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# Nelson's logic $\mathcal{S}$

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## Abstract

Besides the better-known Nelson logic ( $\mathcal{N}3$ ) and paraconsistent Nelson logic ( $\mathcal{N}4$ ), in 1959 David Nelson introduced, with motivations of realizability and constructibility, a logic called  $\mathcal{S}$ . The logic  $\mathcal{S}$  was originally presented by means of a calculus (crucially lacking the contraction rule) with infinitely many rule schemata and no semantics (other than the intended interpretation into Arithmetic). We look here at the propositional fragment of  $\mathcal{S}$ , showing that it is algebraizable (in fact, implicative), in the sense of Blok and Pigozzi, with respect to a variety of three-potent involutive residuated lattices. We thus introduce the first known algebraic semantics for  $\mathcal{S}$  as well as a finite Hilbert-style calculus equivalent to Nelson's presentation; this also allows us to clarify the relation between  $\mathcal{S}$  and the other two Nelson logics  $\mathcal{N}3$  and  $\mathcal{N}4$ .

*Keywords:* Nelson's logics, constructive logics, strong negation, paraconsistent Nelson logic, substructural logics, three-potent residuated lattices, algebraic logic.

## 1 Introduction

In the course of his extensive investigations into the notion of 'constructible falsity', David Nelson introduced a number of systems of non-classical logics that have aroused considerable interest in the logic and algebraic logic community (see, e.g. [24] and the references cited therein). Over the years, the main goal of Nelson's enterprise was to provide logical formalisms that allow for more fine-grained analyses of notions such as 'falsity' and 'negation' than either classical or intuitionistic logic can afford.

Nelson's analysis of the meaning of 'falsity' is in many ways analogous to the intuitionistic analysis of 'truth'. The main property advocated by Nelson—namely, if a formula  $\sim(\phi \wedge \psi)$  is provable, then either  $\sim\phi$  or  $\sim\psi$  is provable—is one that may be regarded as a dual to the well-known disjunction property of intuitionistic logic. In later investigations, just as the intuitionists

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argued against the usual object language formulation of the principle of excluded middle,  $\phi \vee \sim\phi$ , so Nelson was led to introduce logical systems that reject certain object language formulations of the principle of explosion (*ex contradictione quodlibet*). The resulting logics thus combine an intuitionistic approach to truth with a dual-intuitionistic treatment of falsity, not unlike the one of the so-called bi-intuitionistic logic [27, 28].

The systems in the family nowadays known as Nelson's logics share many properties with the positive fragment of intuitionistic logic (in particular, they do not validate Peirce's law  $((\phi \Rightarrow \psi) \Rightarrow \phi) \Rightarrow \phi$ ). They also possess a negation connective with inconsistency-tolerant features, in the sense that formulas such as  $(\phi \wedge \sim\phi) \Rightarrow \psi$  need not be valid.

The oldest and most well-known of Nelson's systems was introduced in [22] and is today known simply as *Nelson logic* (following [24], we shall denote it by  $\mathcal{N}3$ ). This logic is by now well understood from a proof-theoretic (see, e.g. [20]) as well as an algebraic point of view [36, 37], both perspectives allowing us to regard  $\mathcal{N}3$  as a substructural logic in the sense of [15].

*Paraconsistent Nelson logic*  $\mathcal{N}4$  is a weakening of  $\mathcal{N}3$  introduced in [4] (also independently considered in [18] and [33]) as, precisely, a non-explosive version of  $\mathcal{N}3$  suited for dealing with inexact predicates. Our understanding of the proof-theoretic as well as the algebraic properties of  $\mathcal{N}4$  is more recent and still not thorough. However, thanks to recent results of Spinks and Veroff,  $\mathcal{N}4$  can now be viewed as a member of the family of relevance logics; indeed,  $\mathcal{N}4$  can be presented, to within definitional equivalence, as an axiomatic strengthening of the contraction-free relevant logic  $\mathcal{RW}$  (see [38] for a summary of this work).

