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Semi-BCI Algebras

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The notion of semi-BCI algebras is introduced and some of its properties are investigated. This notion is a generalization of BCI algebras, suggested by the process of intervalization of BCI algebras. Semi-BCI algebras have a similar structure to pseudo-BCI algebras, however they are not the same. In this paper we investigate the similarities between these classes of algebras by showing how they relate to the process of intervalization. (Mathematics Subject Classification: **03G25, 06F35**)

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1 INTRODUCTION

One of the most well-known references on the algebraic approach to logics is the book of Rasiowa [21], which dates to the 70s. In this book, at pages 16-17, the notion of implicative algebra, which aims at modelling a simple

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notion of implication, is introduced: An **implicative algebra** is an algebra $\langle A, \Rightarrow, \top\rangle$ of type (2, 0) which satisfies the following properties:

(i-1) $a \Rightarrow a = \top$,
(i-2) If $a \Rightarrow b = \top$ and $b \Rightarrow c = \top$, then $a \Rightarrow c = \top$,
(i-3) If $a \Rightarrow b = \top$ and $b \Rightarrow a = \top$, then $a = b$.

A direct consequence of such definition is the establishment of an order relation "$\leq$" on A, by way of a definition that is known as **Order Property (OP)** of implications:

$$a \leq b \text{ if and only if } a \Rightarrow b = \top. \tag{1}$$

Many implications in the literature indeed satisfy **(OP)**. However, some important implications in the field of fuzzy logics do not satisfy such requirement. For an example, consider the algebra $\langle [0,1], \rightarrow_{\mathrm{YG}}, 1\rangle$, such that:

$$x \rightarrow_{\mathrm{YG}} y = \begin{cases} 1, \text{ if } x = y = 0 \\ y^x, \text{ otherwise} \end{cases} .$$

Note that $x \rightarrow_{\mathrm{YG}} y = 1$ implies $x \leq y$ whereas $0.3 \leq 0.5$ and $0.3 \rightarrow_{\mathrm{YG}} 0.5 \cong 0.81225 \neq 1$. For other examples of well-known fuzzy implications failing **(OP)** check [1, 20, 26].

The interval Łukasiewicz implication introduced by Bedregal and Santiago [2] also fails to satisfy **(OP)**. The authors, however, pointed out that the interval Łukasiewicz implication "$\Rightarrow_{LK}$" (see [2, Lemma 4.1]) does satisfy:

1. if $X \ll Y$, then $X \Rightarrow_{LK} Y = 1$;
2. if $X \Rightarrow_{LK} Y = 1$, then $X \leq_{KM} Y$,

where, given $X = [\underline{X}, \overline{X}]$ and $Y = [\underline{Y}, \overline{Y}]$, we set $X \ll Y$ iff $\overline{X} \leq \underline{Y}$ and set $X \leq_{KM} Y$ iff $\underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$.

Above, the relation "$\ll$" is the auxiliary relation [9] of the usual Kulisch-Miranker order on intervals "$\leq_{KM}$". The relation "$\prec\!\!\prec$" is called an **auxiliary relation** of a partial order "$\preccurlyeq$" if it satisfies the following properties [9]:

1. if $x \prec\!\!\prec y$, then $x \preccurlyeq y$;
2. if $u \preccurlyeq x \prec\!\!\prec y \preccurlyeq z$, then $u \prec\!\!\prec z$;
3. if a smallest element 0 for $\preccurlyeq$ exists, then $0 \prec\!\!\prec x$.

Since **(OP)** connects an implication to the underlying order relation that extends some auxiliary relation, this paper proposes a structure with two implications, $\rightarrow$ and $\twoheadrightarrow$, which define two relations, respectively $\preceq$ and $\ll$. These relations are related in a similar way as auxiliary orders are related to partial orders. The resulting algebraic structure is called **semi-BCI algebra**, and it generalizes both BCI algebras and their interval counterpart.

Another generalization for BCI algebras that contains two implications is called **pseudo-BCI algebra** and was proposed by Dudek and Jun [5]. The connection of such algebras to semi-BCI algebras is investigated here.

The paper is organized in the following way: Section 2 provides a brief review of BCI algebras and their properties. Section 3 shows how some mathematical structures can be extended to their interval counterpart. Section 4 shows the intervalization of a class of BCI algebras and some properties of this interval algebra. Section 5 introduces the notion of semi-BCI algebra and proves some of its properties. Section 6 discusses the relation between semi-BCI algebras and pseudo-BCI algebras. Finally, section 7 provides some concluding remarks.

2 BCI ALGEBRAS

BCI algebras were introduced by Iséki [15] in the 60s and since then have been extensively investigated.

Definition 2.1. *A **BCI algebra** is an algebra $\mathcal{C} = \langle A, \rightarrow, \top \rangle$ that satisfies:*

($\mathcal{C}$-1) $(y \rightarrow z) \rightarrow ((z \rightarrow x) \rightarrow (y \rightarrow x)) = \top$,
($\mathcal{C}$-2) $x \rightarrow ((x \rightarrow y) \rightarrow y) = \top$,
($\mathcal{C}$-3) $x \rightarrow x = \top$,
($\mathcal{C}$-4) *if* $x \rightarrow y = \top$ *and* $y \rightarrow x = \top$ *then* $x = y$.

On any such BCI algebra it is possible to define a partial order "$\preceq$":

($\mathcal{C}$-5) $x \preceq y$ *iff* $x \rightarrow y = \top$.

*Whenever $x \rightarrow \top = \top$, i.e. $x \preceq \top$, the BCI algebra $\mathcal{C} = \langle A, \rightarrow, \top \rangle$ will be called a **BCK algebra**.*

Example 2.1. *The Łukasiewicz implicative algebra* $([0, 1], \rightarrow_{LK}, 1)$*, where* $x \rightarrow_{LK} y = min(1, 1 - x + y)$*, is a BCI algebra.*

Let us now recall some useful properties of BCI algebras (for more details see [12]):

(A-1) $\top \preceq x$ implies $x = \top$,
(A-2) $x \preceq y$ implies $y \rightarrow z \preceq x \rightarrow z$, (First place antitonicity)

(A-3) $x \preceq y$ implies $z \rightarrow x \preceq z \rightarrow y$, (Second place isotonicity)
(A-4) $x \preceq y$ and $y \preceq z$ implies $x \preceq z$, (Transitivity)
(A-5) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$, (Exchange – **EP**)
(A-6) $x \preceq y \rightarrow z$ implies $y \preceq x \rightarrow z$,
(A-7) $x \rightarrow y \preceq (z \rightarrow x) \rightarrow (z \rightarrow y)$,
(A-8) $\top \rightarrow x = x$, (Left Neutrality)
(A-9) $((y \rightarrow x) \rightarrow x) \rightarrow x = y \rightarrow x$,
(A-10) $x \rightarrow y \preceq (y \rightarrow x) \rightarrow \top$,
(A-11) $(x \rightarrow y) \rightarrow \top = (x \rightarrow \top) \rightarrow (y \rightarrow \top)$.

Note that properties (A-7), (A-5) and ($\mathcal{C}$-3) model the combinators B, C and I of BCI Logic [11].

Proposition 2.1. *Let $\langle A, \rightarrow, \top \rangle$ be a BCI algebra. $\langle A, \rightarrow, \top \rangle$ is a BCK algebra if and only if for each $x \in A$ there exists $y \in A$ such that $y \preceq x$ and $y \preceq \top$.*

Proof. ($\Rightarrow$) Straightforward because in BCK algebras $\top$ is the greatest element, i.e. $x \preceq \top$ for each $x \in A$.

($\Leftarrow$) Suppose, by contradiction, that $\langle A, \rightarrow, \top \rangle$ is not a BCK algebra. Then, there exists $a \in A$ such that $a \not\preceq \top$. By hypothesis there exists $b \in A$ such that $b \preceq a$ and $b \preceq \top$. So, by (A-8) and the definition of $\preceq$, $(b \rightarrow \top) \rightarrow ((\top \rightarrow a) \rightarrow (b \rightarrow a)) = \top \rightarrow (a \rightarrow \top) = a \rightarrow \top \neq \top$. Therefore, ($\mathcal{C}$-1) fails. □

The next section introduces the process of intervalization over abstract partial orders.

3 INTERVALIZATION OF STRUCTURES

The limited capacity of machines to store just a finite set of finitely represented objects constrains the automatic calculation (computation) of structures in which a machine representation of some objects exceeds such capacity. In the case of real numbers, although programs often provide highly accurate results, it can happen that rounding errors built up during each step in the computation produce results which are not even meaningful. For more details see the early Forsythe's report [7]. In 1988, Siegfried Rump [23] published the result of a computed function in an IBM S/370 mainframe. The function was:

$$y = 333.75b^6 + a^2(11a^2b^2 - b^6 - 121b^4 - 2) + 5.5b^8 + \frac{a}{2b}. \qquad (2)$$

He calculated for $a = 77617.0$ and $b = 33096.0$, and the result was:

1. single precision: $y = 1.172603\ldots$;
2. double precision: $y = 1.1726039400531\ldots$;
3. extended precision: $y = 1.172603940053178\ldots$.

All results lead any user to conclude that IBM S/370 returned the correct result. However this result is WRONG and the correct result lies in the interval: $-0.827396059946821 35 \pm 5 \times 10^{-17}$. Note that *even the sign is wrong!*

One of the proposals to overcome this problem is due, almost simultaneously, to Ramon Moore [17, 18] and Teruo Sunaga [25]. They developed the so-called **interval arithmetic**. Interval arithmetic is a set of operations on the set of all closed intervals $\mathbb{I}(\mathbb{R}) = \{[a, b] : a, b \in \mathbb{R} \text{ and } a \leq b\}$. The operations are defined in the following way:

1. $[a, b] + [c, d] = [a + c, b + d]$,
2. $[a, b] \cdot [c, d] = [\min P, \max P]$, where $P = \{a \cdot c, a \cdot d, b \cdot c, b \cdot d\}$,
3. $[a, b] - [c, d] = [a - d, b - c]$,
4. $[a, b]/[c, d] = [a, b] \cdot ([1/d, 1/c])$, provided that $0 \notin [c, d]$.

