \neq y$  (DP)

**Proposition 32** *Let  $B$  be an  $f$ -bi-implication and  $\rho$  be an automorphism.  $B$  satisfies the diagonal principle iff  $B^\rho$  also satisfies the diagonal principle.*

*Proof.* ( $\Rightarrow$ ) If  $x \neq y$  then because  $\rho$  is bijective,  $\rho(x) \neq \rho(y)$  and so, because  $B$  satisfies the diagonal principle,  $B(\rho(x), \rho(y)) \neq 1$ . Thus, because  $\rho^{-1}$  is bijective, then  $\rho^{-1}(B(\rho(x), \rho(y))) \neq \rho^{-1}(1)$ , i.e.  $B^\rho(x, y) \neq 1$ .

( $\Leftarrow$ ) Follows straightforward from the ( $\Rightarrow$ ) side and the fact that  $(B^\rho)^{\rho^{-1}} = B$ .

### 3.2 Automorphisms on classes of fuzzy bi-implications based on a defining standard of t-norms and fuzzy implications

In the definitions that follow, we call  $TI$  the defining standard  $B(x, y) = T(I(x, y), I(y, x))$  for fuzzy bi-implications, where we assume that  $T$  is a t-norm and  $I$  an r-implication.

**Definition 33** ([9]) *The class of  $a$ -bi-implications contains all binary operators  $B$  on  $[0, 1]$  following the  $TI$  defining standard and based on an arbitrary t-norm  $T$  and on the residuum  $I^T$  of  $T$ , that is, operators defined by setting*

$$B(x, y) = T(I^T(x, y), I^T(y, x)) \quad (3)$$

**Proposition 33** *Let  $B : [0, 1]^2 \rightarrow [0, 1]$  and  $\rho$  be an automorphism.  $B$  is an  $a$ -bi-implication iff  $B^\rho$  is an  $a$ -bi-implication.*

*Proof.* ( $\Rightarrow$ )

$$B^\rho(x, y) = \rho^{-1}(B(\rho(x), \rho(y))) \quad \text{Eq. (2)}$$

$$= \rho^{-1}(T(I^T(\rho(x), \rho(y)), I^T(\rho(y), \rho(x)))) \quad \text{Eq. (3)}$$

$$= \rho^{-1}(T(\rho \circ \rho^{-1}(I^T(\rho(x), \rho(y))), \rho \circ \rho^{-1}(I^T(\rho(y), \rho(x)))))$$

$$= T^\rho((I^T)^\rho(x, y), (I^T)^\rho(y, x)) \quad \text{Eq. (2)}$$

$$= T^\rho(I^{T^\rho})(x, y), I^{T^\rho}(y, x) \quad \text{By Prop. 22}$$

Therefore,  $B^\rho$  also is an  $a$ -bi-implication.

( $\Leftarrow$ ) Follows straightforward from the ( $\Rightarrow$ ) side and the fact that  $(B^\rho)^{\rho^{-1}} = B$ .

**Definition 34** ([9]) *The class of  $\ell$ -bi-implications contains all binary operators  $B$  on  $[0, 1]$  following the  $TI$  defining standard and based on a left-continuous t-norm  $T$  and the residuum of  $T$ , which is an r-implications of type  $\mathbb{LC}$ , that is, operators defined through the equation  $B(x, y) = T(I^T(x, y), I^T(y, x))$ .*

**Proposition 34** *Let  $B : [0, 1]^2 \rightarrow [0, 1]$  and  $\rho$  be an automorphism.  $B$  is an  $\ell$ -bi-implication iff  $B^\rho$  is an  $\ell$ -bi-implication.*

*Proof.* Straightforward from Proposition 33 and Corollary 21.

## 4 Conclusions

In this paper we considered the action of the automorphism group on the three classes of fuzzy bi-implications that were studied in [9], say:  $f$ -bi-implications,  $a$ -bi-implications and  $\ell$ -bi-implications. In particular, we proved that all three classes are closed under automorphisms and therefore, the action of automorphism induces to a partition of these classes. For example, the equivalence class of the bi-implication  $B_M$  is the singleton set  $\{B_M\}$  but the equivalence class of  $B_L$  is not a countable set (to see it, in the example 5 is sufficient to substitute the automorphism  $\rho(x) = x^2$  by  $\rho(x) = x^r$ , with  $r$  been a positive real number).

It is a preliminary work, and several other aspects of the action of automorphism on bi-implication can be studied in future works. For example, since for one hand, in [9] was proved that the  $f$ -bi-implications properly contains the  $a$ -bi-implications which also properly contain the  $\ell$ -bi-implications, we can explore the action of the automorphism on bi-implications in order to characterize the classe of bi-implication which are in a class (for example in  $f$ -bi-implications) which are not in the lesser class (for example in  $a$ -bi-implications). Notice, that if  $B$  is a bi-implication which is in a classe but is not in some other classes then all of their conjugates also are in this same situation. Other point that we can study is if the conjugate of the natural fuzzy negation induced by a bi-implication coincide with the natural negation induced by the conjugate (with respect the same automorphism) of the bi-implication, i.e. if for each automorphism  $\rho$  and bi-implication  $B$ ,  $(N_B)^\rho = N_{B^\rho}$ , where  $N_B$  is the natural negation induced by  $B$ .

**Acknowledgments.** The authors would like to thank four anonymous referees for their remarks on a preliminary version of this paper.

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