Thanks mainly to the works of Odintsov (see e.g. [24], although the result has been formally stated for the first time only in [30]), we also know that  $\mathcal{N}4$  (like  $\mathcal{N}3$ ) is algebraizable in the sense of Blok and Pigozzi. This means that the consequence relation of  $\mathcal{N}4$  can be completely characterized in terms of the equational consequence of the corresponding algebraic semantics, which consists in a variety of algebras called  $\mathcal{N}4$ -lattices.

For our purposes, these algebraic completeness results entail in particular that we can compare both  $\mathcal{N}3$  and  $\mathcal{N}4$  to the logic  $\mathcal{S}$ —the main object of the present paper—by looking at the corresponding classes of algebras. Before we turn our attention to  $\mathcal{S}$ , let us dwell on another remarkable feature shared by  $\mathcal{N}3$  and  $\mathcal{N}4$ .

For the propositional part (on which we shall exclusively focus in this paper), the language of both  $\mathcal{N}3$  and  $\mathcal{N}4$  comprises a conjunction ( $\wedge$ ), a disjunction ( $\vee$ ), a so-called 'strong' negation ( $\sim$ ) and two implications: a so-called 'weak' ( $\rightarrow$ ) and a 'strong' one ( $\Rightarrow$ ), usually introduced via the following term:  $\phi \Rightarrow \psi := (\phi \rightarrow \psi) \wedge (\sim\psi \rightarrow \sim\phi)$ . The presence of *two* implications is crucial in Nelson's logics: it is this feature that makes, one may argue, the Nelson formalism more fine-grained than classical or intuitionistic logic (or most many-valued logics, for that matter). With the two Nelson implications at hand, one is able to register finer shades of logical discrimination than it is possible in logics that are more 'classically' oriented in nature (see Humberstone [17] for a general discussion of this issue).

In fact, different classical or intuitionistic tautologies may be proved within Nelson's logics using either  $\rightarrow$  or  $\Rightarrow$ , creating a non-trivial interplay between these two implications and with the negation connective; the strong implication exhibits an inconsistency-tolerant behaviour, in that  $(p \wedge \sim p) \Rightarrow q$  is not provable, while the other retains a more 'classical' flavour, in that  $(p \wedge \sim p) \rightarrow q$  turns out to be provable. On the other hand, while the weak implication ( $\rightarrow$ ) allows us to see  $\mathcal{N}3$  and  $\mathcal{N}4$  as conservative expansions of positive intuitionistic logic by a negation connective with certain classical features (De Morgan, involutive laws), the strong implication ( $\Rightarrow$ ) permits us to view their algebraic counterparts as residuated structures and therefore to regard  $\mathcal{N}3$  and  $\mathcal{N}4$  as strengthenings (as a matter of fact, axiomatic ones) of well-known substructural or relevance logics.

We note, in passing, that the overall picture is made more interesting and complex by the fact that other meaningful non-primitive connectives can be defined—for example, an ‘intuitionistic’ negation (distinct from the primitive ‘strong’ negation  $\sim$ ) given by  $\phi \rightarrow 0$  or a ‘multiplicative’ monoidal conjunction given by  $\sim(\phi \Rightarrow \sim\psi)$ —and by the fact that interdefinability results hold even among the primitive connectives (some of these being highly non-trivial to prove). We shall not enter into further details concerning this issue for this is not the main focus of the present paper; instead, we will now turn our attention to the logic  $\mathcal{S}$ , which has so far remained least well-known among the members of the Nelson family.

The logic that we (following Nelson’s original terminology) call  $\mathcal{S}$  was introduced in [23] with essentially the same motivations as  $\mathcal{N}3$ : that is, as a more flexible tool for the analysis of falsity, and in particular as an alternative to both  $\mathcal{N}3$  and intuitionistic logic for interpreting Arithmetic through realizability. The propositional language of  $\mathcal{S}$  comprises a conjunction, a disjunction, a falsity constant and (just one) implication. Whether this implication ought to be regarded as a ‘strong’ or a ‘weak’ one will become apparent as a result of the investigations in the present paper.

Nelson’s presentation of  $\mathcal{S}$  is given by means of a calculus that appears peculiar, to the modern eye, in several respects. It may look like a sequent calculus but it is not. One could say that it is in fact a Hilbert-style calculus, though one with few axioms and many rules—infinitely many, in fact: not just instances, but infinitely many rule schemata. No standard semantics is provided in [23] for the calculus other than the intended interpretation of its (first-order) formulas as arithmetic predicates.

