Fuzzy Bi-implications Generated by t-norms and Fuzzy Negations

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Abstract. In the literature, there are several forms of extensions of the classical bi-implication for the fuzzy logic, as for example, the axiomatization proposed by Fodor and Roubens [1]. Another way to obtain a generalization is to provide a definition based on the classical equivalence $\phi \iff \psi \equiv (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$, in which the classical operators of conjunction and implication are replaced, respectively, by a t-norm (T) and a fuzzy implication (I). In this paper, we investigate a particular class of fuzzy bi-implications $B(x, y) = T(I(x, y), I(y, x))$, in which I is a fuzzy (T, N)-implication introduced by Bedregal [2]. We study several properties satisfied by (T, N)-bi-implications, such as the sufficient conditions that they must satisfy in order to be a f-bi-implication.

1 Introduction

Since the introduction of fuzzy set theory [3], where the crisp membership functions valued in $\{0, 1\}$ were generalized to allow degrees of membership valued in [0,1], the investigation of fuzzy logic began as a family of multivalued logics, referred by Petr Hájek as fuzzy logic in a narrow sense [4, p. 2], which is the object of investigation of the mathematical fuzzy logic community. What differentiates fuzzy logics in a narrow sense from other multivalued logics, is that the former has both truth-functionality and truth degrees in [0,1] as fundamental assumptions (see [5]).

Several generalizations of the classical boolean connectives to the fuzzy setting have been introduced and studied. In particular the classical conjunction was extended in fuzzy logic by the triangular norms (see for instance [6–9]), the disjunction by the triangular conorms (see for instance [6–9]), the negation by the fuzzy negation [10] and the implication by the fuzzy implication (see for instance [11]). All these together have been used in several applications, for example the fuzzy implications have been useful to implement automated decision support systems with "if-then" rules, where depending on the context a suitable fuzzy implication is selected to implement such rules (see for instance [12]).

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These fuzzy operators have been used as truth-functional interpretations of formulas over the unit interval. For instance, Hájek [4] proposed a class of logics, called Basic Logics (BL), based on continuous triangular norms and their residua, were further to be extended to logics based on left-continuous triangular norms [5].

In classical logic the binary operator that semantically is true only when the truth-value of its operands are equal, is called equivalence or bi-implication [13, p. 7] or biconditional when the implication is called conditional [14, p. 70]. Since equivalences are reflexive, symmetric and transitive relations [13, p. 22] and the class of operators introduced in this paper are generated by two occurrences of an implication we prefer the name fuzzy bi-implication, just as in [6, p. 235].

In the literature there is not a consensus upon what a fuzzy bi-implication should be and one may find it under the names of T-indistinguishability operator [15, p. 18], fuzzy bi-implication [16,17], fuzzy equality [18], fuzzy bi-residuation [19], fuzzy equivalence [1,20], T-equivalence [21], fuzzy similarity [4, p. 123] and restricted equivalence function [22].

The first steps made in order to study the relations in between several of these extensions and to provide a few novel extensions, were made in [23–25]. In this paper we propose a novel class of fuzzy bi-implications obtained by the composition of a (T, N)-implication [2,26–28] and a fuzzy negation. We study several of its properties and, determine the sufficient conditions for such a fuzzy bi-implication to constitute a sub-class of the well-known axiomatization proposed by Fodor and Roubens in [1].

Several among these fuzzy bi-implications, and, in particular, those proposed and studied in this paper, can be applied, for example, for image comparison (see for instance [29]) as well as, be used as a truth-functional interpretation of formulas with occurrences of bi-implications, just as the Goedel logic has the conjunction interpreted as the minimum t-norm, probably in other fuzzy logics the bi-implication could be interpreted as a particular fuzzy bi-implication in between those proposed in this paper.

This paper is organized in the following manner: in Sect. 2 we provide the basic concepts needed in order to make this paper self-contained; in Sect. 3 we propose the class of (T, N)-bi-implications, study several of its properties and relate it with the Fodor-Roubens axiomatization; finally in Sect. 4 we provide some conclusions and propose directions for future works.

2 Preliminaries

In this part of the paper, we present some important preliminary notions for the development of this work.