Observe that for each operation $* \in \{+, -, \cdot, /\}$, $[a, b] \circledast [c, d] = \{x * y \in \mathbb{R} : x \in [a, b] \text{ and } y \in [c, d]\}$. Assuming that the latter set always corresponds to an interval, this reveals two important properties of this arithmetic (a) **Correctness** and (b) **Optimality**.

> "*Correctness*. The criterion for correctness of a definition of interval arithmetic is that the "Fundamental Theorem of Interval Arithmetic" holds [I]: when an expression is evaluated using intervals, it yields an interval containing all results of pointwise evaluations based on point values that are elements of the argument intervals.
>
> [...]
>
> *Optimality*. By optimality, we mean that the computed floating-point interval is not wider than necessary."
>
> Hickey *et.al* [10, p.1040]

The application of interval methods follows the following paradigm: Enclosure in intervals the values which are not exact by whatever reason (e.g. the value comes from an imprecise measurement) and applying correct and optimal operations on such intervals in order to obtain the best interval which contains the desired output. This approach will avoid what happened with the Rump's example. Therefore, the notion of correctness is indispensable for such philosophy.

[I] Moore [19, Theorem 3.1, p. 21]: If F is an inclusion monotonic interval extension of f, then $\vec{f}(X_1, ..., X_n) \subseteq F(X_1, ..., X_n)$, where $\vec{f}(X_1, ..., X_n) = \{f(x_1, \ldots, x_n) : x_i \in X_i\}$.

The property of correctness was investigated in 2006 by Santiago *et al* [3,24]. In those papers, instead of correctness the authors used the term **interval representation**, since an interval computation could be understood not just as a machine representation of real numbers, but also as a mathematical representation of real numbers (this idea is confirmed by the Representation Theorems of Euclidean continuous functions in [3, 24]). Also, the notion of optimality was named by the authors as **best interval representation**, or **best representation** for short. And in what follows this notion is shown for binary operations: A binary interval operation $\circledast$ *represents* a binary real operation, $*$, whenever:

$$(x, y) \in [a, b] \times [c, d] \Rightarrow x * y \in [a, b] \circledast [c, d]. \tag{3}$$

This can be easily extended to n-ary operations. The authors showed that this notion is more general than what is stated by the Fundamental Theorem of Interval Arithmetic, given that there are representations which are not inclusion monotonic (see [24, p. 238]).

One noteworthy point which will be taken into account in the present paper is the difference between **representation** and **extension** of a function f. For example, given intervals $X = [\underline{X}, \overline{X}]$ and $Y = [\underline{Y}, \overline{Y}]$, the function $X - Y = [\min(\underline{X} - \underline{Y}, \overline{X} - \overline{Y}), \max(\underline{X} - \underline{Y}, \overline{X} - \overline{Y})]$, presented in [16], extends the subtraction on real numbers, however $[2, 3] - [2, 3] = [0, 0]$, $2.5, 2.1 \in [2, 3]$, but $2.5 - 2.1 \notin [2, 3] - [2, 3]$; in other words, this operation is not correct. So, there are interval extensions which are not correct. They are useless for the proposed philosophy.

The process of giving the correct and optimal interval version F for a function f is called **intervalization**. There are many proposals of intervalization of algebraic structures further than that of real numbers proposed by Moore and Sunaga. In the literature, the reader can find proposals even for the field of Logic. For example: The Łukasiewicz implicative algebra $\langle [0, 1], \rightarrow_{LK}, 1\rangle$, where $x \rightarrow_{LK} y = \min(1, 1 - x + y)$ interprets some many-valued logics and was intervalized by Bedregal and Santiago in [2]. Its MV algebra counterpart was intervalized by Cabrer et al in [4], also, in order to overcome the same problems already stated for $\mathbb{I}(\mathbb{R})$. In both cases, the interval algebras did not satisfy the same properties that are satisfied by the algebras that they came from. The same happened with $\mathbb{I}(\mathbb{R})$.

The following section studies a way of intervalizing of BCI algebras. As in the case of MV algebras and Łukasiewicz algebras the resulting structure does not belong to the same category of its starting algebra. In this paper we provide an investigation of the resulting structures. In order to achieve that, some required concepts, like the abstract notion of intervals are introduced.

Definition 3.1 (Abstract Intervals). *Given a poset $\langle A, \leq \rangle$, the set $[a, b] = \{x \in A \mid a \leq x \leq b\}$ is called* ***the closed interval with endpoints*** *a* ***and*** *b and $\mathbb{A} = \{[a, b] \mid a, b \in A$ and $a \leq b\}$ is the* ***set of all closed intervals of elements of*** *A. For any $X \in \mathbb{A}$ we denote its left and right endpoints by $\underline{X}$ and $\overline{X}$, respectively, i.e. if $X = [a, b]$ then $\underline{X} = a$ and $\overline{X} = b$. When $\underline{X} = \overline{X}$ the interval is called* ***degenerate****. The embedding $i : A \to \mathbb{A}$, such that $i(a) = [a, a]$ is called* ***natural embedding****. On the set $\mathbb{A}$, one may define the following canonical partial order: $X \leq_{KM} Y$ if and only if $\underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$. This relation is called* ***pointwise*** *(or* ***Kulisch-Miranker) order****. Thus, we obtain a new poset, $(\mathbb{A}, \leq_{KM})$.*

Since each BCI algebra is a partially ordered system, $\langle B, \leq \rangle$, one may apply Definition 1 to obtain the poset $\langle \mathbb{B}, \leq_{KM} \rangle$. The natural question is about the implications on $\mathbb{B}$: Is there a correct interval operation on $\mathbb{B}$ that satisfies the BCI axioms? The following section will show that the answer is negative. So, since correctness is indispensable, a price must be paid: *A new theory for $\langle \mathbb{B}, \leq_{KM} \rangle$ must be developed*. This is the goal of this paper.

4 INTERVALIZATION OF BCI ALGEBRAS

This section shows, in Proposition 4.1, that it is possible to build an interval BCI algebra, but with a non-correct implication, and also shows, in Theorem 4.1, that it is not possible to have an interval BCI algebra with a correct implication. Finally, we provide the "BCI algebra intervalization theorem" and some properties of the algebras that emerge from it, which are called interval BCI algebras.

Lemma 4.1. *Let $\langle A, \to, \top \rangle$ be a BCI algebra such that $\langle A, \preceq \rangle$ is a meet-semilattice. For each $a, b, c \in A$, $a \to (b \wedge c) = \top$ iff $a \to b = \top$ and $a \to c = \top$. Moreover, if $a \to c = \top$ and $b \to c = \top$ then $(a \wedge b) \to c = \top$.*

Proof. Straightforward. □

Lemma 4.2. *Let $\langle A, \to, \top \rangle$ be a BCI algebra such that $\langle A, \preceq \rangle$ is a meet-semilattice satisfying:*

$$a \preceq b \to c \text{ iff } a \wedge b \preceq c, \tag{4}$$

for every $a, b, c \in A$. For each $a, b, c, d, e \in A$, if $a \to ((b \to c) \wedge (d \to e)) = \top$ then $a \to ((b \wedge d) \to (c \wedge e)) = \top$.

Proof. If $a \to ((b \to c) \wedge (d \to e)) = \top$ then, by Lemma 4.1 and ($\mathcal{C}$-5), $a \preceq b \to c$ and $a \preceq d \to e$. By (4), $a \wedge b \preceq c$ and $a \wedge d \preceq e$ and therefore, $b \preceq a \to c$ and $d \preceq a \to e$. So, $b \wedge d \preceq a \to c$ and $b \wedge d \preceq a \to e$. Thus, applying again (4), $(b \wedge d) \wedge a \preceq c$ and $(b \wedge d) \wedge a \preceq e$. Hence, $(b \wedge d) \wedge a \preceq c \wedge e$. So, by (4) and ($\mathcal{C}$-5), $a \to ((b \wedge d) \to (c \wedge e)) = \top$. □

Proposition 4.1. *Let $\langle A, \to, \top\rangle$ be a BCI algebra such that $\langle A, \preceq\rangle$ is a meet-semilattice satisfying* (4). *Then $\langle \mathbb{A}, \Mapsto, [\top, \top]\rangle$, where*

$$X \Mapsto Y = [(\underline{X} \to \underline{Y}) \wedge (\overline{X} \to \overline{Y}), \overline{X} \to \overline{Y}], \tag{5}$$

is also a BCI algebra that is a meet-semilattice satisfying (4).

Proof. Notice that in this case, defining $X \trianglelefteq Y$ iff $X \Mapsto Y = [\top, \top]$, then $X \trianglelefteq Y$ iff $\underline{X} \to \underline{Y} = \overline{X} \to \overline{Y} = \top$ iff $\underline{X} \preceq \underline{Y}$ and $\overline{X} \preceq \overline{Y}$.

Thus, clearly, the properties ($\mathcal{C}$-3) to ($\mathcal{C}$-5) are trivially satisfied. In the following, ($\mathcal{C}$-1) is proved for the intervalized algebra. When we refer to the properties of a BCI algebra $\langle A, \to, \top\rangle$ we use the notation $(\mathcal{C}\text{-}i)_\to$ and $(A\text{-}i)_\to$.