The above features may in part explain why  $\mathcal{S}$  has received, to the present day, very little attention in comparison to the other two Nelson’s logics: to the point that, to the best of our knowledge, even the most basic questions about  $\mathcal{S}$  had not yet been asked, let alone answered. One could start by asking, e.g. whether  $\mathcal{S}$  does admit a finite axiomatization. Another basic issue, which is interestingly obscured by Nelson’s presentations of  $\mathcal{S}$  and  $\mathcal{N}3$  in [23], is whether one of these two logics is stronger than the other or else whether they are incomparable. Last but not least, Nelson observes that certain formulas are not provable in his system [23, p. 213]. In the absence of a complete semantics for  $\mathcal{S}$ , it does not seem obvious how one could prove such claims. Having established our algebraic completeness result, however, this will become quite straightforward.

The main motivation for the present paper has been to look at the above questions and, more generally—taking advantage of the modern tools of algebraic logic—to gain a better insight into (the propositional part of)  $\mathcal{S}$  and into its relation to other well-known non-classical logics. As we shall see in the following sections, we have successfully settled all the above-mentioned issues, and the corresponding answers can be summarized as follows.

First of all,  $\mathcal{S}$  may indeed be axiomatized by means of a finite Hilbert-style calculus (having *modus ponens* as its only rule schema) that is a strengthening of the contraction-free fragment of intuitionistic logic and also of the substructural logic known as the Full Lambek Calculus with Exchange and Weakening ( $\mathcal{FL}_{ew}$ ). This follows from our main result that  $\mathcal{S}$  is Blok–Pigozzi algebraizable (and therefore, enjoys a strong completeness theorem) with respect to a certain class of residuated lattices, which are the canonical algebras associated with substructural logics stronger than  $\mathcal{FL}_{ew}$ . Furthermore, we may now say that the implication of  $\mathcal{S}$  is indeed a ‘strong’ Nelson implication in the sense that it can be meaningfully compared with the corresponding strong implications of  $\mathcal{N}3$  and  $\mathcal{N}4$ . From this vantage point, we will see that Nelson’s logic  $\mathcal{N}3$  may be regarded as an axiomatic strengthening of  $\mathcal{S}$ , whereas  $\mathcal{N}4$  is incomparable with  $\mathcal{S}$ . And finally, we will confirm that Nelson was correct in claiming that the formulas listed in [23, p. 213] are actually not provable in  $\mathcal{S}$ .

The paper is organized as follows. In Section 2, we introduce the logic  $\mathcal{S}$  through Nelson’s original presentation (duly amending a number of obvious typos) and employ it to prove a few

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formulas that will be useful in the following sections. In Section 3, we prove that Nelson's calculus is algebraizable, and provide an axiomatization of the corresponding class of algebras, which we call  $\mathcal{S}$ -algebras (Subsection 3.1). Because of the above-mentioned peculiar features of Nelson's calculus, the presentation of  $\mathcal{S}$ -algebras obtained algorithmically via the algebraization process is not very convenient. We introduce then an alternative equational presentation in Subsection 3.2 and show the equivalence of the two. As a result of our own presentation, we establish that  $\mathcal{S}$  is a strengthening of  $\mathcal{FL}_{ew}$ . Taking advantage of this insight, in Section 4 we introduce a finite Hilbert-style calculus for  $\mathcal{S}$  that is simply an axiomatic strengthening of a well-known calculus for  $\mathcal{FL}_{ew}$ . Completeness of our axiomatization, and therefore equivalence with Nelson's calculus, is obtained as a corollary of the algebraizability results. In Section 5, we look at concrete  $\mathcal{S}$ -algebras that provide counterexamples for the formulas Nelson claimed to be unprovable within  $\mathcal{S}$ . We present in Subsection 5.2 an easy way of building an  $\mathcal{S}$ -algebra starting from a residuated lattice, which turns out to be useful later on (Section 7). Section 6 establishes the relation between  $\mathcal{S}$  and the two other Nelson's logics,  $\mathcal{N}3$  (Subsection 6.1) and  $\mathcal{N}4$  (Subsection 6.2). We show in particular that both  $\mathcal{N}3$  and the three-valued Łukasiewicz logic (but no other logic in the Łukasiewicz family) may be seen as axiomatic strengthenings of  $\mathcal{S}$ . In Section 7, we use the algebraic insights gained so far to obtain information on the cardinality of the strengthenings of  $\mathcal{S}$ . Finally, Section 8 contains suggestions for future work.