2.1 t-Norms and Fuzzy Negations

In the literature, there are several operators that extend the classical conjunction for fuzzy logic, as for example the t-norms defined below:

Definition 1 **([6] p. 4).** *A function $T : [0,1]^2 \to [0,1]$ is a* **t-norm** *if for all $x, y, z \in [0,1]$, the following axioms are satisfied:*

1. *(T1) Commutativity: $T(x,y) = T(y,x)$;*
2. *(T2) Associativity: $T(x, T(y,z)) = T(T(x,y), z)$;*
3. *(T3) Monotonicity: $T(x,y) \leq T(x,z)$, whenever $y \leq z$;*
4. *(T4) 1-ident: $T(x,1) = x$.*

Some classical examples of t-norms are:

Example 1 **([6] p. 4).**

1. *The minimum $T_{\mathbf{M}}(x,y) = min(x,y)$;*
2. *The product $T_{\mathbf{P}}(x,y) = x \cdot y$;*
3. *The drastic product* $T_{\mathbf{D}}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1)^2 \\ min(x,y), & \text{otherwise} \end{cases}$

t-norms satisfy the following properties:

Proposition 1 **([6] pp. 5–6).** *If $T : [0,1]^2 \to [0,1]$ is a t-norm, then:*

1. *$T(0,x) = T(x,0) = 0$, for all $x \in [0,1]$;*
2. *$T(1,x) = x$, for all $x \in [0,1]$;*
3. *$T(x_1, y_1) \leq T(x_2, y_2)$, whenever $x_1 \leq x_2$ and $y_1 \leq y_2$;*
4. *$T_{\mathbf{D}}(x,y) \leq T(x,y) \leq T_{\mathbf{M}}(x,y)$, for all $x, y \in [0,1]$;*

The following Remark is a direct consequence of Definition 1 and Proposition 1.

Remark 1.

1. *$T(x,y) = 1 \iff x = 1$ and $y = 1$, for all $x, y \in [0,1]$;*
2. *$T(x,x) = 1 \Rightarrow x = 1$.*

Another important connective of fuzzy logic is the negation, as below:

Definition 2 **([11] pp. 13–14 and [30]).**

1. *A* **fuzzy negation** *is a non-increasing function $N : [0,1] \to [0,1]$ such that $N(1) = 0$ and $N(0) = 1$;*
2. *When a fuzzy negation is involutive, i.e., satisfies the property $N(N(x)) = x$ for each $x \in [0,1]$, we say that N is a* **strong fuzzy negation***;*
3. *A fuzzy negation is* **strict** *if it is continuous and for each $x, y \in [0,1]$ satisfies $N(x) > N(y)$ whenever $x < y$;*
4. *A fuzzy negation is* **non-filling** *if $N(x) = 1 \iff x = 0$;*
5. *A fuzzy negation is* **crisp** *if $N(x) \in \{0,1\}$ for any $x \in [0,1]$.*

Remark 2. *It is relevant for this work to emphasize that every strong fuzzy negation is strict [11, p. 15].*

In the example below we present some fuzzy negations.

Example 2 ([11] pp. 14–15).

1. *The Zadeh's negation* $N_C(x) = 1 - x$ *is strong and non-filling, but is not crisp;*
2. $N_{\mathbf{R}}(x) = 1 - \sqrt{x}$ *is strict, non-filling, but is not crisp and strong;*
3. *The threshold* $N^t(x) = \begin{cases} 1, & \textit{if } x < t \\ 1 \textit{ or } 0, & \textit{if } x = t, t \in (0,1) \\ 0, & \textit{if } x > t \end{cases}$ *is crisp, but is neither strict nor non-filling.*

2.2 Fuzzy Implications

The notion of implication in Fuzzy Logic has many non equivalent extensions. In this paper we are going to use the next one.

