On Some Subclasses of the Fodor-Roubens Fuzzy Bi-implication

Claudio Callejas, João Marcos, and Benjamín René Callejas Bedregal

LoLITA and DIMAp, UFRN, Brazil

Abstract. The paper deals with fuzzy versions of the classical bi-implication, that is, extensions of classical bi-implication to the canonical domain of mathematical fuzzy logics, the real-valued unit interval $[0, 1]$. Our approach to fuzzy bi-implication may be summarized as follows: first, we recall a well-known approach to bi-implications, by Fodor and Roubens, via the direct axiomatization of the properties of the corresponding class of operators; next, we investigate a particular defining standard of bi-implication in terms of t-norms and r-implications. We study four prospective classes of bi-implications based on such defining standard, by varying the properties of its composing operators, and show that these classes collapse into precisely two increasingly weaker subclasses of the Fodor-Roubens bi-implication.

1 Introduction

The investigation of fuzzy logic in a narrow sense, as subfield of multi-valued logic, was initiated by Petr Hájek in [9], much after the introduction of fuzzy set theory by Lotfi Zadeh in [20]. Since then, however, a lot of debate has happened over the 'most reasonable way' of extending the most usual operators from the discrete classical domain $\{0, 1\}$ into the continuum represented by the real-valued unit interval $[0, 1]$. In the meanwhile, if some classical connectives have found well-accepted fuzzy counterparts, others remained largely as a matter of contention. The fuzzy interpretation of conjunction, for instance, is well settled in terms of the triangular norm operator, and similarly for disjunction as its dual [11–14]. Classical negation, in weak and strong forms, also has a reasonably well-studied associated fuzzy operator [6]. Furthermore, there are many competing fuzzy versions of implication, of which [1] seems to be the most widely used in the literature.

Classical bi-implication also does not fall short of fuzzy interpretations, and one may find it studied in the literature under the appellations of T-indistinguishability operator [18], fuzzy bi-implication [2, 4], fuzzy equality [17], fuzzy bi-residuation [11, 15], fuzzy equivalence [7, 8], T-equivalence [16], fuzzy similarity [9] and restricted equivalence function [5].

As it happens, it is relatively common to find in the recent literature investigations that study the relations between different classes of the most common fuzzy operators. This happens for instance with the relation between different classes of triangular norms [12] and with the intersections of different classes of

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fuzzy implications [1]. For fuzzy bi-implication, however, the same investigation still remains to be done. In this paper we contribute to fill this gap, by studying the relation between the more well-known definition proposed by Fodor and Roubens and other appealing definitions, old or new, of fuzzy operators that extend the interpretation of the classical bi-implication.

The plan of the paper is as follows: in section 2 we recall some basic definitions and facts about t-norms and r-implications; subsection 3.1 recalls the definitions and proves some important results about the so-called Fodor-Roubens fuzzy bi-implications; section 3.2 studies four classes of bi-implications produced by the defining standard based on the classical equivalence in between $\alpha \Leftrightarrow \beta$ and $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$, and shows that these classes amount to two distinct classes of which one is a subclass of the other, and both are subclasses of Fodor-Roubens bi-implications; we conclude by some considerations concerning open problems and interesting lines for future research.

2 Conjunction and Implication from a Fuzzy Perspective

There are infinite ways in which the interpretation of *conjunction* $\wedge$ may be extended from the classical $\{0,1\}$ domain to the unit interval $[0,1]$, but not all of them behave as what is intuitively expected from a generalization of the Boolean conjunction to the unit square. The fuzzy logic community, as a matter of fact, has by and large agreed to impose the properties of t-norms to any extension of the classical conjunction.

The following definitions and examples may be found in [12].

Definition 1. *A* triangular norm *(in short* t-norm*) is a binary operator T on the unit interval* $[0,1]$ *that:* (T0) *agrees with classical conjunction on* $\{0,1\}$, (T1) *is commutative,* (T2) *is associative,* (T3) *is monotone on both arguments, and* (T4) *has* 1 *as neutral element.*

In fact, it is easy to check that (T0) follows from the remaining properties.

