On classic-like fuzzy modal logics

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***Abstract*—In this paper we explore classic-like aspects of Kripke models endowed with a fuzzy accessibility relation and a fuzzy notion of satisfaction, and prove a general completeness result concerning the fuzzy semantics of a generous class of normal modal systems enriched with multiple instances of the axiom of confluence.**

I. Introduction

With different goals, several papers in the literature have proposed to 'modalize' fuzzy logics or to 'fuzzify' modal logics. In [2], for instance, the author aims at constructing logical calculi with languages appropriate for specifying dynamical systems whose behavior and structure is only modeled approximately. Other authors are also interested in providing adequate axiomatizations for such logics. For example, in [4] the authors provide an axiomatization for the $\Box$-fragment and the $\Diamond$-fragment of the so-called Gödel modal logics, based on the many-valued Gödel logic and some well-known logics from the literature on modal logics. In [5] the authors characterize minimal many-valued modal logics for a $\Box$ operator defined over finite residuated lattices. All papers cited above have one thing in common: the semantical framework used to characterize the modal systems is based on Kripke-style structures.

The semantics that we utilize here is also a many-valued Kripke-style semantics. Our particular aim, though, is to characterize a generous class of many-valued modal systems with locally bivalent semantics that behave just like the usual boolean-based Kripke semantics for modal logics. In [6], the authors study models for a certain kind of fuzzy modal logics and prove weak completeness results for a couple of modal extensions of classic-like fuzzy models of some traditional normal modal systems, viz. K, T, D, B, $S4$ and $S5$. In [1] we followed a similar thread to prove completeness results for a much more inclusive class of fuzzy normal modal systems which contain instances of the *axiom of confluence* $(G^{k,l,m,n})$ $\Diamond^k\Box^m\varphi \supset \Box^l\Diamond^n\varphi$. It should be clear that the systems $K+G^{k,l,m,n}$ encompass the above traditional systems, and a lot else. Indeed, one may observe that the characteristic modal axioms (T) $\Box\varphi \supset \varphi$, (D) $\Box\varphi \supset \Diamond\varphi$, (B) $\varphi \supset \Box\Diamond\varphi$, (4) $\Box\varphi \supset \Box\Box\varphi$ and (5) $\Diamond\varphi \supset \Box\Diamond\varphi$ are but particular instances of $(G^{k,l,m,n})$ where $\langle k,l,m,n\rangle$ are $\langle 0,0,1,0\rangle$, $\langle 0,0,1,1\rangle$, $\langle 0,1,0,1\rangle$, $\langle 0,2,1,0\rangle$ and $\langle 1,1,0,1\rangle$, respectively.

In our preliminary study, [1], we have followed [6] in producing for the real-valued unit interval $[0,1]$ the 'canonical' binary partition $\{[0,1),[1,1]\}$ and in putting certain restrictions on the fuzzy operators which we have used to interpret the connectives of our language. Notions of satisfactiona and validity of a formula are straighforwardly defined based on this partition. A weak completeness result was then established for a large class of modal systems. In the present paper, our 'crisp semantics' is more general: instead $\{[0,1),[1,1]\}$ we use a partition $\{[0,i),[i,1]\}$, with $i \neq 0$. We are to show, then, how to extend the completeness result for a much larger class of classic-like fuzzy modal logics.

The so-called *Geach axiom* $(G^{1,1,1,1})$ is well-known to characterize, in terms of the associated notion of accessibility $\rightsquigarrow$ (and its inverse $\leftsquigarrow$) in the corresponding Kripke frames, the *diamond property*, namely: if $y \leftsquigarrow x \rightsquigarrow z$, then there is some w such that $y \rightsquigarrow w \leftsquigarrow z$. As noted in [7], where $\overset{i}{\rightsquigarrow}$ denotes an i-long sequence of $\rightsquigarrow$ transitions (and similarly for $\overset{i}{\leftsquigarrow}$ and $\leftsquigarrow$ transitions), the natural generalization of the diamond property is the following $\langle k,l,m,n\rangle$*-confluence property*: if $y \overset{k}{\leftsquigarrow} x \overset{l}{\rightsquigarrow} z$, then there is some w such that $y \overset{m}{\rightsquigarrow} w \overset{n}{\leftsquigarrow} z$. From the logical viewpoint, a general completeness proof based directly on the axiom of confluence, thus, is attractive in having the potential to subsume a denumerable number of particular instances of $(G^{k,l,m,n})$. At any rate, it should be noted that the confluence property has importance on its own. In abstract rewriting systems and type theory, for instance, one deals with frames in which accessibility characterizes some appropriate notion of reduction. There, confluence is used together with termination to guarantee convergence of reductions, which on its turn guarantees the existence of normal forms and has applications on the design of decision procedures. Strong normalization, in particular, is a much desirable property of lambda calculi, and is a property guaranteed by theorems of confluence à la Church-Rosser, with applications to programming language theory. The availability of modal logics of confluence, and in fact of fuzzy versions of such logics, allows one to expect to have a local perspective on rewrite systems and on program evaluation, and this time imbued with varying degrees of uncertainty, customized to the user's discretion.

The plan of the paper is as follows: in section II we introduce the usual fuzzy operators; in section III we present the concept of classic-like fuzzy semantics and show that there exist fuzzy logics with the same set of tautologies of classical propositional logic; in section IV we present a particular kind of fuzzy Kripke semantics for modal logics; in section V we prove completeness results for the modal system K extended with instances of the axiom of confluence.