Definition 3 ([11] p. 2). *A* **fuzzy implication** *is a binary operator* $I : [0,1]^2 \to [0,1]$ *that satisfies:*

(I1) $I(x_1, y) \geq I(x_2, y)$, *whenever* $x_1, x_2, y \in [0,1]$ *and* $x_1 \leq x_2$*;*
(I2) $I(x, y_1) \leq I(x, y_2)$, *whenever* $x, y_1, y_2 \in [0,1]$ *and* $y_1 \leq y_2$*;*
(I3) $I(0,0) = 1$*;*
(I4) $I(1,1) = 1$*;*
(I5) $I(1,0) = 0$.

Properties $(I1)$ and $(I2)$ are called *first place antitonicity* and *second place isotonicity*, respectively The remaining properties together with

(I6) $I(0,1) = 1$,

are called boundary conditions. The boundary conditions guarantee that the class of fuzzy implications extend the classical implication. The next result has $(I6)$ as a particular case. Hence, $(I6)$ is unnecessary in the Definition 3.

Proposition 2 ([11] p. 2). *If* $I : [0,1]^2 \to [0,1]$ *is a fuzzy implication, then*

$$I(x, 1) = 1, \textit{ for all } x \in [0,1]$$

The properties considered in the next two definitions were stated for fuzzy implications in [11, pp. 9,20] and [24], but we will study them for fuzzy bi-implications, which will be presented in Sect. 3.

Definition 4. *A fuzzy operator* $F : [0,1]^2 \to [0,1]$ *is said to satisfy the property of:*

(LNP) **left neutrality** *if:*

$$F(1, y) = y, \textit{ for all } y \in [0,1]$$

(IP) **identity** *if:*

$$F(x, x) = 1, \textit{ for all } x \in [0,1]$$

(LOP) **left-ordering** *if:*

$$F(x, y) = 1, \text{ whenever } x \leq y$$

Definition 5. *Let $N : [0, 1] \to [0, 1]$ be a fuzzy negation. A fuzzy operator $F : [0, 1] \to [0, 1]$ is said to satisfy:*

(CP) the **contraposition law** *with respect to N, if:*

$$F(x, y) = F(N(y), N(x)), \text{ for all } x, y \in [0, 1]$$

(LCP) the **left contraposition law** *with respect to N, if:*

$$F(N(x), y) = F(N(y), x), \text{ for all } x, y \in [0, 1]$$

(RCP) the **right contraposition law** *with respect to N, if:*

$$F(x, N(y)) = F(y, N(x)), \text{ for all } x, y \in [0, 1]$$

If F satisfies the contraposition law (or left contraposition or right contraposition) with respect to N, then we denote these properties by $CP(N)$ (respectively, by $LCP(N)$ or $RCP(N)$).

2.2.1 (T,N)-implications

In [2], the author introduced a class of fuzzy implications obtained by the defining standard based on the classical equivalence $\phi \Rightarrow \psi \equiv \neg(\phi \wedge \neg\psi)$

Definition 6 ([2]). *Let $T : [0, 1]^2 \to [0, 1]$ be a t-norm and $N : [0, 1] \to [0, 1]$ be a fuzzy negation. The function defined by:*

$$I_T^N(x, y) = N(T(x, N(y))), \text{ for every } x, y \in [0, 1]$$

is called N-dual fuzzy implication of T.

In [26–28], the authors called the N-dual fuzzy implications simply as (T, N)-implications and studied properties of these fuzzy operators. For example, they studied the conditions for (T, N)-implications to satisfy (EP), (CP), (LCP), (RCP) and (LNP). For this work, it is important to mention the following result:

Proposition 3 ([27]). *Let $T : [0, 1]^2 \to [0, 1]$ be a t-norm. If $N : [0, 1] \to [0, 1]$ is a crisp fuzzy negation, then I_T^N satisfies (LOP).*

2.3 Fuzzy Bi-implications

In what follows we show an axiomatization of a class of fuzzy bi-implications proposed by Fodor and Roubens.

Definition 7 ([1] p. 33). *A function $B : [0, 1]^2 \to [0, 1]$ is called* **f-bi-implication** *if it satisfies the following axioms:*

(B1) $B(x, y) = B(y, x)$, *for all* $x, y \in [0, 1]$; *(commutativity)*
(B2) $B(0, 1) = B(1, 0) = 0$; *(boundary condition)*
(B3) $B(x, x) = 1$, *for all* $x \in [0, 1]$; *(identity principle)*
(B4) $B(x, y) \leq B(x', y')$, *whenever* $x \leq x' \leq y' \leq y$.