Since, $\langle A, \to, \top\rangle$ is a BCI algebra, by $(\mathcal{C}\text{-}1)_\to$, each $X, Y, Z \in \mathbb{A}$, $(\underline{Y} \to \underline{Z}) \to ((\underline{Z} \to \underline{X}) \to (\underline{Y} \to \underline{X})) = \top$ and $(\overline{Y} \to \overline{Z}) \to ((\overline{Z} \to \overline{X}) \to (\overline{Y} \to \overline{X})) = \top$. Then, $((\underline{Y} \to \underline{Z}) \wedge (\overline{Y} \to \overline{Z})) \to ((\underline{Z} \to \underline{X}) \to (\underline{Y} \to \underline{X})) = \top$ and $((\underline{Y} \to \underline{Z}) \wedge (\overline{Y} \to \overline{Z})) \to ((\overline{Z} \to \overline{X}) \to (\overline{Y} \to \overline{X})) = \top$. So, by Lemma 4.1, $((\underline{Y} \to \underline{Z}) \wedge (\overline{Y} \to \overline{Z})) \to (((\underline{Z} \to \underline{X}) \to (\underline{Y} \to \underline{X})) \wedge ((\overline{Z} \to \overline{X}) \to (\overline{Y} \to \overline{X}))) = \top$. Thus, by Lemma 4.2 and (5), $\underline{Y \Mapsto Z} \to (((\underline{Z} \to \underline{X}) \wedge (\overline{Z} \to \overline{X})) \to ((\underline{Y} \to \underline{X}) \wedge (\overline{Y} \to \overline{X}))) = \top$. Therefore by (5), (*) $\underline{Y \Mapsto Z} \to (\underline{Z \Mapsto X} \to \underline{Y \Mapsto X}) = \top$. On the other hand, by $(\mathcal{C}\text{-}i)_\to$, $(\overline{Y} \to \overline{Z}) \to ((\overline{Z} \to \overline{X}) \to (\overline{Y} \to \overline{X})) = \top$ and so, by (5): (**) $\overline{Y \Mapsto Z} \to (\overline{Z \Mapsto X} \to \overline{Y \Mapsto X}) = \top$. Thus, from (*) and (**) and Lemma 4.1, $(\underline{Y \Mapsto Z} \wedge \overline{Y \Mapsto Z}) \to ((\underline{Z \Mapsto X} \to \underline{Y \Mapsto X}) \wedge (\overline{Z \Mapsto X} \to \overline{Y \Mapsto X})) = \top$. Since, by (5), $(\underline{Y \Mapsto Z} \wedge \overline{Y \Mapsto Z}) = \underline{Y \Mapsto Z}$, then (***) $\underline{Y \Mapsto Z} \to ((\underline{Z \Mapsto X} \to \underline{Y \Mapsto X}) \wedge (\overline{Z \Mapsto X} \to \overline{Y \Mapsto X})) = \top$. Thus, from (***) and (**), $(\underline{Y \Mapsto Z} \to \underline{(Z \Mapsto X) \Mapsto (Y \Mapsto X)}) \wedge (\overline{Y \Mapsto Z} \to \overline{(Z \Mapsto X) \Mapsto (Y \Mapsto X)}) = \top$ and $\overline{Y \Mapsto Z} \to \overline{(Z \Mapsto X) \Mapsto (Y \Mapsto X)} = \top$. Therefore, by (5), $(Y \Mapsto Z) \Mapsto ((Z \Mapsto X) \Mapsto (Y \Mapsto X)) = [\top, \top]$.

Property ($\mathcal{C}$-2): Clearly, $\underline{X} \to \underline{Y} \geq ((\underline{X} \to \underline{Y}) \wedge (\overline{X} \to \overline{Y}))$ and therefore, by $(A\text{-}2)_\to$, $((\underline{X} \to \underline{Y}) \wedge (\overline{X} \to \overline{Y})) \to \underline{Y} \geq (\underline{X} \to \underline{Y}) \to \underline{Y}$. So, by $(A\text{-}3)_\to$ and $(\mathcal{C}\text{-}2)_\to$, $\underline{X} \to (((\underline{X} \to \underline{Y}) \wedge (\overline{X} \to \overline{Y})) \to \underline{Y}) \geq \underline{X} \to ((\underline{X} \to \underline{Y}) \to \underline{Y}) = \top$. So, by (5) and $(A\text{-}1)_\to$, $\underline{X} \to (\underline{(X \Mapsto Y)} \to \underline{Y}) = \top$. On the other hand, by (5) and $(\mathcal{C}\text{-}2)_\to$, (#) $\overline{X} \to (\overline{(X \Mapsto Y)} \to \overline{Y}) = \top$. Therefore, (##) $(\underline{X} \to (\underline{(X \Mapsto Y)} \to \underline{Y})) \wedge (\overline{X} \to (\overline{(X \Mapsto Y)} \to \overline{Y})) = \top$. Hence, from (##),

(#) and (5): $\underline{X} \rightarrow \underline{(X \Mapsto Y) \Mapsto Y} = \top$ and $\overline{X} \rightarrow \overline{(X \Mapsto Y) \Mapsto Y} = \top$. Consequently, $[\underline{X} \rightarrow \underline{(X \Mapsto Y) \Mapsto Y} \wedge \overline{X} \rightarrow \overline{(X \Mapsto Y) \Mapsto Y}, \overline{X} \rightarrow \overline{(X \Mapsto Y) \Mapsto Y}] = [\top, \top]$. Therefore, by (5), $X \Mapsto ((X \Mapsto Y) \Mapsto Y) = [\top, \top]$.

We now check that $\langle \mathbb{A}, \trianglelefteq \rangle$ is a meet-semilattice. In fact, let $X, Y, Z \in \mathbb{A}$. Then, $X \Mapsto (Y \wedge Z) = [\top, \top]$ iff $(\underline{X} \rightarrow (\underline{Y} \wedge \underline{Z})) \wedge (\overline{X} \rightarrow (\overline{Y} \wedge \overline{Z})) = \top$ and $\overline{X} \rightarrow (\overline{Y} \wedge \overline{Z}) = \top$ iff $\underline{X} \rightarrow \underline{Y} = \top$, $\underline{X} \rightarrow \underline{Z} = \top$, $\overline{X} \rightarrow \overline{Y} = \top$ and $\overline{X} \rightarrow \overline{Z} = \top$ iff $X \Mapsto Y = [\top, \top]$ and $X \Mapsto Z = [\top, \top]$. In addition, $X \trianglelefteq Y \Mapsto Z$ iff $\underline{X} \preceq (\underline{Y} \rightarrow \underline{Z}) \wedge (\overline{Y} \rightarrow \overline{Z})$ and $\overline{X} \preceq \overline{Y} \rightarrow \overline{Z}$ iff $\underline{X} \preceq \underline{Y} \rightarrow \underline{Z}$ and $\overline{X} \preceq \overline{Y} \rightarrow \overline{Z}$ iff $\underline{X} \wedge \underline{Y} \preceq \underline{Z}$ and $\overline{X} \wedge \overline{Y} \preceq \overline{Z}$ iff $X \wedge Y \trianglelefteq Z$. Therefore, $\langle \mathbb{A}, \Mapsto, [\top, \top] \rangle$ satisfies (4). □

If A has two different elements, say a and b, such that $a \rightarrow b = \top$, i.e. $a \preceq b$, then $\Mapsto$ **is not an interval representation of** $\rightarrow$. In particular, $[a, b] \Mapsto [a, b] = [\top, \top]$. Nevertheless, by ($\mathcal{C}$-4), $b \rightarrow a \neq \top$ and so $b \rightarrow a \notin [a, b] \Mapsto [a, b]$. This leads us to the following *general* theorem:

Theorem 4.1. *Let $\langle A, \rightarrow, \top \rangle$ be a BCI algebra. If there are $a, b \in A$ such that $a \neq b$ and $a \rightarrow b = \top$, then for any interval $\mathbb{T} \in \mathbb{A}$ there is no interval representation $\rightarrowtail$ for $\rightarrow$ such that $\langle \mathbb{A}, \rightarrowtail, \mathbb{T} \rangle$ be a BCI algebra.*

Proof. Case $\top \notin \mathbb{T}$. Then $a \rightarrow a = \top \notin \mathbb{T} = [a, a] \rightarrowtail [a, a]$ and therefore $\rightarrowtail$ is not an interval representation of $\rightarrow$.

Case $\mathbb{T} = [\top, \top]$, then $[a, b] \rightarrowtail [a, b] = \mathbb{T} = [\top, \top]$. Nevertheless, by ($\mathcal{C}$-4), $b \rightarrow a \neq \top$ and so $b \rightarrow a \notin [a, b] \rightarrowtail [a, b]$. Therefore, also in this case $\rightarrowtail$ is not an interval representation of $\rightarrow$.

Case $\mathbb{T} = [\alpha, \top]$ for some $\alpha < \top$. Then $\top \rightarrow \alpha \neq \top$. However, if $\langle \mathbb{A}, \rightarrowtail, \mathbb{T} \rangle$ is a BCI algebra then, by property (A-8), $\mathbb{T} \rightarrowtail [\top, \top] = [\top, \top]$ and therefore, $\top \rightarrow \alpha \notin [\alpha, \top] \rightarrowtail [\top, \top]$ which means that $\rightarrowtail$ again is not an interval representation of $\rightarrow$. □

In what follows, we introduce a process of intervalization of certain BCI algebras.

Theorem 4.2. *Let $\langle A, \rightarrow, \top \rangle$ be a BCI algebra such that $\langle A, \preceq \rangle$ is a meet-semilattice verifying: for each $x, y, z \in A$, $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$, and let $\mathbb{A} = \{[\underline{X}, \overline{X}] : \underline{X}, \overline{X} \in A$ and $\underline{X} \preceq \overline{X}\}$. For $X, Y \in \mathbb{A}$, define:*

1. $X \Rrightarrow Y = [\overline{X} \rightarrow \underline{Y}, \underline{X} \rightarrow \overline{Y}]$.
2. $X \Rightarrow Y = [(\underline{X} \rightarrow \underline{Y}) \wedge (\overline{X} \rightarrow \overline{Y}), \underline{X} \rightarrow \overline{Y}]$.