The associated notions of continuity are the usual ones. In particular:

Definition 2. *A t-norm T is* left-continuous *if for all non-decreasing sequences* $(x_n)_{n\in\mathbb{N}}$ *we have that* $\lim_{n\to\infty} T(x_n, y) = T(\lim_{n\to\infty} x_n, y)$.

There are uncountably many t-norms, but below we mention some of the most well-known among them.

Example 1.

1. $T_M(x,y) = \min(x,y)$ (minimum t-norm)
2. $T_P(x,y) = x \cdot y$ (product t-norm)
3. $T_L(x,y) = \max(x+y-1, 0)$ (Łukasiewicz t-norm)
4. $T_D(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1)^2 \\ \min(x,y) & \text{otherwise} \end{cases}$ (drastic t-norm)

Notice, in particular, that T_M, T_P and T_L are all left-continuous, yet T_D is not.

The following constitutes a generalization of the transitivity property in the context t-norms:

Definition 3. *Let T be a left-continuous t-norm, and F a binary operator on $[0,1]$. We say that F is T*-transitive *if $T(F(x,y),F(y,z)) \leq F(x,z)$.*

As regards *implication* $\Rightarrow$, there are several competing approaches to what should constitute its fuzzy counterpart (see for example [6, 10, 19]). Below we propose an axiomatization equivalent to the ones that may be found in [1, 8, 10], which characterize the most common fuzzy implication found in the literature.

Definition 4. *A* fuzzy implication *is a binary operator I on the unit interval $[0,1]$ that:* (I0) *agrees with classical implication on $\{0,1\}$,* (I1) *is antitone on the first argument and* (I2) *is monotone on the second argument.*

The following are examples of fuzzy implications:

Example 2.

1. $I_M(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$ (minimum / Gödel implication)
2. $I_P(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$ (product / Goguen implication)
3. $I_L(x,y) = \min(1-x+y, 1)$ (contractionless / Łukasiewicz implication)
4. $I_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ 1 & \text{otherwise} \end{cases}$ (drastic / Weber implication)
5. $I_B^1(x,y) = \begin{cases} 0 & \text{if } x = 1 \text{ and } y \neq 1 \\ 1 & \text{otherwise} \end{cases}$ (boolean 1-implication)

The first four examples are well-known, but for the last one, introduced here, we have to check that the corresponding definition satisfies the properties in Definition 4. In any case, (I0) is obvious. Consider now $x_a < x_b$. So, $x_a < 1$, thus $I_B^1(x_a,y) = 1 \geq I_B^1(x_b,y)$, satisfying thus (I1). Next, assume $y_a < y_b \leq 1$, and suppose that $I_B^1(x,y_a) > I_B^1(x,y_b)$. Notice that this is only possible in case $I_B^1(x,y_a) = 1$ and $I_B^1(x,y_b) = 0$. From $I_B^1(x,y_b) = 0$ one may conclude in particular that $x = 1$, and from this and $I_B^1(x,y_a) = 1$ it follows that $y_a = 1$. Contradiction, for $y_a < y_b$. Thus, (I2) is also satisfied.

Definition 5. *A fuzzy implication I is said to satisfy:*

- *the* identity principle*, if $I(x,x) = 1$* (IP)
- *the* left-ordering property*, if $I(x,y) = 1$ whenever $x \leq y$* (LOP)
- *the* right-ordering property*, if $I(x,y) \neq 1$ whenever $x > y$* (ROP)

Given an arbitrary t-norm T and an arbitrary fuzzy implication I, the pair (T,I) is said to satisfy:

- modus ponens*, if $T(x,I(x,y)) \leq y$* (MP)

Notice that all implications in Ex. 2 satisfy (LOP), and *a fortiori* also (IP). There are well-known examples of implications failing (LOP) (check the first chapter of [1]), but they will be of no particular interest to us here.

The following operation is used to generalize the Deduction Metatheorem:

Definition 6. *The* residuum *of a left-continuous t-norm T is the (unique) operation I such that $I(x, y) \geq z$ iff $T(z, x) \leq y$.*

A particularly interesting class of fuzzy implications is the one based on residua:

Definition 7. *A binary operator I on $[0, 1]$ is called an* r-implication *if there is a t-norm T such that:*

$$I(x, y) = \sup\{z \in [0, 1] \mid T(z, x) \leq y\}$$

In such case we may say also that I is an r-implication based on T*, and denote it by I^T. We say that I^T* is of type $\mathbb{LC}$ *in case T is left-continuous. In such case we also say that (T, I^T) form an* adjoint pair*, or that I^T* is the adjoint companion of T*.*

Given a left-continuous t-norm T, it should be clear from the above that its residuum I^T is the pointwise largest operation such that (T, I^T) satisfies modus ponens.