It is important to note that other two further boundary conditions that an extension of the classical bi-implication needs to satisfy are $B(0, 0) = B(1, 1) = 1$ which are immediate consequences of $(B3)$.

Example 3. *Examples of f-bi-implications are:*

1. $B_{\mathbf{M}}(x, y) = \begin{cases} 1, & \text{if } x = y \\ min(x, y), & \text{otherwise} \end{cases}$, *that satisfies* (LNP), (IP), *but does not satisfy* (LOP);
2. $B_{\mathbf{P}}(x, y) = \begin{cases} 1, & \text{if } x = y \\ \frac{min(x,y)}{max(x,y)}, & \text{otherwise} \end{cases}$, *that satisfies* (LNP), (IP), *but does not satisfy* (LOP);
3. $B_{\mathbf{KP}}(x, y) = 1 - max(x^2, y^2) + xy$, *that satisfies* (IP), *but does not satisfy* (LNP) *and* (LOP).

Note that, by $(B2)$ their does not exist a f-bi-implication that satisfies (LOP).

3 Fuzzy Bi-implications Generated by t-Norms and Fuzzy Negations

In this section, we investigate a special type of fuzzy bi-implications obtained by the defining standard based on the classical logical equivalence $\phi \iff \psi \equiv (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$ Considering the t-norms and fuzzy implications as a generalizations, respectively, of the classical conjunction and classical implication, we have the following function $B(x, y) = T(I(x, y), I(y, x))$, where T is a t-norm and I is a fuzzy implication. In this paper, we study the case in which I is a (T, N)-implication.

Definition 8. *Let* $N : [0, 1] \to [0, 1]$ *be a fuzzy negation and* $T : [0, 1]^2 \to [0, 1]$ *be a t-norm. The* N *dual fuzzy bi-implication of* T *(or fuzzy* (T, N)*-bi-implication or simply* (T, N)*-bi-implication) is a function* $B_T^N : [0, 1]^2 \to [0, 1]$ *of the form:*

$$\begin{aligned} B_T^N(x, y) &= T(I_T^N(x, y), I_T^N(y, x)) \\ &= T(N(T(x, N(y))), N(T(y, N(x)))) \end{aligned}$$

Example 4. *Let* $T_{\mathbf{M}}$ *and* N_C *be the operators defined, respectively, in Examples 1 and 2, then:*

$$\begin{aligned} B_{T_{\mathbf{M}}}^{N_C}(x, y) &= min(1 - min(x, 1 - y), 1 - min(y, 1 - x)) \\ &= min(max(1 - x, y), max(1 - y, x)) \end{aligned}$$

Note that $B_{T_{\mathbf{M}}}^{N_C}$ *is not a f-bi-implication, since it does not satisfy* $(B3)$, *for example,* $B_{T_{\mathbf{M}}}^{N_C}(0.7, 0.7) = 0.7$. *Thus,* (T, N)*-bi-implications fail to constitute a subclass of the class of f-bi-implications.*

3.1 Properties of (T, N)-bi-implications

In this subsection we investigate some properties of (T, N)-bi-implications. In the next result we show that there exist an unique fuzzy negation that generates a given (T, N)-bi-implication.

Proposition 4. *Let $T : [0,1]^2 \to [0,1]$ be a t-norm and $N : [0,1] \to [0,1]$ be a fuzzy negation. Then,*

$$B_T^N(x, 0) = N(x), \text{ for any } x \in [0,1]$$

Proof. Just note that:

$$\begin{aligned} B_T^N(x,0) &= T(N(T(x, N(0))), N(T(0, N(x))) \\ &= T(N(T(x,1)), N(0)) \quad - \quad \text{by 1 of Proposition 1} \\ &= T(N(x), 1) \quad - \quad \text{by (T4) and Definition 2} \\ &= N(x) \quad - \quad \text{by (T4)} \end{aligned}$$

Corollary 1. *If $T : [0,1]^2 \to [0,1]$ is a t-norm and $N : [0,1] \to [0,1]$ is a fuzzy negation, then $B_T^N(\cdot, 0) : [0,1] \to [0,1]$ is a fuzzy negation.*

A natural question is, does there exist an unique t-norm that generates a given (T, N)-bi-implication. For a (T, N)-bi-implication generated by strong fuzzy negations, the following Proposition proves a positive answer for this question.