II. Fuzzy Operators

We first review some useful terminology and easy results:

Definition II.1. *Throughout the paper we shall use $\mathcal{O}$ to denote the* ***boolean*** *domain $\{0,1\}$ of classical logic, and $\mathcal{U}$ to denote the* ***unit*** *interval $[0,1]$, typical of fuzzy logics. By $\leqslant$ we will always denote the natural* ***total order*** *on $\mathcal{U}$. Given a k-ary operator $©_b$ on $\mathcal{O}$ and a k-ary operator $©_u$ on $\mathcal{U}$, we shall say that $©_u$* ***agrees with*** *$©_b$ if $©_u|_{\mathcal{O}} = ©_b$. Given some $i \in \mathcal{U}\backslash\{0\}$, we will use Π to denote the* ***partition*** *$\{\Pi_0, \Pi_1\}$ of $\mathcal{U}$, where $\Pi_0 = [0,i)$ and $\Pi_1 = [i,1]$.*

We list in what follows the defining properties of the most standard fuzzy operators used to interpret their homonymous classical counterparts:

Definition II.2. *A* ***fuzzy conjunction****, or* ***t-norm****, is a binary operation T on $\mathcal{U}$ such that:* (T0) *T agrees with classical conjunction,* (T1) *T is commutative,* (T2) *T is associative,* (T3) *T is monotone, that is, order-preserving, on both arguments, and* (T4) *T has 1 as neutral element. We call $x \in \mathcal{U}$ a Π_0-**divisor** of a t-norm T if there exists some $y \in \mathcal{U}$ such that $T(x,y) \in \Pi_0$; such Π_0-divisor is called* ***non-trivial*** *if both $x, y \in \Pi_1$. We say that T is* ***left-continuous*** *if it preserves limits of non-decreasing sequences, that is, if $\lim_{n\to\infty} T(x_n, y) = T(\lim_{n\to\infty} x_n, y)$, for every non-decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Definition II.3. *A* ***fuzzy disjunction****, or* ***s-norm****, is a binary operation S on $\mathcal{U}$ such that:* (S0) *S agrees with classical disjunction,* (S1) *S is commutative,* (S2) *S is associative,* (S3) *S is monotone on both arguments, and* (S4) *S has 0 as neutral element. We call $x \in \mathcal{U}$ a Π_1-**divisor** of a t-norm T if there exists some $y \in \mathcal{U}$ such that $S(x,y) \in \Pi_1$; such Π_1-divisor is called* ***non-trivial*** *if both $x, y \in \Pi_0$.*

Some easily checkable important derived properties of the above operators include:

Proposition II.1. *For any t-norm T, s-norm S, and every $x, y \in \mathcal{U}$:*
(i) *If $T(x,y) \in \Pi_1$, then $x \in \Pi_1$ and $y \in \Pi_1$.*
(ii) *If $S(x,y) \in \Pi_0$, then $x \in \Pi_0$ and $y \in \Pi_0$.*

Note that small t-norms such as the 'drastic t-norm' (that sets the value of $T(x,y)$ as $\min(x,y)$ if $\max(x,y) = 1$, and as 0 otherwise) fail the converse of Prop. II.1(i). Dually, large s-norms such as the 'drastic s-norm' fail the converse of Prop. II.1(ii). It is interesting to observe that:

Proposition II.2. *Let T be a t-norm, and S be an s-norm. Then:*
(i) *If T lacks non-trivial Π_0-divisors, then $x \in \Pi_1$ and $y \in \Pi_1$ imply $T(x,y) \in \Pi_1$, for every $x, y \in \mathcal{U}$.*
(ii) *If S lacks non-trivial Π_1-divisors, then $x \in \Pi_0$ and $y \in \Pi_0$ imply $S(x,y) \in \Pi_0$, for every $x, y \in \mathcal{U}$.*

Definition II.4. *A* ***fuzzy negation*** *is a unary operation N on $\mathcal{U}$ such that:* (N0) *N agrees with classical negation,* (N1) *N is antitone, that is, order-reversing.*

Definition II.5. *A* ***fuzzy implication*** *is a binary operation I on $\mathcal{U}$ such that:* (I0) *I agrees with classical implication,* (I1) *I is antitone on the first argument, and* (I2) *I is monotone on the second argument.*

Given that the unit interval $\mathcal{U} = [0,1]$ is closed and bounded, the Bolzano-Weierstrass theorem guarantees that:

Proposition II.3. *The image of a left-continuous t-norm is complete (in the sense that its subsets contain their own suprema).*

Residuation allows us to define a particularly interesting kind of fuzzy implication:

Proposition II.4. *The residuum I of a left-continuous t-norm is a fuzzy implication. Moreover, $I(x,y) = 1$ iff $x \leqslant y$.*

III. Fuzzy Semantics

Let P be a denumerable set of propositional variables, and let the set of formulas of classical propositional logic, L_P, be inductively defined by:

$$\varphi ::= p \mid (\neg\varphi) \mid (\varphi_1 \wedge \varphi_2) \mid (\varphi_1 \vee \varphi_2) \mid (\varphi_1 \supset \varphi_2)$$

where p ranges over elements of P.

The following definition employs the standard fuzzy operators in interpreting the above symbols for the classical connectives:

Definition III.1. *A* ***fuzzy evaluation*** *of the propositional variables is any total function $e : P \longrightarrow \Pi_0 \cup \Pi_1$. The structure $\mathbb{S} = \langle N, T, S, I\rangle$ will be called a* ***fuzzy semantics*** *for the propositional connectives $\langle\neg, \wedge, \vee, \supset\rangle$. By way of a fuzzy semantics, an evaluation e may be recursively extended to a* ***fuzzy valuation*** *$e^{\mathbb{S}} : L_P \longrightarrow \Pi_0 \cup \Pi_1$ as follows:*

$$\begin{array}{lcl} e^{\mathbb{S}}(p) & = & e(p) \text{ for each } p \in P \\ e^{\mathbb{S}}(\neg\alpha) & = & N(e^{\mathbb{S}}(\alpha)) \\ e^{\mathbb{S}}(\alpha \wedge \beta) & = & T(e^{\mathbb{S}}(\alpha), e^{\mathbb{S}}(\beta)) \\ e^{\mathbb{S}}(\alpha \vee \beta) & = & S(e^{\mathbb{S}}(\alpha), e^{\mathbb{S}}(\beta)) \\ e^{\mathbb{S}}(\alpha \supset \beta) & = & I(e^{\mathbb{S}}(\alpha), e^{\mathbb{S}}(\beta)) \end{array}$$