Then $\Rrightarrow$ is the best representation of $\rightarrow$ and the algebra $\langle \mathbb{A}, \Rrightarrow, \Rightarrow, [\top, \top] \rangle$ satisfies:

(IBCI1) $X \Rrightarrow (Y \Rrightarrow Z) = Y \Rrightarrow (X \Rrightarrow Z)$, *(EP)*
(IBCI2) $X \Rightarrow (Y \Rightarrow Z) = Y \Rightarrow (X \Rightarrow Z)$, *(EP)*
(IBCI3) $X \Rrightarrow Y \precsim (Z \Rrightarrow X) \Rightarrow (Z \Rrightarrow Y)$,
(IBCI4) $[\top, \top] \Rrightarrow X = X$, *(Left Neutrality)*
(IBCI5) $X \ll Y \precsim Z$ *implies* $X \ll Z$,
(IBCI6) $X \precsim Y \ll Z$ *implies* $X \ll Z$,
(IBCI7) $X \precsim Y$ *and* $Y \precsim X$ *imply* $X = Y$, *(Antisymmetry)*

where $X \ll Y$ *iff* $X \Rrightarrow Y = [\top, \top]$ *and* $X \precsim Y$ *iff* $X \Rightarrow Y = [\top, \top]$. *However, when A has at least one element different from* $\top$, *then* $\langle \mathbb{A}, \Rrightarrow, [\top, \top] \rangle$ *is not a BCI algebra.*

Proof. According to [2, Proposition 4.4] the operation $\Rrightarrow$ is the best representation of $\to$. Note that:

1. $X \ll Y \Leftrightarrow X \Rrightarrow Y = [\top, \top] \Leftrightarrow \overline{X} \to \underline{Y} = \top$ and $\underline{X} \to \overline{Y} = \top \Leftrightarrow \overline{X} \preceq \underline{Y}$ and $\underline{X} \preceq \overline{Y} \Leftrightarrow \overline{X} \preceq \underline{Y}$.
2. $X \precsim Y \Leftrightarrow X \Rightarrow Y = [\top, \top] \Leftrightarrow \underline{X} \to \underline{Y} = \top$ and $\overline{X} \to \overline{Y} = \top$ and $\underline{X} \to \overline{Y} = \top \Leftrightarrow \underline{X} \preceq \underline{Y}$ and $\overline{X} \preceq \overline{Y}$ and $\underline{X} \preceq \overline{Y} \Leftrightarrow \underline{X} \preceq \underline{Y}$ and $\overline{X} \preceq \overline{Y}$, i.e. $\precsim$ is the Kulisch-Miranker order.

Note that (IBCI7) is satisfied, since $\langle A, \preceq \rangle$ is a poset and "$\precsim$" is the Kulisch-Miranker order.

(IBCI1): $X \Rrightarrow (Y \Rrightarrow Z) = X \Rrightarrow [\overline{Y} \to \underline{Z}, \underline{Y} \to \overline{Z}] = [\overline{X} \to (\overline{Y} \to \underline{Z}), \underline{X} \to (\underline{Y} \to \overline{Z})]$. According to property (A-5) of BCI algebras this term is also equal to $[\overline{Y} \to (\overline{X} \to \underline{Z}), \underline{Y} \to (\underline{X} \to \overline{Z})] = Y \Rrightarrow [\overline{X} \to \underline{Z}, \underline{X} \to \overline{Z}] = Y \Rrightarrow (X \Rrightarrow Z)$.

(IBCI2): $X \Rightarrow (Y \Rightarrow Z) = X \Rightarrow [(\underline{Y} \to \underline{Z}) \wedge (\overline{Y} \to \overline{Z}), \underline{Y} \to \overline{Z}] = [(\underline{X} \to ((\underline{Y} \to \underline{Z}) \wedge (\overline{Y} \to \overline{Z}))) \wedge (\overline{X} \to (\underline{Y} \to \overline{Z})), \underline{X} \to (\underline{Y} \to \overline{Z})]$. On the other hand, $Y \Rightarrow (X \Rightarrow Z) = Y \Rightarrow [(\underline{X} \to \underline{Z}) \wedge (\overline{X} \to \overline{Z}), \underline{X} \to \overline{Z}] = [(\underline{Y} \to ((\underline{X} \to \underline{Z}) \wedge (\overline{X} \to \overline{Z}))) \wedge (\overline{Y} \to (\underline{X} \to \overline{Z})), \underline{Y} \to (\underline{X} \to \overline{Z})] =$ $[((\underline{Y} \to (\underline{X} \to \underline{Z})) \wedge (\underline{Y} \to (\overline{X} \to \overline{Z}))) \wedge (\overline{Y} \to (\underline{X} \to \overline{Z})), \underline{Y} \to (\underline{X} \to \overline{Z})]$. By property (A-5), the last term is equal to: $[((\underline{X} \to (\underline{Y} \to \underline{Z})) \wedge (\overline{X} \to (\underline{Y} \to \overline{Z}))) \wedge (\underline{X} \to (\overline{Y} \to \overline{Z})), \underline{X} \to (\underline{Y} \to \overline{Z})]$ which, by associativity and commutativity of meet, is equal to $[((\underline{X} \to (\underline{Y} \to \underline{Z})) \wedge (\underline{X} \to (\overline{Y} \to \overline{Z}))) \wedge (\overline{X} \to (\underline{Y} \to \overline{Z})), \underline{X} \to (\underline{Y} \to \overline{Z})] = [(\underline{X} \to ((\underline{Y} \to \underline{Z}) \wedge (\overline{Y} \to \overline{Z}))) \wedge (\overline{X} \to (\underline{Y} \to \overline{Z})), \underline{X} \to (\underline{Y} \to \overline{Z})] = X \Rightarrow (Y \Rightarrow Z)$.

(IBCI3): By definition, $X \Rrightarrow Y = [\overline{X} \to \underline{Y}, \underline{X} \to \overline{Y}]$ and $(Z \Rrightarrow X) \Rightarrow (Z \Rrightarrow Y) = [((\overline{Z} \to \underline{X}) \to (\overline{Z} \to \underline{Y})) \wedge ((\underline{Z} \to \overline{X}) \to (\underline{Z} \to \overline{Y})), (\overline{Z} \to \underline{X}) \to (\underline{Z} \to \overline{Y})]$. Since, by (A-2), (A-3) and (A-7), $\overline{X} \to \underline{Y} \preceq \underline{X} \to \underline{Y} \preceq (\overline{Z} \to \underline{X}) \to (\overline{Z} \to \underline{Y})$ and $\overline{X} \to \underline{Y} \preceq \overline{X} \to \overline{Y} \preceq (\underline{Z} \to \overline{X}) \to (\underline{Z} \to \overline{Y})$, then $\overline{X} \to \underline{Y} \preceq ((\overline{Z} \to \underline{X}) \to (\overline{Z} \to \underline{Y})) \wedge ((\underline{Z} \to \overline{X}) \to (\underline{Z} \to \overline{Y}))$. On the other hand, by (A-2), (A-3) and (A-7), $\underline{X} \to \overline{Y} \preceq (\overline{Z} \to \underline{X}) \to (\overline{Z} \to \overline{Y}) \preceq (\overline{Z} \to \underline{X}) \to (\underline{Z} \to \overline{Y})$. Therefore, $X \ggg Y \precsim (Z \ggg X) \Rightarrow (Z \ggg Y)$.

(IBCI4): Given (A-8) we see that $[\top, \top] \ggg [\underline{X}, \overline{X}] = [\top \to \underline{X}, \top \to \overline{X}] = [\underline{X}, \overline{X}]$.

(IBCI5): Note that $X \ll Y \precsim Z$ implies $\overline{X} \preceq \underline{Y} \preceq \underline{Z}$. Hence, $X \ll Z$.

(IBCI6): It is analogous to the latter.

For each BCI algebra $\langle A, \to, \top\rangle$ with at least an element $a \in A$ such that $a \neq \top$, the algebra $\langle \mathbb{A}, \ggg, [\top, \top]\rangle$ *is not a BCI algebra*. In fact, it fails to satisfy ($\mathcal{C}$-3), since $[a \wedge \top, \top] \ggg [a \wedge \top, \top] = [\top \to (a \wedge \top), (a \wedge \top) \to \top] = [a \wedge \top, (a \wedge \top) \to \top] \neq [\top, \top]$. □

Observe that $\ggg$ is the best interval representation of $\to$, but $\Rightarrow$ is not an interval representation of $\to$.

Definition 4.1. *Given a BCI(BCK) algebra $\langle A, \to, \top\rangle$ such that $\langle A, \preceq\rangle$ is a meet-semilattice verifying: $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, for any $x, y, z \in A$, the algebra obtained by the method used in Theorem 4.2, $\langle \mathbb{A}, \ggg, \Rightarrow, [\top, \top]\rangle$, is called* ***interval BCI(BCK) algebra****,* ***IBCI(IBCK) algebra****, for short.*

As we will see in the following results, the process of intervalization destroys some basic properties of BCI algebras, like **(OP)**, and some properties are generalized.

Theorem 4.3. *Given an IBCI(IBCK) algebra, $\mathbb{A}$, and $X, Y \in \mathbb{A}$, the following properties are satisfied:*

($\mathcal{G}$-1) $X \ggg X = [\overline{X} \to \underline{X}, \top]$.
($\mathcal{G}$-2) $X \ggg Y = [\top, \top]$ *iff* $\overline{X} \preceq \underline{Y}$.

Proof. ($\mathcal{G}$-1): Note that $X \ggg X = [\overline{X} \to \underline{X}, \underline{X} \to \overline{X}] \overset{(\mathcal{C}\text{-}5)}{=} [\overline{X} \to \underline{X}, \top]$.

($\mathcal{G}$-2): Note that $X \ggg Y = [\top, \top]$ iff $\overline{X} \to \underline{Y} = \top$ and $\underline{X} \to \overline{Y} = \top$ iff $\overline{X} \preceq \underline{Y}$. □

Corollary 4.1. $X \ggg X = [\top, \top]$ *iff X is degenerate.*

We conclude that the relation "$\ll$" corresponding to the operator "$\ggg$" will be reflexive (and hence a partial order) only if it is restricted to the subset of degenerate intervals of $\mathbb{A}$.

The operation "$\ggg$" (operation "$\Rightarrow$") is said to satisfy the **Weak Order Property (WOP)** whenever the relation "$\ll$" (relation "$\precsim$") fails to be a partial order.

The next proposition provides another situation in which an intervalized BCI algebra behaves like a BCI algebra.

Proposition 4.2. *Given an IBCI(IBCK) algebra, $\mathbb{A}$, for any $X, Y, Z \in \mathbb{A}$ and for any degenerate $U_d = [u, u]$, $V_d = [v, v]$ for $u, v \in A$, the following properties are satisfied:*

($\mathcal{C}_d$-1) $(Y \Rrightarrow Z) \precsim ((Z \Rrightarrow U_d) \Rrightarrow (Y \Rrightarrow U_d))$;
($\mathcal{C}_d$-2) $U_d \Rrightarrow ((U_d \Rrightarrow V_d) \Rrightarrow V_d) = [\top, \top]$;
($\mathcal{C}_d$-3) $X \Rrightarrow U_d = X \Rightarrow U_d$.