Proposition 5. *If N is a strong fuzzy negation and T is t-norm, then it does not exist a t-norm $T' \neq T$ such that $B_T^N = B_{T'}^N$.*

Proof. Just note that, for any strong negation N and $x, y \in [0,1]$

$$\begin{aligned} I(x,y) = N(T(x, N(y))) &\Rightarrow N(I(x,y)) = T(x, N(y)) \\ &\Rightarrow T(x,y) = N(I(x, N(y))) \end{aligned}$$

Note that when a (T, N)-bi-implication is generated by a strong fuzzy negation, both the fuzzy negation and the t-norm, are unique. The next two results are immediately obtained by the Definition 8 and the Corollary 1.

Proposition 6. *If $T : [0,1]^2 \to [0,1]$ is a t-norm and $N : [0,1] \to [0,1]$ is a fuzzy negation, then B_T^N satisfies $(B1)$.*

Corollary 2. *If $T : [0,1]^2 \to [0,1]$ is a t-norm and $N : [0,1] \to [0,1]$ is a fuzzy negation, then $B_T^N(0, \cdot) : [0,1] \to [0,1]$ is a fuzzy negation.*

Since there is a unique fuzzy negation that generates a (T, N)-bi-implication, the fuzzy negation of Corollaries 1 and 2 coincide. In the following results we investigate the sufficient conditions for a (T, N)-bi-implication to be a f-bi-implication.

Proposition 7. *If $T : [0,1]^2 \to [0,1]$ is a t-norm and $N : [0,1] \to [0,1]$ is a fuzzy negation, then B_T^N satisfies $(B2)$.*

Proof. Follows from $(B1)$ and by the equality $B_T^N(1,0) = N(1) = 0$ of Proposition 4.

Proposition 8. *If N is a non-filling fuzzy negation, then B_T^N satisfies $(B3)$ if, and only if, the pair (T, N) satisfies the law of non-contradiction.*[1]

Proof. Since,

$$\begin{aligned} B_T^N(x,x) = 1 &\iff T(N(T(x,N(x))), N(T(x,N(x)))) = 1 \ - \text{ by Definition 8} \\ &\iff N(T(x,N(x))) = 1 \qquad\qquad\qquad - \text{ by Remark 1} \\ &\iff T(x,N(x)) = 0 \qquad\qquad\qquad\quad - \text{ by non-filling condition} \end{aligned}$$

Corollary 3. *If N is a strict fuzzy negation, then B_T^N satisfies $(B3)$ if, and only if, the pair (T, N) satisfies the law of non-contradiction.*

Proof. It follows from the fact that all strict fuzzy negation are injective hence, non-filling.

Example 5. *If $T = T_{\mathbf{P}}$ and $N_\perp(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$, then (T, N) satisfies the law of non-contradiction, $N_\perp$ is a non-filling fuzzy negation that fails to be strict and $B_{T_{\mathbf{P}}}^{N_\perp}$ satisfies $(B3)$, by Proposition 7.*

Theorem 1. *Let $T : [0,1]^2 \to [0,1]$ be a t-norm and $N : [0,1] \to [0,1]$ be a fuzzy negation. If I_T^N satisfies (LOP), then B_T^N satisfies $(B4)$.*

Proof. First let's see that:

$$B_T^N(x,y) = \begin{cases} N(T(x,N(y))), & \text{if } x \geq y \\ N(T(y,N(x))), & \text{if } x \leq y \end{cases} \tag{1}$$

Indeed, as for all $x, y \in [0,1]$ we have

$$x \leq y \text{ or } y \leq x$$

Then, by (LOP) of I_T^N, we have

$$I_T^N(x,y) = 1 \text{ or } I_T^N(y,x) = 1$$

Thus, the equality

$$\begin{aligned} B_T^N(x,y) &= T(N(T(x,N(y))), N(T(y,N(x)))) \\ &= T(I_T^N(x,y), I_T^N(y,x)) \end{aligned}$$

[1] If $T : [0,1]^2 \to [0,1]$ is a t-norm and $N : [0,1] \to [0,1]$ is a fuzzy negation, then we say that the pair (T, N) satisfies the law of non-contradiction if $T(x, N(x)) = 0$, for all $x \in [0,1]$ (this law is equivalently stated in [11, p. 55]).

ensures the Eq. (1) is valid.