*A formula $\alpha \in L_P$ is called an $\mathbb{S}$-**tautology**, denoted by $\vDash_{\mathbb{S}} \alpha$, if for every fuzzy evaluation e we have $e^{\mathbb{S}}(\alpha) \in \Pi_1$. We shall denote by $\mathbb{T}(L_P)$ the set of all classical tautologies in L_P and by $\mathbb{T}^{\mathbb{S}}(L_P)$ the set of all $\mathbb{S}$-tautologies in L_P.*

The fact that each fuzzy operator agrees with the corresponding classical operator immediately guarantees the following result:

Proposition III.1. *All fuzzy tautologies are classical tautologies, that is, $\mathbb{T}^{\mathbb{S}}(L_P) \subseteq \mathbb{T}(L_P)$, for any fuzzy semantics $\mathbb{S}$.*

The following definitions, from [8], and the subsequent result aim at capturing the core of classical semantics from within the context of fuzzy semantics:

Definition III.2. *$\mathbb{S}$ is a* ***classic-like fuzzy semantics*** *if $\mathbb{T}(L_P) \subseteq \mathbb{T}^{\mathbb{S}}(L_P)$.*

Definition III.3. *Let $\mathbb{S} = \langle N, T, S, I\rangle$ be a fuzzy semantics and Π be a partition for $\mathcal{U}$. We say that: (1) N is* ***crisp with***

respect to Π *when* $N(x) \in \Pi_0$ *if and only if* $x \in \Pi_1$*; (2)* T *is* ***crisp with respect to*** Π *when* $T(x,y) \in \Pi_1$ *if and only if* $x, y \in \Pi_1$*; (3)* S *is* ***crisp with respect to*** Π *when* $S(x,y) \in \Pi_0$ *if and only if* $x, y \in \Pi_0$*; (4)* I *is* ***crisp with respect to*** Π *when* $I(x,y) \in \Pi_0$ *if and only if* $x \in \Pi_1$ *and* $y \in \Pi_0$*. When the above conditions are all satisfied we say that* $\mathbb{S}$ *is* Π*-**crisp**.*

Notice in particular how crisp t-norms and crisp s-norms are fully characterized by Prop. II.1 and Prop. II.2. Part of what it takes for a fuzzy implication to be crisp is also guaranteed by Prop. II.4. To show now that a Π-crisp fuzzy semantics is a classic-like fuzzy semantics we prove first the following result.

Proposition III.2. *Given a fuzzy valuation* $e^{\mathbb{S}}$ *of a* Π*-crisp fuzzy semantics* $\mathbb{S}$*, there is a classical valuation* $v : L_P \longrightarrow \mathcal{O}$ *that simulates it, that is, such that*

$$v(\varphi) = 1 \quad \textit{iff} \quad e^{\mathbb{S}}(\varphi) \in \Pi_1$$

holds good for every $\varphi \in L_P$*.*

Proof: Let $\flat : \mathcal{U} \longrightarrow \mathcal{O}$ be such that $\flat(x) = 1$ if $x \in \Pi_1$ and $\flat(x) = 0$ otherwise. We will show that $v = \flat \circ e^{\mathbb{S}}$ defines a standard boolean valuation. The base step is trivial. In the inductive step, for the case of a negated formula $\neg\psi$, note that $v(\neg\psi) = 1$ iff $\flat(e^{\mathbb{S}}(\neg\psi)) = 1$ iff $e^{\mathbb{S}}(\neg\psi) \in \Pi_1$ iff $N(e^{\mathbb{S}}(\psi)) \in \Pi_1$. As $\mathbb{S}$ is Π-crisp, $N(e^{\mathbb{S}}(\psi)) \in \Pi_1$ iff $e^{\mathbb{S}}(\psi) \in \Pi_0$. The induction hypothesis applies to ψ, thus we conclude that $e^{\mathbb{S}}(\psi) \in \Pi_0$ iff $v(\psi) = 0$. From all this we conclude that $v(\neg\psi) = 1$ iff $v(\psi) = 0$, exactly as one would expect of the standard classical semantics of negation. The cases of the remaining operators are analogous. ∎

Corollary III.1. *All classical tautologies are tautologies of a* Π*-crisp fuzzy semantics, that is,* $\mathbb{S}$ *is a classic-like fuzzy semantics whenever* $\mathbb{S} = \langle N, T, S, I\rangle$ *is* Π*-crisp.*

Proof: Consider a classical tautology φ, and pick an arbitrary fuzzy valuation $e^{\mathbb{S}}$. In view of Prop. III.2, there is a classical valuation v that simulates $e^{\mathbb{S}}$. But the formula φ is a tautology, so v must satisfy it, hence $e^{\mathbb{S}}$ must equally satisfy this formula. ∎

IV. Fuzzy Kripke Semantics

The set of modal formulas, LM_P, is defined by adding $(\Diamond\phi)$ to the inductive clauses defining L_P. The connective $\Box$ may be introduced by definition, setting $\Box\alpha := \neg\Diamond\neg\alpha$.

Definition IV.1. *Generalizing the notion of a characteristic function to the domain of fuzzy logic, a* ***fuzzy*** n***-ary relation*** B *over a universe* A *is characterized by a membership function* $\mu_B : A^n \longrightarrow \mathcal{U}$ *which associates to each tuple* $\overrightarrow{x} \in A^n$ *its degree of membership* $\mu_B(\overrightarrow{x})$ *in* B*. In this context, a* ***fuzzy subset*** *is characterized by a fuzzy unary relation, or the corresponding unary membership function. A* ***crisp*** n*-ary relation is any fuzzy* n*-ary relation* B *over a given* A *such that* $\mu_B(A^n) \subseteq \mathcal{O}$*, and crisp sets are defined analogously.*

In the following definitions the standard Kripke models are fuzzified:

Definition IV.2. *A* ***fuzzy frame*** $\mathbb{F}$ *is a structure* $\langle W, \rightsquigarrow\rangle$*, where* W *is a non-empty crisp set (of 'objects', 'worlds', or 'states') and* $\rightsquigarrow$ *is a fuzzy binary ('reduction', 'accessibility', or 'transition') relation over* W*. As expected, to characterize* m*-step accessibility,* $\overset{m}{\rightsquigarrow}$*, we set:*