Proof. ($\mathcal{C}_d$-1): Since $(Y \Rrightarrow Z) = [\overline{Y} \to \underline{Z}, \underline{Y} \to \overline{Z}]$ and $((Z \Rrightarrow U_d) \Rrightarrow (Y \Rrightarrow U_d)) = [(\underline{Z} \to \overline{U_d}) \to (\overline{Y} \to \underline{U_d}), (\overline{Z} \to \underline{U_d}) \to (\underline{Y} \to \overline{U_d})] = [(\underline{Z} \to u) \to (\overline{Y} \to u), (\overline{Z} \to u) \to (\underline{Y} \to u)]$. By property ($\mathcal{C}$-1), $\overline{Y} \to \underline{Z} \preceq (\underline{Z} \to u) \to (\overline{Y} \to u)$ and $\underline{Y} \to \overline{Z} \preceq (\overline{Z} \to u) \to (\underline{Y} \to u)$. Therefore, $(Y \Rrightarrow Z) \precsim ((Z \Rrightarrow U_d) \Rrightarrow (Y \Rrightarrow U_d))$.

($\mathcal{C}_d$-2): $U_d \Rrightarrow ((U_d \Rrightarrow V_d) \Rrightarrow V_d) = U_d \Rrightarrow ([\overline{U_d} \to v, \underline{U_d} \to v] \Rrightarrow V_d) = U_d \Rrightarrow ([(\underline{U_d} \to v) \to v, (\overline{U_d} \to v) \to v]) = [\overline{U_d} \to ((\underline{U_d} \to v) \to v), \underline{U_d} \to ((\overline{U_d} \to v) \to v)] = [u \to ((u \to v) \to v), u \to ((u \to v) \to v)] = [\top, \top]$.

($\mathcal{C}_d$-3): Note that $X \Rrightarrow U_d = [\overline{X} \to u, \underline{X} \to u] = [(\underline{X} \to u) \wedge (\overline{X} \to u), \underline{X} \to u] = X \Rightarrow U_d$. □

In what follows a list of properties of IBCI algebras is provided.

Theorem 4.4. *An IBCI algebra has the following properties: For all $U_d = [u, u]$, $X, Y, Z \in \mathbb{A}$,*

(B-1) $[\top, \top] \precsim X$ *implies* $X = [\top, \top]$,
(B-2) $X \precsim Y$ *implies* $Y \Rrightarrow Z \precsim X \Rrightarrow Z$,
(B-3) $X \precsim Y$ *implies* $Z \Rrightarrow X \precsim Z \Rrightarrow Y$,
(B-4) $X \precsim Y$ *and* $Y \precsim Z$ *implies* $X \precsim Z$,
(B-5) $X \precsim Y \Rrightarrow U_d$ *implies* $Y \precsim X \Rrightarrow U_d$,
(B-6) $X \Rrightarrow Y \precsim (U_d \Rrightarrow X) \Rrightarrow (U_d \Rrightarrow Y)$,
(B-7) $((Y \Rrightarrow U_d) \Rrightarrow U_d) \Rrightarrow U_d = Y \Rrightarrow U_d$,
(B-8) $X \Rrightarrow Y \precsim (Y \Rrightarrow X) \Rrightarrow [\top, \top]$,
(B-9) $(X \Rrightarrow Y) \Rrightarrow [\top, \top] = (X \Rrightarrow [\top, \top]) \Rrightarrow (Y \Rrightarrow [\top, \top])$.

Proof.
(B-1) Suppose $[\top, \top] \precsim X$, then $\top \precsim \underline{X}$ and $\top \precsim \overline{X}$. By (A-1), $\underline{X} = \top$ and $\overline{X} = \top$. Therefore, $X = [\top, \top]$.
(B-2) Follows from the relation $\precsim$ and (A-2).
(B-3) Follows from the relation $\precsim$ and (A-3).

(B-4) Straightforward.

(B-5) Suppose $X \precsim Y \Rrightarrow U_d$, then $\underline{X} \preceq \overline{Y} \to u$ and $\overline{X} \preceq \underline{Y} \to u$. By (A-6) $\overline{Y} \preceq \underline{X} \to u$ and $\underline{Y} \preceq \overline{X} \to u$. Hence, $Y \precsim X \Rrightarrow U_d$.

(B-6) $(X \Rrightarrow Y) \precsim (U_d \Rrightarrow X) \Rrightarrow (U_d \Rrightarrow Y)$ iff $(X \Rrightarrow Y) \precsim [u \to \underline{X}, u \to \overline{X}] \Rrightarrow [u \to \underline{Y}, u \to \overline{Y}]$ iff $(X \Rrightarrow Y) \precsim [(u \to \overline{X}) \to (u \to \underline{Y}), (u \to \underline{X}) \to (u \to \overline{Y})]$ iff $[\overline{X} \to \underline{Y}, \underline{X} \to \overline{Y}] \precsim [(u \to \overline{X}) \to (u \to \underline{Y}), (u \to \underline{X}) \to (u \to \overline{Y})]$. By Property (A-7), $\overline{X} \to \underline{Y} \preceq (u \to \overline{X}) \to (u \to \underline{Y})$ and $\underline{X} \to \overline{Y} \preceq (u \to \underline{X}) \to (u \to \overline{Y})$.

(B-7) $((Y \Rrightarrow U_d) \Rrightarrow U_d) \Rrightarrow U_d = ([\overline{Y} \to \underline{U_d}, \underline{Y} \to \overline{U_d}] \Rrightarrow U_d) \Rrightarrow U_d = [(\underline{Y} \to u) \to u, (\overline{Y} \to u) \to u] \Rrightarrow U_d = [((\overline{Y} \to u) \to u) \to u, ((\underline{Y} \to u) \to u) \to u] \overset{(A-9)}{=} [\overline{Y} \to u, \underline{Y} \to u] = [\overline{Y} \to \underline{U_d}, \underline{Y} \to \overline{U_d}] = Y \Rrightarrow U_d$.

(B-8) Since $\overline{X} \to \underline{Y} \preceq (\underline{Y} \to \overline{X}) \to \top$ and $\underline{X} \to \overline{Y} \preceq (\overline{Y} \to \underline{X}) \to \top$, then by definition $X \Rrightarrow Y \precsim (Y \Rrightarrow X) \Rrightarrow [\top, \top]$.

(B-9) $(X \Rrightarrow [\top, \top]) \Rrightarrow (Y \Rrightarrow [\top, \top]) = [\overline{X} \to \top, \underline{X} \to \top] \Rrightarrow [\overline{Y} \to \top, \underline{Y} \to \top] = [(\underline{X} \to \top) \to (\overline{Y} \to \top), (\overline{X} \to \top) \to (\underline{Y} \to \top)] \overset{(A-11)}{=} [(\underline{X} \to \overline{Y}) \to \top, (\overline{X} \to \underline{Y}) \to \top] = [\overline{X} \to \underline{Y}, \underline{X} \to \overline{Y}] \Rrightarrow [\top, \top] = (X \Rrightarrow Y) \Rrightarrow [\top, \top]$. □

Proposition 4.3. *If $X \precsim Y$, then $X \Rrightarrow Y = [\overline{X} \to \underline{Y}, \top]$.*

Proof. Suppose $X \precsim Y$, then $\underline{X} \preceq \underline{Y} \preceq \overline{Y}$. Therefore $\underline{X} \to \overline{Y} = \top$ and hence $X \Rrightarrow Y = [\overline{X} \to \underline{Y}, \top]$. □

Theorem 4.5. *The properties:*

($\mathbf{OP}_{M_1}$) $X \ll Y$ *if and only if* $X \Rightarrow Y = [\top, \top]$ *and*
($\mathbf{OP}_{M_2}$) $X \precsim Y$ *if and only if* $X \Rrightarrow Y = [\top, \top]$

are not satisfied for all non-degenerate X and Y. However, the following holds for all X and Y:

($\mathbf{OP}_1$) *If* $X \ll Y$, *then* $X \Rightarrow Y = [\top, \top]$ *and*
($\mathbf{OP}_2$) *If* $X \Rrightarrow Y = [\top, \top]$, *then* $X \precsim Y$.

Proof. ($\mathbf{OP}_{\mathbf{M_1}}$) is not satisfied. In fact, if $X \Rightarrow Y = [\top, \top]$ then $\underline{X} \preceq \underline{Y}$ and $\overline{X} \preceq \overline{Y}$, and this does not mean that $\overline{X} \preceq \underline{Y}$.

($\mathbf{OP}_{\mathbf{M_2}}$) is also not satisfied, since for a given non-degenerate interval $X = [\underline{X}, \overline{X}]$, $X \precsim X$ and $X \Rrightarrow X = [\overline{X} \to \underline{X}, \underline{X} \to \overline{X}]$ is not necessarily equal to $[\top, \top]$.

($\mathbf{OP}_a$): Suppose $\overline{X} \preceq \underline{Y}$. Then $\underline{X} \preceq \overline{X} \preceq \underline{Y} \preceq \overline{Y}$ and therefore $X \Rightarrow Y = [\top, \top]$.

($\mathbf{OP}_b$): Suppose $X \Rrightarrow Y = [\top, \top]$. Then $[\overline{X} \to \underline{Y}, \underline{X} \to \overline{Y}] = [\top, \top]$; i.e. $\overline{X} \to \underline{Y} = \top$ and $\underline{X} \to \overline{Y} = \top$, so, $\overline{X} \preceq \underline{Y}$. Therefore $\underline{X} \preceq \underline{Y}$ and $\overline{X} \preceq \overline{Y}$. □

Proposition 4.4. *The operators "$\Rrightarrow$" and "$\Rightarrow$", defined in Theorem 4.2, map degenerate intervals to degenerate intervals.*

Proof. Straightforward, since $[u, u] \Rightarrow [v, v] = [u \to v, u \to v] = [w, w]$ and by ($\mathcal{C}_d$-3) $[u, u] \Rrightarrow [v, v] = [w, w]$. □

As we have seen, the mathematical structure that arises from the intervalization of a BCI algebra is a new mathematical structure. This structure will here be named 'semi-BCI algebra', and we will study it in detail in what follows.

5 SEMI-BCI ALGEBRAS

This paper showed that some implications do not satisfy the order property **(OP)** and the correct intervalization of algebras leads to more general structures. This section introduces a new algebra which aims to capture both situations.