Now, given $x \leq x' \leq y' \leq y$, then

$$B_T^N(x,y) = N(T(y,N(x))) \text{ and } B_T^N(x',y') = N(T(y',N(x')))$$

Therefore, by the monotonicity conditions of T and N, we conclude that

$$B_T^N(x,y) \leq B_T^N(x',y')$$

Corollary 4. *If $T : [0,1]^2 \rightarrow [0,1]$ is a t-norm and $N : [0,1] \rightarrow [0,1]$ is a non-filling fuzzy negation such that the pair (T, N) satisfies the law of non-contradiction and I_T^N satisfies (LOP), then B_T^N is a f-bi-implication.*

Proof. Follows from Propositions 5, 6 and 7, and Theorem 1.

Corollary 5. *If N is a crisp fuzzy negation and T is a t-norm, then B_T^N satisfies $(B4)$.*

Proof. By Proposition 2, I_T^N satisfies (LOP). Therefore, by Theorem 1, B_T^N satisfies $(B4)$.

There is only one crisp fuzzy negation that is non-filling, that is $N_\perp$, which in turn also satisfies the law of non-contradiction with any t-norm. Consequently, by Corollaries 4 and 5, every (T, N)-bi-implication generated by $N_\perp$ and a t-norm is a f-bi-implication.

There are (T, N)-implications that fail to satisfy (LOP). For example, for $T_\mathbf{M}$ and N_C we have that $I_{T_\mathbf{M}}^{N_C}(0.3, 0.5) = 0.7$. But in the next result the (T, N)-bi-implication generated by $T_\mathbf{M}$ and a strong fuzzy negation satisfies $(B4)$.

Theorem 2. *If $T = T_\mathbf{M}$ and N is a strong fuzzy negation, then $B_{T_\mathbf{M}}^N$ satisfies $(B4)$.*

Proof. For any $x, y \in [0,1]$, either $x \leq N(y)$ or $x \geq N(y)$. In addition, as N is a strong fuzzy negation:

$$x \leq N(y) \iff y \leq N(x)$$

and

$$x \geq N(y) \iff y \geq N(x)$$

If $x \leq N(y)$, then $y \leq N(x)$ and so,

$$\begin{aligned} B_{T_\mathbf{M}}^N(x,y) &= T_\mathbf{M}(N(T_\mathbf{M}(x,N(y))), N(T_\mathbf{M}(y,N(x)))) \\ &= min(N(min(x,N(y))), N(min(y,N(x)))) \\ &= min(N(x), N(y)) \end{aligned}$$

If $x \geq N(y)$, then $y \geq N(x)$ and so

$$\begin{aligned} B_{T_\mathbf{M}}^N(x,y) &= T_\mathbf{M}(N(T_\mathbf{M}(x,N(y))), N(T_\mathbf{M}(y,N(x)))) \\ &= min(N(min(x,N(y))), N(min(y,N(x)))) \\ &= min(N(N(y)), N(N(x))) \\ &= min(x,y) \qquad - \quad \text{because } N \text{ is strong} \end{aligned}$$

Thus,

$$B^N_{T_{\mathbf{M}}}(x,y) = \begin{cases} min(N(x), N(y)), & \text{if } x \le N(y) \\ min(x,y), & \text{if } x \ge N(y) \end{cases}$$