Note that:

Proposition 1. *(T_X, I^{T_X}) form adjoint pairs, for each $X \in \{M, P, L\}$.*

It is also the case that (cf. [1, 2]):

Proposition 2. *Let I^T be an r-implication. Then:* (i) *$I^T(1, y) \geq y$;* (ii) *I^T satisfies the identity principle;* (iii) *I^T satisfies the left-ordering property. Assume I^T to be of type $\mathbb{LC}$. Then:* (iv) *I^T is T-transitive;* (v) *I^T satisfies the right-ordering property.*

Even though we will not need the following results here, it is interesting to mention that for r-implications we can immediately count on $I^T(1, y) \leq y$ as well, thus validating (MP), and to mention also the characteristic strengthening of the above result according to which any r-implication that satisfies both (LOP) and (ROP) is of type $\mathbb{LC}$. It might also be interesting to notice how Prop. 2(v) shows that I_B^1 cannot be the residuum of a left-continuous t-norm, as it obviously fails (ROP). While neither I_D nor I_B^1 are of the type $\mathbb{LC}$, and on the one hand it is easy to see that I_D is indeed the residuum of T_D, on the other hand it is not at all obvious which t-norm, if any, would I_B^1 be the residuum of.

3 Fuzzy Bi-implication

3.1 Via Axiomatization

As it happens with implication, for the bi-implication $\Leftrightarrow$ there is also no universal agreement on what should constitute its fuzzy counterpart. The most well-known class of fuzzy bi-implications was investigated by Fodor and Roubens and is characterized by the following properties (see [8]):

Definition 8. *The class of* f-bi-implications *contains binary operators* B *on the unit interval* $[0,1]$ *respecting the following axioms:*

(B1) $B(x, y) = B(y, x)$ (B-commutativity)
(B2) $B(x, x) = 1$ (B-identity)
(B3) $B(0, 1) = 0$
(B4) *If* $w \leq x \leq y \leq z$, *then* $B(w, z) \leq B(x, y)$

In view of (B1), (B2) *and* (B3), *it is easy to see that any Fodor-Roubens fuzzy bi-implication is bound to agree with classical bi-implication on* $\{0, 1\}$. *We will refer to this 'boundary' property as* (B0).

Here are some examples of f-bi-implications:

Example 3

1. $B_M(x, y) = \begin{cases} 1 & \text{if } x = y \\ \min(x, y) & \text{otherwise} \end{cases}$
2. $B_P(x, y) = \begin{cases} 1 & \text{if } x = y \\ \frac{\min(x,y)}{\max(x,y)} & \text{otherwise} \end{cases}$
3. $B_L(x, y) = 1 - |x - y|$
4. $B_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 1 & \text{otherwise} \end{cases}$
5. $B_B^{TI1}(x, y) = \begin{cases} 1 & \text{if } x = y \text{ or } x, y \neq 1 \\ 0 & \text{otherwise} \end{cases}$

Definition 9. *A fuzzy bi-implication* B *is said to satisfy:*

– *the* diagonal principle, *if* $B(x, y) \neq 1$ *whenever* $x \neq y$ (DP)

It should be clear that:

Theorem 1. (i) *Not all* f*-bi-implications satisfy the diagonal principle.* (ii) *There are* f*-bi-implications that are* T*-intransitive, that is, that fail* T*-transitivity for every t-norm* T. *Indeed,* B_D *is a convenient witness to both these facts.*

Proof. Part (i). If $x < y < 1$ then $B_D(x, y) = 1$. Since we have $x \neq y$ and $B_D(x, y) = 1$ then B_D does not satisfy (DP). Part (ii). Let T be an arbitrary t-norm. Then, in view of (T4) and the definition of B_D, $T(B_D(1, .9), B_D(.9, .8)) = T(.9, 1) = .9 \not\leq .8 = B_D(1, .8)$. Therefore, B_D fails to be T-transitive. □