Definition 5.1 (Semi-BCI algebra). *Given a set A endowed with two binary operations "$\twoheadrightarrow$" and "$\to$", and $\top \in A$, an algebra $\langle A, \twoheadrightarrow, \to, \top \rangle$ is called a **semi-BCI algebra**, or **SBCI algebra** for short, whenever for all $x, y, z \in A$,*

(SBCI1) $x \twoheadrightarrow (y \twoheadrightarrow z) = y \twoheadrightarrow (x \twoheadrightarrow z)$,
(SBCI2) $x \to (y \to z) = y \to (x \to z)$,
(SBCI3) $x \twoheadrightarrow y \preceq (z \twoheadrightarrow x) \to (z \twoheadrightarrow y)$,
(SBCI4) $\top \twoheadrightarrow x = x$,
(SBCI5) *if* $x \ll y \preceq z$ *then* $x \ll z$,
(SBCI6) *if* $x \preceq y \ll z$ *then* $x \ll z$,
(SBCI7) *if* $x \preceq y$ *and* $y \preceq x$ *then* $x = y$,

where $x \ll y \Leftrightarrow x \twoheadrightarrow y = \top$ *and* $x \preceq y \Leftrightarrow x \to y = \top$.

*An SBCI algebra that satisfies $x \ll \top$ for all $x \in A$ is called a **semi-BCK algebra**, or **SBCK algebra** for short.*

Example 5.1.

1. *Any IBCI algebra is a SBCI algebra.*
2. *Consider the following algebra:* $\mathbf{A} = \langle [\mathbf{0}, \mathbf{1}], \twoheadrightarrow_{\mathbf{R}}, \to_{\mathbf{LK}}, \mathbf{1} \rangle$, *where* $x \twoheadrightarrow_R y = 1 - x + xy$ *(see Reichenbach [22]), and* $x \to_{LK} y =$

$\min\{1, 1-x+y\}$. *Thus, since* $\to_{LK}$ *is the Łukasiewicz implication, then the corresponding relation* $\leq$ *is the usual order and* $x \ll y$ *if and only if* $x \twoheadrightarrow_R y = 1$ *if and only if* $x = 0$ *or* $y = 1$. *It is straightforward to check the satisfaction of (SBCI5)-(SBCI7).*

(SBCI1): $x \twoheadrightarrow_R (y \twoheadrightarrow_R z) = 1 - xy + xyz = y \twoheadrightarrow_R (x \twoheadrightarrow_R z)$.

(SBCI2): For $x \to_{LK} (y \to_{LK} z) = min(1, 1-x+min(1, 1-y+z))$ *and* $y \to_{LK} (x \to_{LK} z) = min(1, 1-y+min(1, 1-x+z))$, *consider the following cases:* ***(1)*** *If* $x \to_{LK} (y \to_{LK} z) = 1$, *then* $1-x+min(1, 1-y+z) \geq 1 \Leftrightarrow min(1, 1-y+z) \geq x \Leftrightarrow 1-y+z \geq x \Leftrightarrow 1-x+z \geq y \Leftrightarrow min(1, 1-x+z) \geq y \Leftrightarrow 1-y+min(1, 1-x+z) \geq 1 \Leftrightarrow y \to_{LK} (x \to_{LK} z) = 1$. ***(2)*** *If* $x \to_{LK} (y \to_{LK} z) \neq 1$, *then* $1-x+min(1, 1-y+z) < 1 \Leftrightarrow min(1, 1-y+z) < x \Leftrightarrow 1-y+z < x \Leftrightarrow 1-x+z < y \Leftrightarrow min(1, 1-x+z) < y \Leftrightarrow 1-y+min(1, 1-x+z) < 1$. *Therefore,* $y \to_{LK} (x \to_{LK} z) = 1-y+min(1, 1-x+z) = 1-y+1-x+z = 1-x+1-y+z = 1-x+min(1, 1-y+z) = x \to_{LK} (y \to_{LK} z)$.

(SBCI3): $x \twoheadrightarrow_R y = 1-x+xy$ *and* $(z \twoheadrightarrow_R x) \to_{LK} (z \twoheadrightarrow_R y) = \min\{1, 1-(1-z+zx)+(1-z+zy)\} = \min\{1, 1-zx+zy\}$. *Since* $1-x+xy \leq 1-zx+zy$ *then* $1-x+xy \leq_{LK} \min\{1, 1-zx+zy\}$ *and therefore* $(x \twoheadrightarrow_R y) \leq_{LK} ((z \twoheadrightarrow_R x) \to_{LK} (z \twoheadrightarrow_R y))$.

(SBCI4): $1 \twoheadrightarrow_R y = 1-1+1 \cdot y = y$.

Therefore $A = \langle [0,1], \twoheadrightarrow_R, \to_{LK}, 1\rangle$ *is an SBCI algebra.*

Proposition 5.1. *In an SBCI algebra* $\langle A, \twoheadrightarrow, \to, \top\rangle$ *the following hold:*

(SBCI8) *If* $x \ll y$ *and* $y \ll z$ *then* $x \ll z$,
(SBCI9) *If* $x \ll y$ *and* $y \ll x$ *then* $x = y$,
(SBCI10) $(y \twoheadrightarrow z) \preceq (z \twoheadrightarrow x) \to (y \twoheadrightarrow x)$,
(SBCI11) *If* $\top \preceq x$, *then* $x = \top$,
(SBCI12) $x \to x = \top$,
(SBCI13) *If* $x \ll y$ *then* $x \preceq y$,
(SBCI14) $x \twoheadrightarrow y \preceq x \to y$,
(SBCI15) $x \to ((x \to y) \to y) = \top$,
(SBCI16) *If* $x \ll y$ *then* $x \twoheadrightarrow ((x \twoheadrightarrow y) \twoheadrightarrow y) = \top$,
(SBCI17) *If* $x \ll y$, *then* $z \twoheadrightarrow x \preceq z \twoheadrightarrow y$,
(SBCI18) *If* $x \ll y$, *then* $y \twoheadrightarrow z \preceq x \twoheadrightarrow z$.

Proof. The proof of items (SBCI8)-(SBCI10) and (SBCI14)-(SBCI16) are straightforward.

(SBCI11): By (SBCI3), $(\top \twoheadrightarrow x) \to ((\top \twoheadrightarrow \top) \to (\top \twoheadrightarrow x)) = \top \overset{(SBCI4)}{\Rightarrow} x \to (\top \to x) = \top$, as $\top \to x = \top$, then $x \to \top = \top$. So, by (SBCI7), $x = \top$.

(SBCI12): By (SBCI10), $(\top \twoheadrightarrow \top) \to ((\top \twoheadrightarrow x) \to (\top \twoheadrightarrow x)) = \top \overset{(SBCI4)}{\Rightarrow} \top \to (x \to x) = \top \overset{(SBCI11)}{\Rightarrow} x \to x = \top$.

(SBCI13): By (SBCI3), $(x \twoheadrightarrow y) \to ((\top \twoheadrightarrow x) \to (\top \twoheadrightarrow y)) = \top$, but as $x \twoheadrightarrow y = \top$ then $\top \to ((\top \twoheadrightarrow x) \to (\top \twoheadrightarrow y)) = \top \overset{(SBCI4)}{\Rightarrow} \top \to (x \to y) = \top \overset{(SBCI11)}{\Rightarrow} x \to y = \top$, so $x \preceq y$.

(SBCI17): By (SBCI3), $(x \twoheadrightarrow y) \to ((z \twoheadrightarrow x) \to (z \twoheadrightarrow y)) = \top$, as $x \twoheadrightarrow y = \top$ then by (SBCI11), $(z \twoheadrightarrow x) \to (z \twoheadrightarrow y) = \top$, thus $(z \twoheadrightarrow x) \preceq (z \twoheadrightarrow y)$.

(SBCI18): By (SBCI10), $(x \twoheadrightarrow y) \to ((y \twoheadrightarrow z) \to (x \twoheadrightarrow z)) = \top$, as $x \twoheadrightarrow y = \top$ then by (SBCI11), $(y \twoheadrightarrow z) \to (x \twoheadrightarrow z) = \top$, thus $(y \twoheadrightarrow z) \preceq (x \twoheadrightarrow z)$. □

We conclude from (SBCI8) and (SBCI9) that the relation "$\ll$" is transitive and antisymmetric, but it is not necessarily reflexive, whereas the relation "$\preceq$" is antisymmetric and reflexive by (SBCI7) and (SBCI12) respectively, but it is not necessarily transitive. The following proposition provides a condition for them to be partial orders.

Proposition 5.2. *The relation "$\ll$" coincides with "$\preceq$" if and only if "$\ll$" is reflexive.*

Proof. Suppose $\ll$ is reflexive, then $x \preceq y$ implies $x \ll x \preceq y$, by (SBCI5) $x \ll y$. The remainder of the proof is straightforward. □

Example 5.2. *The algebra $A = \langle [0,1], \twoheadrightarrow_{GD}, \to_{FD}, 1\rangle$ is a SBCI algebra, where:*

$$x \twoheadrightarrow_{GD} y = \begin{cases} 1, \text{ if } x \leq y \\ y, \text{ if } x > y \end{cases}$$

and

$$x \to_{FD} y = \begin{cases} 1, \text{ if } x \leq y \\ max(1-x, y), \text{ if } x > y \end{cases}$$

and the following relations coincide: (1) $x \ll y$ if and only if $x \twoheadrightarrow_{GD} y = 1$ if and only if $x \leq y$ and (2) $x \preceq y$ if and only if $x \to_{FD} y = 1$ if and only if $x \leq y$. In fact, "$\ll$" is reflexive and according to Baczyński et al [1][p.10, Table 1.4], $\twoheadrightarrow_{GD}$ satisfies (SBCI1) and $\to_{FD}$ satisfies (SBCI2). (SBCI3) is also satisfied. In fact, if $x \leq y$, then $x \twoheadrightarrow_{GD} y = 1$ and therefore, if (i) $z \leq x \leq y$, $z \twoheadrightarrow_{GD} x = z \twoheadrightarrow_{GD} y = 1$; (ii) $x < z \leq y$, $z \twoheadrightarrow_{GD} x = x$

and $z \twoheadrightarrow_{GD} y = 1$*; (iii)* $x \leq y < z$, $z \twoheadrightarrow_{GD} x = x$ *and* $z \twoheadrightarrow_{GD} y = y$*. Now, if* $x > y$*, then* $x \twoheadrightarrow_{GD} y = y$ *and therefore, if (i)* $z \leq y < x$, $z \twoheadrightarrow_{GD} x = z \twoheadrightarrow_{GD} y = 1$*; (ii)* $y < z \leq x$, $z \twoheadrightarrow_{GD} x = 1$ *and* $z \twoheadrightarrow_{GD} y = y$*; (iii)* $y < x \leq z$, $z \twoheadrightarrow_{GD} x = x$ *and* $z \twoheadrightarrow_{GD} y = y$*. In any case,* $x \twoheadrightarrow_{GD} y \leq (z \twoheadrightarrow_{GD} x) \rightarrow_{FD} (z \twoheadrightarrow_{GD} y)$*. The remaining axioms of SBCI algebras are straightforward to check.*