Let $x \le x' \le y' \le y$. Then,

- for $x \le N(y)$, we have
$$B^N_{T_{\mathbf{M}}}(x,y) = N(y)$$
Thus, if $x' \le N(y')$, then $B^N_{T_{\mathbf{M}}}(x',y') = N(y') \ge N(y) = B^N_{T_{\mathbf{M}}}(x,y)$. If $x' \ge N(y')$, then $B^N_{T_{\mathbf{M}}}(x',y') = x' \ge N(y') \ge N(y) = B^N_{T_{\mathbf{M}}}(x,y)$. So, $B^N_{T_{\mathbf{M}}}(x,y) \le B^N_{T_{\mathbf{M}}}(x',y')$.
- for $x \ge N(y)$, we have
$$B^N_{T_{\mathbf{M}}}(x,y) = x$$
Thus, if $x' \le N(y')$, then $B^N_{T_{\mathbf{M}}}(x',y') = N(y') \ge x' \ge x = B^N_{T_{\mathbf{M}}}(x,y)$. If $x' \ge N(y')$, then $B^N_{T_{\mathbf{M}}}(x',y') = x' \ge x = B^N_{T_{\mathbf{M}}}(x,y)$. So, $B^N_{T_{\mathbf{M}}}(x,y) \le B^N_{T_{\mathbf{M}}}(x',y')$.

Therefore, $B^N_{T_{\mathbf{M}}}$ satisfies $(B4)$.

Even though $B^N_{T_{\mathbf{M}}}$ of Theorem 2 satisfies $(B4)$ it is worthy to mention that the pair $(T_{\mathbf{M}}, N)$, where N is a strong fuzzy negation, does not satisfy the law of non-contradiction, because $min(x, N(x)) = 0 \iff x = 0$ or $N(x) = 0$. Hence $B^N_{T_{\mathbf{M}}}$ fails to be a f-implication. The next Propositions show sufficient conditions for a (T,N)-bi-implication to satisfy (LNP), $LCP(N)$, $RCP(N)$ and $CP(N)$).

Proposition 9. *Let $T : [0,1]^2 \to [0,1]$ be a t-norm and $N : [0,1] \to [0,1]$ be a fuzzy negation. If N is strong, then B^N_T satisfies (LNP).*

Proof. For (LNP) we have:

$$\begin{aligned} B^N_T(1,x) &= T(N(T(1,N(x))), N(T(x,N(1))) && - \text{ by Definition 8} \\ &= T(N(N(x)), N(T(x,0))) && - \text{ by (T1) and (T4)} \\ &= T(x, N(0)) && - \text{ by Proposition 1 and because } N \text{ is strong} \\ &= x && - \text{ by Definition 2 and (T4)} \end{aligned}$$

Proposition 10. *If N is a strong fuzzy negation and T is a t-norm, then B^N_T satisfies $LCP(N), RCP(N)$ and $CP(N)$).*

Proof. For $LCP(N)$, just see that for all $x, y \in [0,1]$:

$$\begin{aligned} B^N_T(N(x),y) &= T(N(T(N(x),N(y))), N(T(y,N(N(x))))) \\ &= T(N(T(N(x),N(y))), N(T(y,x))) \\ &= T(N(T(N(y),N(x))), N(T(x,N(N(y))))) \\ &= B^N_T(N(y),x) \end{aligned}$$

The property $RCP(N)$ follows from Proposition 9 and $(B1)$, and for $CP(N)$, just note that for all $x, y \in [0,1]$:

$$\begin{aligned} B_T^N(N(x), N(y)) &= T(N(T(N(x), N(N(y)))), N(T(N(y), N(N(x))))) \\ &= T(N(T(N(x), y)), N(T(N(y), x))) \\ &= T(N(T(x, N(y))), N(T(y, N(x)))) \\ &= B_T^N(x, y) \end{aligned}$$

4 Conclusions and Future Works

In this paper, we introduce a new class of binary operators that extend the classical bi-implications, called (T, N)-bi-implications. We show that the class of (T, N)-bi-implications is not contained in the class of f-bi-implications and that these two classes of functions have a non-empty intersection. We also obtain sufficient conditions for a (T, N)-bi-implication to be a f-bi-implication. Some open questions are: if the class of f-bi-implications is a subclass of the class of (T, N)-bi-implications and investigate other properties satisfied by the (T, N)-bi-implications, as the exchange principle.

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