Moreover:

Theorem 2. *The following property holds good for any r-implication* I^T*:*

– $\min(I^T(x, y), I^T(y, x)) = I^T(\max(x, y), \min(x, y))$

Proof. Let I^T be of type $\mathbb{LC}$. Assume without loss of generality that $x \leq y$. Recall that, by Prop. 2(iii), I^T satisfies (LOP), thus $\min(I^T(x, y), I^T(y, x)) = \min(1, I^T(y, x)) = I^T(y, x)$. Notice, in addition, that $y = \max(x, y)$ and $x = \min(x, y)$, once $x \leq y$. □

3.2 Via a Defining Standard over t-Norms and Fuzzy Implications

Inspired by the classical (in fact, intuitionistic) equivalence in between $\alpha \Leftrightarrow \beta$ and $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$, in this section we explore the fuzzy bi-implication obtained by setting as defining standard $B(x, y) = T(I(x, y), I(y, x))$. We shall call this *TI defining standard* for bi-implication. As a matter of fact, a very general result may be proven about such defining standard, when r-implications are involved:

Theorem 3. *Given $B(x, y) = T(I(x, y), I(y, x))$, where I is an r-implication, the specific choice of t-norm T is inconsequential. Indeed, the following property holds good in general:*

– $B(x, y) = \min(I(x, y), I(y, x))$

Proof. Assume without loss of generality that $x \leq y$. Since, by Prop. 2(iii), the r-implication I satisfies (LOP), we have that $B(x, y) = T(I(x, y), I(y, x)) = T(1, I(y, x))$, and by (T4), we know that $T(1, I(y, x)) = I(y, x)$. Using (LOP) again, we conclude that $I(y, x) = \min(I(x, y), I(y, x))$. □

In the definitions that follow, we fix the TI defining standard for fuzzy bi-implications, assume that T is a t-norm and I an r-implication, and experiment with properties associated to left-continuity. We start by proposing the following very generous class of fuzzy bi-implications:

Definition 10. *The class of* aa-bi-implications *contains all binary operators B on $[0, 1]$ following the TI defining standard and based on arbitrary t-norms and arbitrary r-implications, that is, operators defined by setting $B(x, y) = T_1(I^{T_2}(x, y), I^{T_2}(y, x))$, where T_1 and T_2 are arbitrary t-norms.*

One may readily prove that:

Theorem 4. *Every aa-bi-implication B satisfies the equation $B(1, y) \geq y$.*

Proof. By Theor. 3, we know that $B(1, y) = \min(I^T(1, y), I^T(y, 1))$, for some appropriate r-implication I_T, which by Prop. 2(iii) must satisfy (LOP). From the latter property we conclude that $I^T(y, 1) = 1$, thus, $B(1, y) = I^T(1, y)$. The proof is completed by recalling from Prop. 2(i) that $I^T(1, y) \geq y$. □

Theorem 5. *Every aa-bi-implication is an f-bi-implication.*

Proof. Let T be a t-norm, I be an r-implication and B be the aa-bi-implication based on T and I. It is obvious that B satisfies (B1) and (B3), and (B2) follows from Prop. 2(ii). Now, recall by Prop. 2(iii) that I satisfies (LOP), and assume $w \leq x \leq y \leq z$. So:

$$
\begin{aligned}
B(w, z) &= T(I(w, z), I(z, w)) \\
&= T(1, I(z, w)) && \text{by (LOP), once } w \leq z \\
&= I(z, w) && \text{by (T4)} \\
&\leq I(y, x) && \text{by (I1), once } z \geq y \text{, and (I2), once } w \leq x \\
&= T(I(x, y), I(y, x)) && \text{by (LOP), once } x \leq y \\
&= B(x, y)
\end{aligned}
$$

Therefore, B satisfies (B4). □

A restricted version of the above definition may be found in [2]:

Definition 11. *The class of* a-bi-implications *contains all aa-bi-implications in which* $T_1 = T_2$*, that is, in which* I^{T_2} *is precisely the residuum of* T_1*.*

Example 4. The 'drastic' bi-implication B_D (Ex. 3.4) is an a-bi-implication. Indeed, $B_D(x, y) = T_D(I_D(x, y), I_D(y, x))$.