- $\mu_{\overset{0}{\rightsquigarrow}}(w_i, w_j) \in \Pi_1$ *means that* $w_i = w_j$
- $\mu_{\overset{n+1}{\rightsquigarrow}}(w_i, w_j) \in \Pi_1$ *means that there is some* w_k *such that* $\mu_{\overset{n}{\rightsquigarrow}}(w_i, w_k) \in \Pi_1$ *and* $\mu_{\rightsquigarrow}(w_k, w_j) \in \Pi_1$

Furthermore, $w_i \overset{m}{\leftsquigarrow} w_j$ *is used to denote* $w_j \overset{m}{\rightsquigarrow} w_i$*.*

Definition IV.3. *Given a fuzzy frame* $\mathbb{F}$*, an* $\mathbb{F}$*-**evaluation** is any total function* $\rho : P \times W \longrightarrow \mathcal{U}$*. A* ***fuzzy Kripke model*** *is a structure* $\mathcal{K} = \langle \mathbb{F}, \mathbb{S}, V\rangle$*, where* $\mathbb{F}$ *is a fuzzy frame,* $\mathbb{S}$ *is a classic-like fuzzy semantics where* T *is a left-continuous t-norm and* V *is an* $\mathbb{F}$*-valuation. Given a fuzzy Kripke model* $\mathcal{K}$*, the associated* ***degree of satisfiability*** *is a total function* $\Vdash_{\mathcal{K}} : W \times LM_P \longrightarrow \mathcal{U}$ *recursively defined as follows (in infix notation, we write* $w \Vdash_{\mathcal{K}} \varphi$ *where* $w \in W$ *and* $\varphi \in LM_P$*; when there is no risk of ambiguity, we use more simply* $w \Vdash \varphi$ *instead of* $w \Vdash_{\mathcal{K}} \varphi$*):*

$$\begin{array}{lcl}
w \Vdash \alpha & = & V(\alpha, w), \textit{ if } \alpha \in P \\
w \Vdash \neg\alpha & = & N(w \Vdash \alpha) \\
w \Vdash \alpha \wedge \beta & = & T(w \Vdash \alpha, w \Vdash \beta) \\
w \Vdash \alpha \vee \beta & = & S(w \Vdash \alpha, w \Vdash \beta) \\
w \Vdash \alpha \supset \beta & = & I(w \Vdash \alpha, w \Vdash \beta) \\
w \Vdash \Diamond\alpha & = & \sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash \alpha) : w' \in W\} \\
w \Vdash \Box\alpha & = & N(w \Vdash \Diamond\neg\alpha)
\end{array}$$

A formula $\varphi \in LM_P$ *is said to be* ***true*** *in a fuzzy Kripke model* $\mathcal{K}$*, denoted by* $\vDash_{\mathcal{K}} \alpha$*, if* $(w \Vdash \varphi) \in \Pi_1$ *for every* $w \in W$*. Given a collection* $\mathfrak{K}$ *of fuzzy Kripke models, a formula* $\varphi \in LM_P$ *is said to be a* $\mathfrak{K}$*-**tautology** (denoted by* $\vDash_{\mathfrak{K}} \varphi$*), if* φ *is true in every model in* $\mathfrak{K}$*.*

Note that the above notion of satisfaction coincides with the standard interpretation in modal logics based on the standard bivalent semantics, with the fuzzy operators collapsing into their counterparts in classical logic, and with the interpretations of $\Diamond$ and $\Box$ coinciding with their standard interpretations in Kripke semantics.

Many standard properties of binary relations have natural fuzzy counterparts, among which we may mention:

Definition IV.4. *We say the fuzzy accessibility relation* $\rightsquigarrow$ *is:*

- Π*-**reflexive** if* $\mu_{\rightsquigarrow}(x,x) \in \Pi_1$*, for every* $x \in W$
- Π*-**symmetric** if* $\mu_{\rightsquigarrow}(x,y) \in \Pi_1$ *implies* $\mu_{\rightsquigarrow}(y,x) \in \Pi_1$*, for every* $x, y \in W$
- Π*-**transitive** if* $\mu_{\overset{2}{\rightsquigarrow}}(x,y) \in \Pi_1$ *implies* $\mu_{\rightsquigarrow}(x,y) \in \Pi_1$ *for every* $x, y \in W$
- Π*-**euclidean** if* $\mu_{\rightsquigarrow}(x,y) \in \Pi_1$ *and* $\mu_{\rightsquigarrow}(x,z) \in \Pi_1$ *imply* $\mu_{\rightsquigarrow}(y,z) \in \Pi_1$ *for every* $x, y, z \in W$

In general, given natural numbers k, l, m, n*, we say that* $\rightsquigarrow$ *is* Π*-**(k,l,m,n)-confluent** if for each* $x, y, z \in W$ *such that* $\mu_{\overset{k}{\rightsquigarrow}}(x,y) \in \Pi_1$ *and* $\mu_{\overset{l}{\rightsquigarrow}}(x,z) \in \Pi_1$ *there exists* $w \in W$ *such that* $\mu_{\overset{m}{\rightsquigarrow}}(y,w) \in \Pi_1$ *and* $\mu_{\overset{n}{\rightsquigarrow}}(z,w) \in \Pi_1$*.*

V. Modal Systems based on Instances of $G^{k,l,m,n}$

We will show that normal modal systems based on instances of $G^{k,l,m,n}$ can be characterized by adequate fuzzy Kripke models. First of all, we will prove the completeness of the K-Modal System with respect to the class all fuzzy Kripke models. Next, we will enrich this system with one or more instances of $G^{k,l,m,n}$ and prove a general completeness result for the systems thereby obtained.