Proposition 5.3. *Let* $\langle A, \rightarrow_1, \top\rangle$ *and* $\langle A, \rightarrow_2, \top\rangle$ *be BCI algebras with* $\leq_1$ *and* $\leq_2$ *as their respective partial orders. If* $\leq_2$ *contains* $\leq_1$ *and* $x \rightarrow_1 y \leq_2 x \rightarrow_2 y$ *and*

$$w \leq_2 x \leq_1 y \leq_2 z \text{ implies } w \leq_1 z \tag{6}$$

for each $w, x, y, z \in A$*, then* $\langle A, \rightarrow_1, \rightarrow_2, \top\rangle$ *is an SBCI algebra. The same applies to BCK algebras and SBCK algebras.*

Proof.

(SBCI1): Straightforward from (A-5).

(SBCI2): Straightforward from (A-5).

(SBCI3): By ($\mathcal{C}$-1), $(z \rightarrow_1 x) \rightarrow_1 ((x \rightarrow_1 y) \rightarrow_1 (z \rightarrow_1 y)) = \top$. So, by (A-5), $(x \rightarrow_1 y) \rightarrow_1 ((z \rightarrow_1 x) \rightarrow_1 (z \rightarrow_1 y)) = \top$, i.e. $(x \rightarrow_1 y) \leq_1 ((z \rightarrow_1 x) \rightarrow_1 (z \rightarrow_1 y))$. Thus, since $\leq_2$ contains $\leq_1$ and $\rightarrow_1 \leq_2 \rightarrow_2$, then $(x \rightarrow_1 y) \leq_2 ((z \rightarrow_1 x) \rightarrow_2 (z \rightarrow_1 y))$.

(SBCI4): Straightforward from (A-8).

(SBCI5): Straightforward from (6) by taking $w = x$.

(SBCI6): Straightforward from (6) by taking $y = z$.

(SBCI7): Straightforward from (A-4). □

Corollary 5.1. *If* $\langle A, \rightarrow, \top\rangle$ *is a BCI algebra, then* $\langle A, \rightarrow, \rightarrow, \top\rangle$ *is an SBCI algebra.*

Proof. Straightforward from Prop. 5.3, as (6) holds in view of (A-4). □

Remark. The same applies to BCK algebras and SBCK algebras.

Proposition 5.4. *Consider the SBCI algebra of the form* $\langle A, \rightarrow, \rightarrow, \top\rangle$*. Then the reduct* $\langle A, \rightarrow, \top\rangle$ *is a BCI algebra.*

Proof. To check (C-1), note that by (SBCI3), $z \rightarrow x \preceq (y \rightarrow z) \rightarrow (y \rightarrow x)$, i.e. $(z \rightarrow x) \rightarrow ((y \rightarrow z) \rightarrow (y \rightarrow x)) = \top$. Therefore, by (SBCI1), $(y \rightarrow z) \rightarrow ((z \rightarrow x) \rightarrow (y \rightarrow x)) = \top$. To check (C-2), note that by (SBCI1) and

(SBCI12), $x \rightarrow ((x \rightarrow y) \rightarrow y) = (x \rightarrow y) \rightarrow (x \rightarrow y) = \top$. (C-3) follows by (SBCI12), and (C-4) by (SBCI7). □

By Corollary 5.1 and Proposition 5.4, we obtain:

Theorem 5.1. $(A, \rightarrow, \top)$ *is a BCI algebra iff* $(A, \rightarrow, \rightarrow, \top)$ *is an SBCI algebra.*

Proposition 5.5. *There are SBCI algebras* $\langle A, \twoheadrightarrow, \rightarrow, \top\rangle$ *such that the reduct* $\langle A, \twoheadrightarrow, \top\rangle$ *is not BCI algebra.*

Proof. Consider the SBCI algebra $A = \langle [0, 1], \twoheadrightarrow_R, \rightarrow_{LK}, 1\rangle$ given in Example 5.1. The reduct $\langle [0, 1], \twoheadrightarrow_R, 1\rangle$ is not a BCI algebra, since $x \twoheadrightarrow_R x = 1 - x + x^2 \neq 1$, for all $x \in (0, 1)$. Therefore, ($\mathcal{C}$-3) does not hold. □

Remark. There are algebras $A = (A, \rightarrow, \top)$ in which the exchange principle (EP) is satisfied, but A is not a BCI algebra. Indeed, there are the BCH algebras, the BZ algebras, the CI algebras etc. (see for example [13, 14].)

6 COMPARING SEMI-BCI AND PSEUDO-BCI ALGEBRAS

The generalization of BCI/BCK algebras is not new. In fact, G. Georgescu and A. Iorgulescu [8] proposed a generalization of BCK algebras, and later W. A. Dudek and Y. B. Jun [5] proposed a generalization of BCI algebras. The first was called **pseudo-BCK algebras** and the second was called **pseudo-BCI algebras**. Like we did for SBCI algebras these authors proposed two operations that generalize the primitive operation of BCK/BCI algebras. Therefore a natural question arises: Are SBCI algebras just a new presentation for those other algebras? This section shows that the answer to this question is negative and shows how these algebras are related to one another.

Definition 6.1 ([6]). *A **pseudo-BCI algebra**, or **PBCI algebra** for short, is a structure* $\langle A, \leq, \rightarrow, \rightsquigarrow, \top\rangle$ *such that "*$\leq$*" is a binary relation on the set* A*, "*$\rightarrow$*" and "*$\rightsquigarrow$*" are binary operations on* A*,* $\top \in A$ *and for all* $x, y, z \in A$*:*

(PB-1) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$,
(PB-2) $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$,
(PB-3) $x \leq (x \rightarrow y) \rightsquigarrow y$,
(PB-4) $x \leq (x \rightsquigarrow y) \rightarrow y$,
(PB-5) $x \leq x$,
(PB-6) *if* $x \leq y$ *and* $y \leq x$*, then* $x = y$,
(PB-7) $x \leq y \Leftrightarrow x \rightarrow y = \top \Leftrightarrow x \rightsquigarrow y = \top$.

Example 6.1. *The structure* $\mathcal{A} = \langle \mathbb{R}^2, \preceq, \twoheadrightarrow, \rightarrow, (0,0)\rangle$, *where* $(x_1, y_1) \twoheadrightarrow (x_2, y_2) = (x_2 - x_1, (y_2 - y_1)e^{-x_1})$ *and* $(x_1, y_1) \rightarrow (x_2, y_2) = (x_2 - x_1, y_2 - y_1 e^{x_2 - x_1})$, *is a PBCI algebra.*

The next proposition shows that PBCI algebras do not model the intervalization of BCI algebras.

Proposition 6.1. *Let* $\langle A, \rightarrow, \top\rangle$ *be a BCI algebra such that* $\langle A, \preceq\rangle$ *is a meet-semilattice and for each* $x, y, z \in A$, $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$. *Then, the algebra* $\langle \mathbb{A}, \Rrightarrow, \Rightarrow, [\top, \top]\rangle$, *where* $\Rrightarrow$ *and* $\Rightarrow$ *are defined in Theorem 4.2, is not a PBCI algebra.*

Proof. By (PB-5) and (PB-7), for all $X \in \mathbb{A}$, $X \Rrightarrow X = [\top, \top]$ should hold, however by Corollary 4.1 this only applies if X is degenerate. $\square$

Given a PBCI algebra, if the relations: (a) $x \leq_1 y \Leftrightarrow x \rightarrow y = \top$ and (b) $x \leq_2 y \Leftrightarrow x \rightsquigarrow y = \top$ are defined, then the axiom (PB-7) imposes that they must coincide. In the case of SBCI algebras the relations $\ll$ and $\preceq$ does not necessarily coincide. Moreover, even if they coincide there are SBCI algebras which are not PBCI algebras — see Proposition 6.2. Finally, there are SBCI algebras in which the relation "$\ll$" can be irreflexive refuting the axiom (PB-5) (see Proposition 5.5). Therefore, this leads us to conclude that SBCI and PBCI algebras are different structures.