In view of Ex. 4 and Theor. 1 we know that there are a-bi-implications (thus, *a fortiori*, *aa*-bi-implications) that are intransitive and fail the diagonal principle.

As a corollary of Theor. 3, however, it is easy to see that the classes of *aa*-bi-implications and a-bi-implications are coextensive, even though the former class might have initially seemed to be more inclusive than the latter. So:

Theorem 6. *Every aa-bi-implication is an a-bi-implication.*

In what follows we restrict a bit further the preceding definitions of fuzzy bi-implication.

Definition 12. *The class of* $a\ell$-bi-implications *contains all binary operators* B *on* $[0, 1]$ *following the* TI *defining standard and based on arbitrary t-norms and r-implications of type* $\mathbb{LC}$*, that is, operators defined through the equation* $B(x, y) = T_1(I^{T_2}(x, y), I^{T_2}(y, x))$*, where* T_1 *is an arbitrary t-norm and* I^{T_2} *an r-implication of type* $\mathbb{LC}$*.*

The following specialization of $a\ell$-bi-implications was studied in [11]:

Definition 13. *The class of* ℓ-bi-implications *contains all* $a\ell$*-bi-implications in which* $T_1 = T_2$*, that is, in which* I^{T_2} *is precisely the adjoint companion of* T_1*.*

Example 5. B_M, B_P and B_L are ℓ-bi-implications.

Again, as an immediate corollary of Theor. 3, we know that the two latter classes of bi-implications are coextensive, that is:

Theorem 7. *Every* $a\ell$*-bi-implication is an* ℓ*-bi-implication.*

As a more interesting side-effect of Theor. 3, the following results from [3] on ℓ-bi-implications may also be generalized to $a\ell$-bi-implications:

Theorem 8. *Every* $a\ell$*-bi-implication based on an r-implication* I^T *of type* $\mathbb{LC}$ *enjoys both the diagonal principle and* T*-transitivity.*

Proof. Let T_1 be a t-norm, I^T be an r-implication of type $\mathbb{LC}$ and B be the $a\ell$-bi-implication $B(x, y) = T_1(I^T(x, y), I^T(y, x))$ based on T_1 and I^T. By Theor. 3, we know that $B(x, y) = \min(I^T(x, y), I^T(y, x))$. So, $B(x, y) = 1$ iff both $I^T(x, y) = 1$ and $I^T(y, x) = 1$. Given Prop. 2(v), I^T satisfies (ROP), so $I^T(x, y) = 1$ and $I^T(y, x) = 1$ imply that $x \leq y$ and $y \leq x$. It follows that $x = y$ whenever $B(x, y) = 1$, in other words, that B satisfies (DP).

Now we are going to check that $T(B(x,y),B(y,z)) \leq I^T(x,z)$. Recall that, by Prop. 2(iv), I^T is T-transitive, once T is left-continuous. So:

$$\begin{aligned}
&T(B(x,y),B(y,z)) = \\
&\quad = T(\min(I^T(x,y),I^T(y,x)),\min(I^T(y,z),I^T(z,y))) && \text{by Theor. 3} \\
&\quad \leq T(I^T(x,y),I^T(y,z)) && \text{by (T3)} \\
&\quad \leq I^T(x,z) && \text{by } T\text{-transitivity of } I^T
\end{aligned}$$

For analogous reasons, it is also true that $T(B(x,y),B(y,z)) \leq I^T(z,x)$. Therefore, $T(B(x,y),B(y,z)) \leq \min(I^T(x,z),I^T(z,x))$ and again by Prop. 2(iii) and Theor. 3, it follows that $\min(I^T(x,z),I^T(z,x)) = B(x,z)$. Thus, B is T-transitive. □

Last but not least, for the sake of comparing the above classes of bi-implication, we may observe that:

Theorem 9. *Not all a-bi-implications are ℓ-bi-implications. Again, B_D bears witness to this fact.*