Given a fuzzy Kripke model $\mathcal{M} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$, in what follows we shall denote by $\mathcal{M}^C$ the model $\langle W, \rightsquigarrow^C, \mathbb{S}, V\rangle$, where $\mu_{\rightsquigarrow^C} : W \times W \longrightarrow \mathcal{O}$ is such that

$$\mu_{\rightsquigarrow^C}(w, w') = \begin{cases} 1, & \text{if } \mu_{\rightsquigarrow}(w, w') \in \Pi_1 \\ 0, & \text{if } \mu_{\rightsquigarrow}(w, w') \in \Pi_0 \end{cases}$$

and $V^C : P \times W \longrightarrow \mathcal{O}$ is such that $V^C(p, w) = 1$ if $V(p, w) \in \Pi_1$ and $V^C(p, w) = 0$ otherwise.

The following result shows that each fuzzy modal semantics may be assumed to be based on a convenient crisp accessibility relation.

Proposition V.1. *Let $\mathcal{M} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ be a fuzzy Kripke model. Given an arbitrary $w \in W$ and $\alpha \in LM_P$, then $(w \Vdash_{\mathcal{M}} \alpha) \in \Pi_1$ iff $(w \Vdash_{\mathcal{M}^C} \alpha) = 1$.*

Proof: This is checked by induction on α.
[Base step] α is some p $\in P$
$(w \Vdash_{\mathcal{M}} p) \in \Pi_1$ iff, by Def. IV.3, $V(p, w) \in \Pi_1$ iff, by definition of V^C, $V^C(p, w) = 1$ iff $(w \Vdash_{\mathcal{M}^C} p) = 1$.
[Step] Suppose, by Induction Hypothesis, that $(w \Vdash_{\mathcal{M}} \beta) \in \Pi_1$ iff $(w \Vdash_{\mathcal{M}^C} \beta) = 1$. We will check in detail the case where $\alpha = \Diamond\beta$. Suppose first that $(w \Vdash_{\mathcal{M}} \Diamond\beta) \in \Pi_1$. Then, $\sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash_{\mathcal{M}} \beta) : w' \in W\} \in \Pi_1$. There exists w^* such that $T(\mu_{\rightsquigarrow}(w, w^*), w^* \Vdash_{\mathcal{M}} \beta) \in \Pi_1$. By Prop. II.1 $\mu_{\rightsquigarrow}(w, w^*) \in \Pi_1$ and $(w^* \Vdash_{\mathcal{M}} \beta) \in \Pi_1$. By definition of $\mathcal{M}^{\mathcal{C}}$ it's the case that $\mu_{\rightsquigarrow^C}(w, w^*) = 1$ and by Induction Hypothesis $(w^* \Vdash_{\mathcal{M}^C} \beta) = 1$. By the standard interpretation of $\Diamond$, it follows that $(w \Vdash_{\mathcal{M}^C} \Diamond\beta) = 1$. Conversely, using the fact that T is crisp with respect to Π, we can prove that if $(w \Vdash_{\mathcal{M}^C} \Diamond\beta) = 1$, then $(w \Vdash_{\mathcal{M}} \Diamond\beta) \in \Pi_1$. ■

As a straightforward consequence, it follows that:

Corollary V.1. *Given an arbitrary fuzzy Kripke model $\mathcal{M}$ and $\alpha \in LM_P$, then $\vDash_{\mathcal{M}} \alpha$ iff $\vDash_{\mathcal{M}^C} \alpha$.*

A. The K-Modal System

Definition V.1. *The K-modal system is the triple $\langle LM_P, \Delta \cup \{\text{(K)}\}, \{\text{(MP)}, \text{(Nec)}\}\rangle$, where Δ is an axiomatization of Classical Propositional Logic, where* (K) *is the axiom*

$$\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$$

and where (MP) *and* (Nec) *are respectively the rules of Modus Ponens and Necessitation, namely:*

$$\text{(MP)} : \frac{\alpha, \alpha \supset \beta}{\beta} \qquad \textit{and} \qquad \text{(Nec)} : \frac{\vdash \alpha}{\vdash \Box\alpha}$$

Proposition V.2. *Let $\alpha \in LM_P$. Then, α is a theorem in the K-modal system iff $\vDash_{\mathcal{K}} \alpha$ for each fuzzy Kripke model $\mathcal{K} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$.*

Proof: ($\Rightarrow$) We already know, by Corollary III.1, that the theorems of classical logic are all valid in any classic-like fuzzy semantics. It remains to be proven that the axiom (K) is valid and that the inferences rules preserve validity. Suppose that there exists a $w \in W$ such that $(w \Vdash \Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)) \in \Pi_0$. So by Def. $III.3$ it follows that

$$(w \Vdash \Box(\alpha \supset \beta)) \in \Pi_1 \tag{1}$$

and

$$(w \Vdash \Box\alpha \supset \Box\beta) \in \Pi_0 \tag{2}$$

By (2) and Def. $III.3$, we have

$$(w \Vdash \Box\alpha) \in \Pi_1 \quad (3) \qquad \text{and} \qquad (w \Vdash \Box\beta) \in \Pi_0 \quad (4)$$

By (4) and Def. IV.3,

$$N(\sup\{T(\mu_{\rightsquigarrow}(w, w'), N(w' \Vdash \beta))/w' \in W\}) \in \Pi_0 \tag{5}$$

By (5) and Def. $III.3$, we have

$$\sup\{T(\mu_{\rightsquigarrow}(w, w'), N(w' \Vdash \beta))/w' \in W\} \in \Pi_1 \tag{6}$$

By (6) and Prop. II.3 there exists a $w^* \in W$ such that

$$T(\mu_{\rightsquigarrow}(w, w^*), N(w^* \Vdash \beta)) \in \Pi_1 \tag{7}$$

By (7) and the Prop. $II.1$, we have

$$\mu_{\rightsquigarrow}(w, w^*) \in \Pi_1 \quad (8) \qquad \text{and} \qquad N(w^* \Vdash \beta) \in \Pi_1 \quad (9)$$

From (9), by Def. $III.3$ we know that

$$(w^* \Vdash \beta) \in \Pi_0 \tag{10}$$

By (1) and Def. IV.3,

$$\sup\{T(\mu_{\rightsquigarrow}(w, w'), N(w' \Vdash \alpha \supset \beta))/w' \in W\} \in \Pi_0 \tag{11}$$

By (11) and (8) in particular when $w' = w^*$ we have $N(w^* \Vdash \alpha \supset \beta)) \in \Pi_0$, by Def. $III.3$, that is,

$$(w^* \Vdash \alpha \supset \beta) \in \Pi_1 \tag{12}$$

Using (3), (8) and Def. $III.3$, analogously we conclude that

$$(w^* \Vdash \alpha) \in \Pi_1 \tag{13}$$

By (12), (13) and the interpretation of classic-like fuzzy implication it follows that

$$(w^* \Vdash \beta) \in \Pi_1 \tag{14}$$

But (14) contradicts (10) given that $\{\Pi_0, \Pi_1\}$ is a partition.