The present paper is an expanded and improved version of [21], to which we shall refer whenever doing so allows us to omit or shorten our proofs. Let us highlight the main differences and present novelties. From Section 2 to Theorem 4.4 of Section 4, we follow essentially Sections 2–4 of [21]. The remaining part of Section 4 (dealing with EDPC and WBSO varieties) is new, as is Section 5. In particular, Subsection 5.1 contains a proof of the claim made in [21] that the Distributivity axiom (as well as the other formulas mentioned in Proposition 5.2) is not valid in  $\mathcal{S}$ . The usage of the Galatos–Raftery doubling construction (in both Section 5 and Section 6) is entirely new. Subsection 6.1 is essentially an expanded version of Section 5.1 from [21]; on the other hand, the results from Proposition 6.5 to the end of Section 6 are new. Section 7 is also entirely new.

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In this section, we recall Nelson's original presentation of the propositional fragment of  $\mathcal{S}$ , modulo the correction of a number of typos that appear in [23].

We denote by  $\mathbf{Fm}$  the formula algebra over a given similarity type, freely generated by a denumerable set of propositional variables  $\{p, q, r, \dots\}$ . We denote by  $Fm$  the carrier of  $\mathbf{Fm}$  and use  $\varphi, \psi$  and  $\gamma$ , possibly decorated with subscripts, to refer to arbitrary elements of  $Fm$ . A logic is then defined as a substitution-invariant consequence relation  $\vdash$  on  $Fm$ .

### DEFINITION 2.1

Nelson's logic  $\mathcal{S} := \langle \mathbf{Fm}, \vdash_{\mathcal{S}} \rangle$  is the sentential logic in the language  $\langle \wedge, \vee, \Rightarrow, \sim, 0 \rangle$  of type  $\langle 2, 2, 2, 1, 0 \rangle$  defined by the Hilbert-style calculus with the rule schemata in Table 1 and the following axiom schemata. We shall henceforth use the abbreviations  $\phi \Leftrightarrow \psi := (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$  and  $1 := \sim 0$ .

Axioms

- (A1)  $\phi \Rightarrow \phi$
- (A2)  $0 \Rightarrow \psi$
- (A3)  $\sim\phi \Rightarrow (\phi \Rightarrow 0)$
- (A4)  $1$