Proposition 6.2. *There are SBCI algebras* $\langle A, \twoheadrightarrow, \rightarrow, \top\rangle$ *whose relations "*$\ll$*" and "*$\preceq$*" coincide but that are not PBCI algebras.*

Proof. The SBCI algebra $\mathcal{A} = \langle [0,1], \twoheadrightarrow_{GD}, \rightarrow_{FD}, 1\rangle$ provided at Example 5.2 is not a PBCI algebra. In fact, take $x = \dfrac{3}{4}$, $y = \dfrac{1}{2}$ and $z = \dfrac{1}{5}$, then $x \rightarrow_{FD} y = \dfrac{1}{2}$ and $(y \rightarrow_{FD} z) \twoheadrightarrow_{GD} (x \rightarrow_{FD} z) = \dfrac{1}{4}$, but since the relations "$\ll$" and "$\preceq$" coincide with the usual order, (PB-2) is not satisfied. $\square$

Proposition 6.3. *There are PBCI algebras* $\langle A, \preceq, \twoheadrightarrow, \rightarrow, \top\rangle$ *which are not SBCI algebras.*

Proof. The PBCI algebra $\mathcal{A} = \langle \mathbb{R}^2, \preceq, \twoheadrightarrow, \rightarrow, (0,0)\rangle$ presented in Example 6.1 is not a SBCI algebra. In fact, take $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ such that

$y_2(e^{x_1} - 1) \neq y_1(e^{x_2} - 1)$ and any $(x_3, y_3) \in \mathbb{R}^2$. Note that:

$$\begin{array}{ll}(x_1, y_1) \twoheadrightarrow ((x_2, y_2) \twoheadrightarrow (x_3, y_3)) & \\ \quad = & (x_1, y_1) \twoheadrightarrow (x_3 - x_2, y_3 - y_2 e^{x_3 - x_2}) \\ \quad = & ((x_3 - x_2) - x_1, (y_3 - y_2 e^{x_3 - x_2}) - y_1 e^{(x_3 - x_2) - x_1}) \\ \quad = & (x_3 - x_2 - x_1, y_3 - y_2 e^{x_3 - x_2} - y_1 e^{x_3 - x_2 - x_1})\end{array}$$

and

$$\begin{array}{ll}(x_2, y_2) \twoheadrightarrow ((x_1, y_1) \twoheadrightarrow (x_3, y_3)) & \\ \quad = & (x_2, y_2) \twoheadrightarrow (x_3 - x_1, y_3 - y_1 e^{x_3 - x_1}) \\ \quad = & ((x_3 - x_1) - x_2, (y_3 - y_1 e^{x_3 - x_1}) - y_2 e^{(x_3 - x_1) - x_2}) \\ \quad = & (x_3 - x_2 - x_1, y_3 - y_1 e^{x_3 - x_1} - y_2 e^{x_3 - x_2 - x_1}).\end{array}$$

Since $y_2(e^{x_1} - 1) \neq y_1(e^{x_2} - 1)$, then $y_3 - y_2 e^{x_3 - x_2} - y_1 e^{x_3 - x_2 - x_1} \neq y_3 - y_1 e^{x_3 - x_1} - y_2 e^{x_3 - x_2 - x_1}$, and so we conclude that $\mathcal{A}$ does not satisfy the axiom (SBCI1). □

The next proposition ensures that the intersection between the class of PBCI algebras and the class of SBCI algebras contains only BCI algebras.

Proposition 6.4. *Let $\mathcal{A} = \langle A, \twoheadrightarrow, \rightarrow, \top\rangle$ be a SBCI algebra such that the relations: "$\ll$" and "$\preceq$" correspond to the operations: "$\twoheadrightarrow$" and "$\rightarrow$", respectively. If both $\langle A, \ll, \twoheadrightarrow, \rightarrow, \top\rangle$ and $\langle A, \preceq, \twoheadrightarrow, \rightarrow, \top\rangle$ are PBCI algebras, then $\langle A, \rightarrow, \top\rangle$ is a BCI algebra.*

Proof. Since both $\langle A, \ll, \twoheadrightarrow, \rightarrow, \top\rangle$ and $\langle A, \preceq, \twoheadrightarrow, \rightarrow, \top\rangle$ are PBCI algebras, then the relations "$\preceq$" and "$\ll$" coincide. Now, by (PB-4),

$$\begin{array}{rcl}x \preceq (x \rightarrow y) \twoheadrightarrow y & \Rightarrow & x \ll (x \rightarrow y) \twoheadrightarrow y \\ & \Rightarrow & x \twoheadrightarrow ((x \rightarrow y) \twoheadrightarrow y) = \top \\ & \overset{(SBCI1)}{\Rightarrow} & (x \rightarrow y) \twoheadrightarrow (x \twoheadrightarrow y) = \top \\ & \Rightarrow & (x \rightarrow y) \ll (x \twoheadrightarrow y) \\ & \Rightarrow & (x \rightarrow y) \preceq (x \twoheadrightarrow y).\end{array}$$

Therefore, since $x \twoheadrightarrow y \preceq x \rightarrow y$ (SBCI14) for all $x, y \in A$ then, from axiom (SBCI7), follows $x \twoheadrightarrow y = x \rightarrow y$, for all $x, y \in A$. We conclude by Proposition 5.4 that $\langle A, \rightarrow, \top\rangle$ is a BCI algebra. □

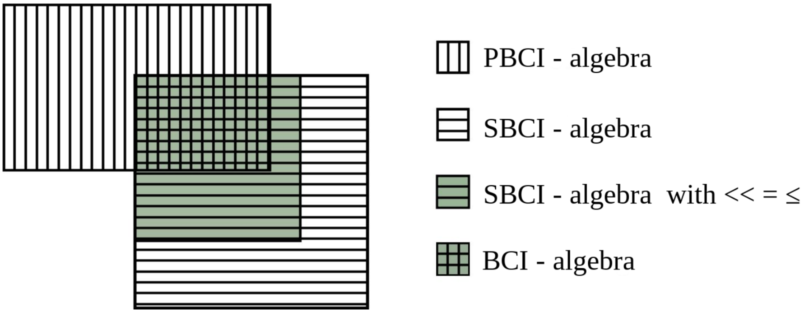

FIGURE 1
Relation between SBCI and PBCI algebras

Figure 1 shows how PBCI and SBCI algebras are related.

7 FINAL REMARKS

This paper proposes a new algebra which generalizes the notion of BCI algebra. It is an algebra which captures the most important properties of a Fuzzy Implication after it has been intervalized in a correct way. The resulting algebras, called **semi-BCI algebras**, capture the properties of the algebras which arise from the intervalization of BCI algebras. In other words, as BCI algebras generalize the Łukasiewicz algebras, the SBCI algebras generalize the respective intervalization of the Łukasiewicz algebras. The paper also discusses the relation of such algebras to PBCI algebras.

As future research, the authors aim to investigate more closely the new algebra hereby introduced and its connection with interval-valued fuzzy logic. Entities like filters, ideals, category and others are to be studied in this context.

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REFERENCES

[1] Baczyński, M. and Jayaram, B. (2008). Fuzzy Implications. Springer, Berlin.

[2] Bedregal, B. C. and Santiago, R. H. N. (2013). Interval representations, Łukasiewicz implicators and Smets-Magrez axioms. *Information Sciences*, 221:192–200.

[3] Bedregal, B. and Santiago, R. (2013). Some continuity notions for interval functions and representation. *Computational and Applied Mathematics*, 32(3):435–446.

[4] Cabrer, L. M. and Mundici, D. (2014). Interval MV-algebras and generalizations. *International Journal of Approximate Reasoning*, 55(8):1623–1642.

[5] Dudek, W. A. and Jun, Y. B. (2008). Pseudo-BCI algebras. *East Asian Mathematical Journal*, 24:187–190.

[6] Dymek, G. (2013). On the category of pseudo-BCI-algebras. Demonstratio Mathematica. Warsaw Technical University Institute of Mathematics, Vol. 46, nr 4:631–644.

[7] Forsythe, G. E. (1970). Pitfalls in computation, or why a math book isn't enough. Technical report, Stanford University, Stanford, CA, USA.

[8] Georgescu, G. and Iorgulescu, A. (2001). Pseudo-BCK algebras: An extension of BCK algebras. In C. S. Calude, M. J. Dinneen, and S. Sburlan, editors, Combinatorics, Computability and Logic: Proceedings of the Third International Conference on Combinatorics, Computability and Logic, (DMTCS'01), pages 97–114, London. Springer London.

[9] Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. and Scott, D. S. (2003). Continuous Lattices and Domains. Encyclopedia of Mathematics and its Applications. Cambridge University Press.

[10] Hickey, T., Ju, Q. and Emdem, M. (2001). Interval arithmetic: from principles to implementation. *Journal of the ACM*, 48(5):1038–1068.

[11] Hindley, J. R. and Seldin, J. P. (1986). Introduction to Combinators and Lambda-Calculus. Cambridge University Press.

[12] Huang, Y. (2006). BCI-Algebra. Science Press, Beijing.

[13] Iorgulescu, A. (2016). New generalizations of BCI, BCK and Hilbert algebras. *J. of Mult.-Valued Logic and Soft Computing*, Vol. 27, 4: 353–406.

[14] Iorgulescu, A. (2016). New generalizations of BCI, BCK and Hilbert algebras - Part II. *J. of Mult.-Valued Logic and Soft Computing*, Vol. 27, 4: 407–456.

[15] Iséki, K. (1966). An algebra related with a propositional calculus. *Proc. Japan Acad.*, 42(1):26–29.

[16] Markov, S. (1977). A non-standard subtraction of intervals. *Serdica*, 3:359–370.

[17] Moore, R. E. (1959). Automatic error analysis in digital computation. Technical Report Space Div. Report LMSD84821, Lockheed Missiles and Space Co., Sunnyvale, CA, USA.

[18] Moore, R. E. (1962). Interval Arithmetic and Automatic Error Analysis in Digital Computing. Ph.D. dissertation, Department of Mathematics, Stanford University, Stanford, CA, USA. Also published as Applied Mathematics and Statistics Laboratories Technical Report No. 25.

[19] Moore, R.E. (1979). Methods and Applications of Interval Analysis. SIAM, Philadelphia.

[20] Pinheiro, J., Bedregal, B., Santiago, R. and Santos, H. (2018). A study of (T,N)-implications and its use to construct a new class of fuzzy subsethood measure. *International Journal of Approximate Reasoning*, 97:1–16

[21] Rasiowa, H. (1974). An Algebraic Approach to Non-classical Logics. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Company.

[22] Reichenbach, H. (1935). Wahrscheinlichkeitslogik. Erkenntnis, 5(1):37–43.

[23] Rump, S. M. (1988). Algorithms for verified inclusions: Theory and practice. In Ramon E. Moore, editor, Reliability in Computing, pages 109–126. Academic Press.

[24] Santiago, R. H. N., Bedregal, B. C. and Acióly, B. M. (2006). Formal aspects of correctness and optimality in interval computations. *Formal Aspects of Computing*, 18(2):231–243.

[25] Sunaga, T. (2009). Theory of an interval algebra and its application to numerical analysis [reprint of res. assoc. appl. geom. mem. 2 (1958), 29–46]. *Japan J. Indust. Appl. Math.*, 26(2-3):125–143.

[26] Yager, R. R. (1980). An approach to inference in approximate reasoning. *International Journal of Man-Machine Studies*, 13(3):323–338.