Assume now that $\vDash_{\mathcal{K}} \beta$. Suppose by contradiction that $\vDash_{\mathcal{K}} \Box\beta$ is not the case. So there exists a $w \in W$ such that $(w \Vdash \Box\beta) \in \Pi_0$, that is, $N(\sup\{T(\mu_{\rightsquigarrow}(w, w'), N(w' \Vdash \beta))/w' \in W\}) \in \Pi_0$. It follows by Def. $III.3$ that $\sup\{T(\mu_{\rightsquigarrow}(w, w'), N(w' \Vdash \beta))/w' \in W\} \in \Pi_1$. For some $w^* \in W$ it is the case that $T(\mu_{\rightsquigarrow}(w, w^*), N(w^* \Vdash \beta)) \in \Pi_1$. From this we conclude that $(w' \Vdash \beta) \in \Pi_0$, contradicting the assumption.

Assume for an arbitrary w that $(w \Vdash \varphi) \in \Pi_1$ and $(w \Vdash \varphi \supset \psi) \in \Pi_1$. Suppose again by contradiction that $(w \Vdash \psi) \in \Pi_0$. Since $(w \Vdash \varphi) \in \Pi_1$, by Def. $III.3$ it follows that $I(w \Vdash \varphi, w \Vdash \psi) \in \Pi_0$, that is $(w \Vdash \varphi \supset \psi) \in \Pi_0$. This is an absurd. ($\Leftarrow$) The K system is known to be complete with respect the class of all Kripke models. So, by Corollary V.1, if $\vDash_{\mathcal{K}} \alpha$ then $\vdash_K \alpha$. ■

B. Completeness of $KG^{k,l,m,n}$

In what follows, we shall prove a sequence of lemmas which are used to establish a soundness result in Theorem V.1.

Lemma V.1. *Let $\mathcal{M} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ be a fuzzy Kripke model. If $(w \Vdash \Diamond^z \varphi) \in \Pi_1$, then there exists a w_z such that both $\mu_{\overset{z}{\rightsquigarrow}}(w, w_z) \in \Pi_1$ and $(w_z \Vdash \varphi) \in \Pi_1$.*

Proof: The proof proceeds by induction on z.
$[Basis]$ $z = 1$
If $(w \Vdash \Diamond\beta) \in \Pi_1$, then $\sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash \beta)/w' \in W\} \in \Pi_1$, by Def. IV.3. So, by Prop. II.3 there is a $w_1 \in W$ such that $T(\mu_{\rightsquigarrow}(w, w_1), w_1 \Vdash \beta) \in \Pi_1$. By Prop. $II.1$ we have $\mu_{\rightsquigarrow}(w, w_1) \in \Pi_1$ and $(w_1 \Vdash \beta) \in \Pi_1$.
$[Step]$ Suppose by Induction Hypothesis that for $z = k$ the property is valid. Note that if $(w \Vdash \Diamond^{k+1}\beta) \in \Pi_1$, then, by Def. IV.3,

$$\sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash \Diamond^k\beta)/w' \in W\} \in \Pi_1 \qquad (15)$$

From (15) and Prop. II.3 there exists a w_1 such that

$$T(\mu_{\rightsquigarrow}(w, w_1), w_1 \Vdash \Diamond^k\beta) \in \Pi_1 \qquad (16)$$

By (16) and Prop. $II.1$ we have:

$$\mu_{\rightsquigarrow}(w, w_1) \in \Pi_1 \quad (17) \qquad \text{and} \qquad (w_1 \Vdash \Diamond^k\beta) \in \Pi_1 \quad (18)$$

By (18) and Induction Hypothesis it follows that there exists a $w_{k'}$ such that $\mu_{\overset{k}{\rightsquigarrow}}(w_1, w_{k'}) \in \Pi_1$ and $(w_{k'} \Vdash \beta) \in \Pi_1$. Using (17) and setting $w_{k+1} = w_{k'}$ we conclude that $\mu_{\overset{k+1}{\rightsquigarrow}}(w, w_{k+1}) \in \Pi_1$ and $(w_{k+1} \Vdash \beta) \in \Pi_1$. ■

Lemma V.2. *Let $\mathcal{M} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ be a fuzzy Kripke model. If $\mu_{\overset{m}{\rightsquigarrow}}(w, v) \in \Pi_1$ and $(w \Vdash \Box^m\varphi) \in \Pi_1$, then $(v \Vdash \varphi) \in \Pi_1$.*

Proof: The proof is carried out by induction on m.
$[Basis]$ $m = 1$. Assume that:

$$\mu_{\rightsquigarrow}(w, v) \in \Pi_1 \quad (19) \qquad \text{and} \qquad (w \Vdash \Box\beta) \in \Pi_1 \quad (20)$$

By (20), Def. III.3 and Def. IV.3 we have that

$$T(\mu_{\rightsquigarrow}(w, v), N(v \Vdash \beta)) \in \Pi_0 \qquad (21)$$

By (19), (21) and Def. III.3 we have that $N(v \Vdash \beta) \in \Pi_0$. By Def. III.3 it follows that $(v \Vdash \beta) \in \Pi_1$.
$[Step]$ $m = k + 1$
The (IH) Induction Hypothesis states that for $m = k$, if $\mu_{\overset{k}{\rightsquigarrow}}(w, v) \in \Pi_1$ and $(w \Vdash \Box^k\beta) \in \Pi_1$ then $(v \Vdash \beta) \in \Pi_1$.
Given $\mu_{\overset{k+1}{\rightsquigarrow}}(w, v) \in \Pi_1$ and $(w \Vdash \Box^{k+1}\beta) \in \Pi_1$, we can prove, using (IH) and Definitions $III.3$ and $IV.3$, that $(v \Vdash \beta) \in \Pi_1$. ■

The following result concerns equivalences between formulas with nested modalities.