TABLE 1 Rules of  $\mathcal{S}$

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$\frac{\Gamma \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \gamma))}{\Gamma \Rightarrow (\psi \Rightarrow (\varphi \Rightarrow \gamma))} \text{ (P)}$	$\frac{\varphi \Rightarrow (\varphi \Rightarrow (\varphi \Rightarrow \gamma))}{\varphi \Rightarrow (\varphi \Rightarrow \gamma)} \text{ (C)}$	$\frac{\Gamma \Rightarrow \varphi \quad \varphi \Rightarrow \gamma}{\Gamma \Rightarrow \gamma} \text{ (E)}$
$\frac{\Gamma \Rightarrow \varphi \quad \psi \Rightarrow \gamma}{\Gamma \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow \gamma)} \text{ (\Rightarrow l)}$	$\frac{\gamma}{\varphi \Rightarrow \gamma} \text{ (\Rightarrow r)}$	$\frac{\varphi \Rightarrow \gamma}{(\varphi \wedge \psi) \Rightarrow \gamma} \text{ (\wedge l1)}$
$\frac{\psi \Rightarrow \gamma}{(\varphi \wedge \psi) \Rightarrow \gamma} \text{ (\wedge l2)}$	$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow (\varphi \wedge \psi)} \text{ (\wedge r)}$	$\frac{\varphi \Rightarrow \gamma \quad \psi \Rightarrow \gamma}{(\varphi \vee \psi) \Rightarrow \gamma} \text{ (\vee l1)}$
$\frac{\varphi \Rightarrow^2 \gamma \quad \psi \Rightarrow^2 \gamma}{(\varphi \vee \psi) \Rightarrow^2 \gamma} \text{ (\vee l2)}$	$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow (\varphi \vee \psi)} \text{ (\vee r1)}$	$\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow (\varphi \vee \psi)} \text{ (\vee r2)}$
$\frac{(\varphi \wedge \sim\psi) \Rightarrow \gamma}{\sim(\varphi \Rightarrow \psi) \Rightarrow \gamma} \text{ (\sim\Rightarrow l)}$	$\frac{\Gamma \Rightarrow^2 (\varphi \wedge \sim\psi)}{\Gamma \Rightarrow^2 \sim(\varphi \Rightarrow \psi)} \text{ (\sim\Rightarrow r)}$	$\frac{(\sim\varphi \vee \sim\psi) \Rightarrow \gamma}{\sim(\varphi \wedge \psi) \Rightarrow \gamma} \text{ (\sim\wedge l)}$
$\frac{\Gamma \Rightarrow (\sim\varphi \vee \sim\psi)}{\Gamma \Rightarrow \sim(\varphi \wedge \psi)} \text{ (\sim\wedge r)}$	$\frac{(\sim\varphi \wedge \sim\psi) \Rightarrow \gamma}{\sim(\varphi \vee \psi) \Rightarrow \gamma} \text{ (\sim\vee l)}$	$\frac{\Gamma \Rightarrow (\sim\varphi \wedge \sim\psi)}{\Gamma \Rightarrow \sim(\varphi \vee \psi)} \text{ (\sim\vee r)}$
$\frac{\varphi \Rightarrow \gamma}{\sim\sim\varphi \Rightarrow \gamma} \text{ (\sim\sim l)}$	$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \sim\sim\varphi} \text{ (\sim\sim r)}$	

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(A5)  $(\phi \Rightarrow \psi) \Leftrightarrow (\sim\psi \Rightarrow \sim\phi)$

In Table 1 below, following Nelson's notation,  $\Gamma$  denotes an arbitrary finite list  $(\phi_1, \dots, \phi_n)$  of formulas, and the following abbreviations are used:

$$\Gamma \Rightarrow \phi = \phi_1 \Rightarrow (\phi_2 \Rightarrow (\dots \Rightarrow (\phi_n \Rightarrow \phi) \dots)).$$

If  $\Gamma$  is empty, then  $\Gamma \Rightarrow \phi$  is just  $\phi$ . Moreover, we let

$$\phi \Rightarrow^2 \psi = \phi \Rightarrow (\phi \Rightarrow \psi)$$

and

$$\Gamma \Rightarrow^2 \phi = \phi_1 \Rightarrow^2 (\phi_2 \Rightarrow^2 (\dots \Rightarrow^2 (\phi_n \Rightarrow^2 \phi) \dots)).$$

We have fixed obvious typos in the rules  $(\wedge l2)$ ,  $(\wedge r)$  and  $(\sim\Rightarrow r)$  as they appear in [23, p. 214–5]. For example, rule  $(\sim\Rightarrow r)$  from Nelson's paper reads as:

$$\frac{\Gamma \Rightarrow^2 (\phi \wedge \psi)}{\Gamma \Rightarrow^2 \sim(\phi \Rightarrow \psi)}.$$

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This is not even classically valid. One might consider correcting the rule as follows:

$$\frac{\Gamma \Rightarrow^2 (\phi \wedge \sim\psi)}{\Gamma \Rightarrow^2 \sim(\phi \Rightarrow \psi)}$$

but this does not seem consistent with the convention used by Nelson for the other rules: the  $\Rightarrow$  connective should appear on the right-hand side at the bottom and  $\wedge$  at the top. We assume thus that this corrected version was intended to have been written upside-down. The rule (C), called *weak condensation* by Nelson, replaces (and is indeed a weaker form of) the *contraction* rule:

$$\frac{\phi \Rightarrow (\phi \Rightarrow \psi)}{\phi \Rightarrow \psi}.$$

This rule is also known in the literature as '3-2 contraction' [29, p. 389] and corresponds, on algebraic models, to the property of *three-potency* (see Section 3.2). Notice also that the usual rule of *modus ponens* (from  $\phi$  and  $\phi \Rightarrow \psi$ , infer  $\psi$ ) is an instance of (E) for  $\Gamma = \emptyset$ . Lastly, let us highlight that every rule schema involving  $\Gamma$  is actually a shorthand for a denumerably infinite set of rule schemata. For instance, the schema:

$$\frac{\Gamma \Rightarrow \phi \quad \psi \Rightarrow \gamma}{\Gamma \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow \gamma)} (\Rightarrow 1)$$

stands for the following collection of rule schemata:

$$\begin{array}{l} \frac{\phi \quad \psi \Rightarrow \gamma}{(\phi \Rightarrow \psi) \Rightarrow \gamma} \\ \frac{\gamma_1 \Rightarrow \phi \quad \psi \Rightarrow \gamma}{\gamma_1 \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow \gamma)} \\ \dots \\ \frac{\gamma_1 \Rightarrow (\gamma_2 \Rightarrow (\gamma_3 \Rightarrow \phi)) \quad \psi \Rightarrow \gamma}{\gamma_1 \Rightarrow (\gamma_2 \Rightarrow (\gamma_3 \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow \gamma)))} \\ \dots \end{array}$$

Thus, Nelson's calculus employs not just infinitely many axiom and rule instances but actually infinitely many rule schemata. Notice, nonetheless, that defining as usual a derivation as a finite sequence of formulas, we have that the consequence relation of  $\mathcal{S}$  is finitary.

## 3 Algebraic semantics

In this section, we show that  $\mathcal{S}$  is algebraizable (and, in fact, is *implicative* in Rasiowa's sense [14, Definition 2.3]), and we give two equivalent presentations for its equivalent algebraic semantics (that we shall call  *$\mathcal{S}$ -algebras*). The first presentation is obtained via the algorithm of [7, Theorem 2.17], while the second one is closer to the usual axiomatizations of classes of residuated lattices, which constitute the algebraic counterparts of many logics in the substructural family. In fact, the latter presentation of  $\mathcal{S}$ -algebras will allow us to see at a glance that they form an equational class and will also make it easier to compare them with other known classes of algebras related to substructural logics.

Following standard usage, we denote by  $\mathbf{A}$  (in boldface) an algebra and by  $A$  (italics) its carrier set. Given the formula algebra  $\mathbf{Fm}$ , the associated set of *equations*,  $Fm \times Fm$ , will henceforth be denoted by  $Eq$ . To say that  $\langle \phi, \psi \rangle \in Eq$ , we will write  $\phi \approx \psi$ , as usual. We say that a valuation  $\nu: Fm \rightarrow A$  satisfies  $\phi \approx \psi$  in  $\mathbf{A}$  when  $\nu(\phi) = \nu(\psi)$ . We say that an algebra  $\mathbf{A}$  satisfies  $\phi \approx \psi$  when all valuations over  $\mathbf{A}$  satisfy it.

It will be convenient for us to work with the following definition of *algebraizable logic*, which is not the original one [7, Definition 2.1] but an equivalent so-called intrinsic characterization [7, Theorem 3.21] of it:

DEFINITION 3.1

A logic  $\mathcal{L}$  is *algebraizable* if and only if there are equations  $E(x) \subseteq Eq$  and formulas  $\Delta(x, y) \subseteq Fm$  such that:

- (R)**  $\emptyset \vdash_{\mathcal{L}} \Delta(\phi, \phi)$
- (Sym)**  $\Delta(\phi, \psi) \vdash_{\mathcal{L}} \Delta(\psi, \phi)$
- (Trans)**  $\Delta(\phi, \psi) \cup \Delta(\psi, \gamma) \vdash_{\mathcal{L}} \Delta(\phi, \gamma)$
- (Rep)**  $\bigcup_{i=1}^n \Delta(\phi_i, \psi_i) \vdash_{\mathcal{L}} \Delta(\bullet(\phi_1, \dots, \phi_n), \bullet(\psi_1, \dots, \psi_n))$ ,  
for each  $n$ -ary connective  $\bullet$
- (Alg3)**  $\phi \Vdash_{\mathcal{L}} \Delta(E(\phi))$ .