Proof. Recall from Ex. 4 that B_D is an a-bi-implication and that in Theor. 1 we proved that B_D does not satisfy neither the diagonal principle nor the T-transitivity property for no t-norm T. Since by the definition of the class of ℓ-bi-implications, any creature from this class is in particular an $a\ell$-bi-implication, and in Theor. 8 we have seen that every $a\ell$-bi-implication satisfies both (DP) and T-transitivity, the drastic bi-implication B_D gives us two good reasons to conclude that not every a-bi-implication is an ℓ-bi-implication. □

While the latter distinguishing result should be contrasted with the ordinary facts mentioned in Ex. 5, the next result should be contrasted with Ex. 3.5:

Theorem 10. *Not all f-bi-implications are a-bi-implications. Indeed, this statement has B_B^{TI1} as witness.*

Proof. Consider any $y \in (0,1)$. Then, $B_B^{TI1}(1,y) = 0$. Yet, in view of Theor. 4 we know that $B(1,y) \geq y$ for any aa-bi-implication B.

4 Conclusions

There are basically three classes of fuzzy bi-implications to be found in this paper: ($\mathcal{B}1$) f-bi-implications; ($\mathcal{B}2$) a-bi-implications (which we have shown to be coextensive with the apparently more general class of aa-bi-implications); ($\mathcal{B}3$) ℓ-bi-implications (which we have shown to be coextensive with the apparently more general class of $a\ell$-bi-implications) We have seen that ($\mathcal{B}3$) is a proper subclass of ($\mathcal{B}2$), and that ($\mathcal{B}2$) is a proper subclass of ($\mathcal{B}1$). The full picture may be appreciated in Fig. 1.

In [5] a class $\mathcal{B}4$ of 'restricted equivalence functions' is introduced via axiomatization as a subclass of $\mathcal{B}1$. It has not been shown, however this consists

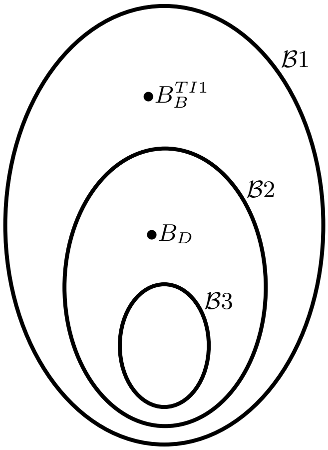

Fig. 1. Subclasses of Fodor-Roubens fuzzy bi-implication

in a proper subclass. An interesting line of investigation would be thus to determine which are the relations that hold between $\mathcal{B}4$ and the other classes of fuzzy bi-implications that we have studied here.

Finally, to get a better view over the possibilities it is also very important to investigate other defining standards for bi-implication, such as the one that sets $B(x, y) = I(S(x, y), T(x, y))$, where T is a t-norm, I a convenient fuzzy implication, and S a t-conorm (the dual of a t-norm, used for interpreting disjunction). Some results have already been found that characterize some classes, based on such an alternative defining standard, that properly extend $\mathcal{B}3$ yet are not extended by $\mathcal{B}1$, providing thus a legitimate alternative to the Fodor-Roubens paradigm. Presenting these results in detail is left as matter for future work.

To the authors of this paper, the class of Fodor-Roubens implications is too inclusive. In particular, satisfaction of the equation $I(1, y) \leq y$ is not enforced, and that seems to us rather inadvisable, at least if one wants to count on (fuzzy) modus ponens. This defect is appropriately fixed by r-implications. In exporting the intuitions behind (Fodor-Roubens) fuzzy implications into the class of f-bi-implications, the defects of the former are inherited, and there will be no way of guaranteeing that, say, $B(1, y) \leq y$. Not by coincidence, the alternative classes of bi-implications we have studied here are based precisely on r-implications, and the fact that they turned out to define proper subclasses of the f-bi-implications containing the most natural examples of fuzzy bi-implications from the literature would seem to lend support to our decision of concentrating our attention on such classes. However, if one takes into account, on a closer look, the additional fact that our main theorems concerning both the class of bi-implications following the TI defining standard and the class of a-bi-implications are based directly on the left-ordering property, rather than on other properties of fuzzy implications, there seems to be some chance that an interesting class of bi-implications might still lurk somewhere in between $\mathcal{B}1$ and $\mathcal{B}2$. We close our present study by leaving the investigation of this thread open for the interested researcher.

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