Lemma V.3. *If $\mathcal{M} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ is a fuzzy Kripke model, and w is a element of W, then $(w \Vdash \neg\Diamond^m\varphi) \in \Pi_1$ iff $(w \Vdash \Box^m\neg\varphi) \in \Pi_1$.*

Proof: It is not hard to check this by induction on m. The basis and inductive steps use Def. $IV.3$ and Def. $III.3$. ■

Lemma V.4. *Let $\mathcal{M} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ be a fuzzy Kripke model. If $(w \Vdash \neg\Diamond^n\varphi) \in \Pi_1$ and $\mu_{\overset{n}{\rightsquigarrow}}(w, v) \in \Pi_1$, then $(v \Vdash \varphi) \in \Pi_0$.*

Proof: This is a straightforward consequence of the previous results. Indeed, note first that by Lemma $V.3$ we have $(v \Vdash \neg\Diamond^m\varphi) \in \Pi_1$ iff $(v \Vdash \Box^m\neg\varphi) \in \Pi_1$. So we know that $(w \Vdash \Box^m\neg\varphi) \in \Pi_1$ and $\mu_{\overset{n}{\rightsquigarrow}}(w, v) \in \Pi_1$, and applying Lemma V.2 it follows that $(v \Vdash \neg\varphi) \in \Pi_1$. By Def. III.3 we conclude that $(v \Vdash \varphi) \in \Pi_0$. ■

Lemma V.5. *Let $\mathcal{M} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ be a fuzzy Kripke model. If $(w \Vdash \Box^n\varphi) \in \Pi_0$, then there exists some w_n such that $\mu_{\overset{n}{\rightsquigarrow}}(w, w_n) \in \Pi_1$ and $(w_n \Vdash \neg\varphi) \in \Pi_1$.*

Proof: This is checked by induction on n. The basis is straightforward using Def. $IV.3$. If $(w \Vdash \Box^{k+1}\varphi) \in \Pi_0$ we have for some w_1 that $\mu_{\rightsquigarrow}(w, w_1) \in \Pi_1$ and $(w_1 \Vdash \Box^k\varphi) \in \Pi_0$. So, using the Induction Hypothesis it follows that $\mu_{\overset{k+1}{\rightsquigarrow}}(w, w_{k+1}) \in \Pi_1$ and $(w_{k+1} \Vdash \neg\varphi) \in \Pi_1$ ■

The following lemma shows that the axiom $G^{k,l,m,n}$ is sound with respect fuzzy Kripke models in which $\rightsquigarrow$ is Π-(k, l, m, n)-confluent:

Lemma V.6 (Soundness Lemma)**.** *If α is a formula of form $G^{k,l,m,n}$ and $\mathcal{G}$ is a fuzzy Kripke model where $\rightsquigarrow$ is Π_1-(k, l, m, n)-confluent, then $\vDash_{\mathcal{G}} \alpha$.*

Proof: Let α be $\Diamond^k\Box^m\beta \supset \Box^l\Diamond^n\beta$. Suppose that $(w \Vdash_{\mathcal{G}} \Diamond^k\Box^m\beta \supset \Box^l\Diamond^n\beta) \in \Pi_0$ for some $w \in W$. Then by Def. III.1

$$(w \Vdash_{\mathcal{G}} \Diamond^k\Box^m\beta) \in \Pi_1 \qquad (22)$$

and

$$(w \Vdash_{\mathcal{G}} \Box^l\Diamond^n\beta) \in \Pi_0 \qquad (23)$$

By (22) and Lemma $V.1$ there exists a w_k such that

$$\mu_{\overset{k}{\rightsquigarrow}}(w, w_k) \in \Pi_1 \quad (24) \qquad \text{and} \qquad (w_k \Vdash_{\mathcal{G}} \Box^m\beta) \in \Pi_1 \quad (25)$$

By (23) and Lemma $V.5$ there exists a w_l such that

$$\mu_{\overset{l}{\rightsquigarrow}}(w, w_l) \in \Pi_1 \qquad (26)$$

and

$$(w_l \Vdash_{\mathcal{G}} \neg\Diamond^n\beta) \in \Pi_1 \qquad (27)$$

By (24), (26) and the appropriate instance of the Π-confluence property of $\rightsquigarrow$ there exists a $x \in W$ such that

$$\mu_{\overset{m}{\rightsquigarrow}}(w_k, x) \in \Pi_1 \quad (28) \qquad \text{and} \qquad \mu_{\overset{n}{\rightsquigarrow}}(w_l, x) \in \Pi_1 \quad (29)$$

By (25), (28) and Lemma $V.2$ we conclude that

$$(x \Vdash_{\mathcal{G}} \beta) \in \Pi_1 \tag{30}$$

By (27), (29) and Lemma $V.4$, on the other hand, we conclude that

$$(x \Vdash_{\mathcal{G}} \beta) \in \Pi_0 \tag{31}$$

Note that (31) contradicts (30). ■

Theorem V.1. *For any $\alpha \in LM_P$, we have that α is a theorem of $KG^{k,l,m,n}$ iff $\vDash_{\mathcal{KG}} \alpha$ for each fuzzy Kripke model $\mathcal{KG} = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ such that $\rightsquigarrow$ is Π-(k,l,m,n)-confluent.*

Proof: (⇒) Let α be a theorem of the $KG^{k,l,m,n}$ and $\mathcal{KG}^{k,l,m,n}$ be a fuzzy Kripke model where $\rightsquigarrow$ is Π-(k,l,m,n)-confluent. We will prove that $\vDash_{\mathcal{KG}} \alpha$. In view of Prop. $V.2$, however, it is sufficient to check the case where α is an instance of the $G^{k,l,m,n}$-axiom, i.e., to check that $(w \Vdash_{\mathcal{KG}} \Diamond^k \Box^m \beta \supset \Box^l \Diamond^n \beta) \in \Pi_1$ for each $w \in W$ and $\beta \in LM_P$, but from the Lemma $V.6$ it is immediate.
(⇐) In [7] the completeness of system $KG^{k,l,m,n}$ with respect the class of models that satisfies the confluence accessibility relation is established. By Corollary V.1 it follows that the system $\mathcal{KG}$ is complete with respect the $KG^{k,l,m,n}$ system. So, if $\vDash_{\mathcal{KG}} \beta$ then $\vdash_{KG} \beta$. ■

The completeness results proven in Prop. $V.1$ can be shown to hold not only for singular instances of $G^{k,l,m,n}$, but also for several such instances combined. Indeed:

Proposition V.3. *Let $G^{k_1,l_1,m_1,n_1}, \ldots, G^{k_p,l_p,m_p,n_p}$ be instances of the schema $G^{k,l,m,n}$. Let $K + G^{k_1,l_1,m_1,n_1} + \ldots + G^{k_p,l_p,m_p,n_p}$ be the system which results from extending K with $G^{k_1,l_1,m_1,n_1}, \ldots, G^{k_p,l_p,m_p,n_p}$. A formula α is a theorem of $K + G^{k_1,l_1,m_1,n_1} + \ldots + G^{k_p,l_p,m_p,n_p}$ iff $\vDash_{\mathcal{KG}^+} \alpha$ for each fuzzy Kripke model $\mathcal{KG}^+ = \langle W, \rightsquigarrow, \mathbb{S}, V\rangle$ such that $\rightsquigarrow$ is Π-(k_1,l_1,m_1,n_1)-confluent,…,(k_p,l_p,m_p,n_p)-confluent.*

Proof: (⇒) By Theorem $V.1$ this result is valid for $K + G^{k_1,l_1,m_1,n_1}$. If we add G^{k_2,l_2,m_2,n_2} and use Lemma $V.6$ we can conclude that $K + G^{k_1,l_1,m_2,n_2} + G^{k_2,l_2,m_2,n_2}$ is sound in all fuzzy Kripke models such that $\rightsquigarrow$ is (k_1,l_1,m_1,n_1)-confluent and (k_2,l_2,m_2,n_2)-confluent. Using the same reasoning we can extend the result for each system $K + G^{k_1,l_1,m_1,n_1} + \ldots + G^{k_p,l_p,m_p,n_p}$. (⇐) From Corollary V.1 this proof is analogous to the proof of completeness for extensions of K with finitely many instances of $G^{k,l,m,n}$, as done, e.g., in [9]. ■

Notice that the completeness of the modal systems KT, KB and KD are direct consequences of Prop. $V.1$, while the completeness of B, $S4$ and $S5$ follows from Prop. $V.3$. To illustrate, here is how we may obtain completeness for $S5$ (we use below ⇒ and & for the classical metalinguistic implication and conjunction).

Example V.1. *S5 is complete with respect all Π-reflexive and Π-euclidean fuzzy Kripke models.* The modal system S5 is axiomatized by K, T and 5, i.e. $K + \langle 0,0,1,0\rangle + \langle 1,1,0,1\rangle$. But $\rightsquigarrow$ is Π-$\langle 1,1,0,1\rangle$-confluent iff (by Definition $IV.4$) $\forall x \forall y \forall z((\mu_{\rightsquigarrow}(x,y) \in \Pi_1 \ \&\ \mu_{\rightsquigarrow}(x,z) \in \Pi_1) \Rightarrow \exists w(y = w \ \&\ \mu_{\rightsquigarrow}(z,w) \in \Pi_1))$ iff for arbitrary $x,y,z \in W$ we have that $(\mu_{\rightsquigarrow}(x,y) \in \Pi_1 \ \&\ \mu_{\rightsquigarrow}(x,z) \in \Pi_1) \Rightarrow (\mu_{\rightsquigarrow}(z,y) \in \Pi_1)$ iff $\forall x \forall y \forall z(\mu_{\rightsquigarrow}(x,y) \in \Pi_1 \ \&\ \mu_{\rightsquigarrow}(x,z) \in \Pi_1) \rightarrow (\mu_{\rightsquigarrow}(z,y) \in \Pi_1))$ iff (by Definition $IV.4$) $\rightsquigarrow$ is Π-euclidean. Furthermore, using a similar reasoning we note that $\rightsquigarrow$ is Π_1-$\langle 0,0,1,0\rangle$-confluent iff $\forall x \forall y \forall z((x = y \ \&\ x = z) \rightarrow \exists w(\mu_{\rightsquigarrow}(y,w) \in \Pi_1 \ \&\ z = w))$ iff $\forall x(\mu_{\rightsquigarrow}(x,x) \in \Pi_1)$ iff $\rightsquigarrow$ is Π-reflexive. So, by Theorem $V.3$ follows the completeness of $K + \langle 0,0,1,0\rangle + \langle 1,1,0,1\rangle$ with respect all fuzzy Kripke models that are Π-reflexive and Π-euclidean.

VI. Final Remarks

The paper generalizes scattered results from [6] to a much more inclusive collection of modal logics, and also greatly generalizes our previous approach in [1] by the consideration of other classic-like partitions of the interval $[0,1]$ as $[0,i) \cup [i,1]$. The partition presupposed by most fuzzy logics in the literature takes $i = 1$, a constraint which seems by all means unnecessary. Furthermore, adapting the previous results to partitions of the form $[0,i] \cup (i,1]$ requires straightforward modifications to the above.

We believe it is possible to study a multimodal (diamond) version of the axiom of confluence by adding appropriate indices to the modalities, at the linguistic level, and adding corresponding fuzzy accessibility relations, at the semantic level (in such case, the initial case with iterated modalities will accordingly be reduced to distinct one-step modalities). Completeness should in this case be attainable, as in the case of normal modal logics extending classical logic, by adding appropriate interaction axioms. We also conjecture that the above results on the axiom of confluence and its corresponding collection of frames may be extended to every Sahlqvist-definable frame class. This thread of investigation, however, shall be left as matter for future work.

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