Here, the notation  $\Gamma \vdash \Delta$ , where  $\Delta$  is a set of formulas, means that  $\Gamma \vdash \phi$  for each  $\phi \in \Delta$ . The set  $E(x)$  is said to be the set of *defining equations* and  $\Delta(x, y)$  is said to be the set of *equivalence formulas*. We say that  $\mathcal{L}$  is *implicative* when it is algebraizable with  $E(x) := \{x \approx \alpha(x, x)\}$  and  $\Delta(x, y) := \{\alpha(x, y), \alpha(y, x)\}$ , where  $\alpha(x, y)$  denotes a binary term in the language of  $\mathcal{L}$ . In such a case, the term  $\alpha(x, x)$  determines an algebraic constant on every algebra belonging to the algebraic counterpart of  $\mathcal{L}$  (see [14, Lemma 2.6]) and is usually denoted accordingly.

THEOREM 3.2 ([21], Theorem 1).

The logic  $\mathcal{S}$  is implicative, and thus algebraizable, with defining equation  $E(x) := \{x \approx 1\}$  — or, equivalently,  $E(x) := \{x \approx x \Rightarrow x\}$  — and equivalence formulas  $\Delta(x, y) := \{x \Rightarrow y, y \Rightarrow x\}$ .

3.1 *S*-algebras

By Blok and Pigozzi's algorithm ([7, Theorem 2.17], see also [13, Theorem 30], [14, Proposition 3.44]), the equivalent algebraic semantics of  $\mathcal{S}$  is the quasivariety of algebras [9, Definition V.2.24] given by the following definition:

DEFINITION 3.3

An *S*-algebra is a structure  $\mathbf{A} := \langle A, \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$  of type  $\langle 2, 2, 2, 1, 0, 0 \rangle$  that satisfies the following equations and quasiequations:

1. For each axiom  $\varphi$  of  $\mathcal{S}$ , the equation  $E(\varphi)$  defined as  $E(\varphi) := \varphi \approx 1$ .
2.  $x \Rightarrow x \approx 1$ .
3. For each rule

$$\frac{\varphi_1 \cdots \varphi_n}{\phi} \mathbf{(R)}$$

of  $\mathcal{S}$ , the quasiequation  $Q(\mathbf{R})$  defined as follows:

$$Q(\mathbf{R}) = [\varphi_1 \approx 1 \ \& \ \dots \ \& \ \varphi_n \approx 1] \Longrightarrow \phi \approx 1.$$

4.  $[x \Rightarrow y \approx 1 \ \& \ y \Rightarrow x \approx 1] \Longrightarrow x \approx y$ .

We shall henceforth denote by  $E(A_n)$  the equation given in Definition 3.3.1 for the axiom  $A_n$  (for  $1 \leq n \leq 5$ ) of  $\mathcal{S}$  and by  $Q(\mathbf{R})$  the quasiequation given in Definition 3.3.3 for the rule  $\mathbf{R}$  of  $\mathcal{S}$ . We will also use the following abbreviations:  $a * b := \sim(a \Rightarrow \sim b)$ ,  $a^2 := a * a$  and  $a^n := a * (a^{n-1})$  for  $n > 2$ .

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As the notation suggests, the defined connective  $*$  may be regarded as a ‘multiplicative conjunction’ in the sense of substructural logics. On  $\mathcal{S}$ -algebras, the operation  $*$  will be interpreted as a monoid operation having the implication ( $\Rightarrow$ ) as its residuum, whereas the ‘additive conjunction’  $\wedge$  will be interpreted as the meet of the underlying lattice structure. We list next a few properties of  $\mathcal{S}$ -algebras that will help us in viewing them, later on, as a class of residuated structures:

PROPOSITION 3.4 ([21], Proposition 3).

Let  $\mathbf{A} := \langle A, \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$  be an  $\mathcal{S}$ -algebra and  $a, b, c \in A$ . Then:

1.  $\langle A, \wedge, \vee, 0, 1 \rangle$  is a bounded lattice whose order  $\leq$  is given by  $a \leq b$  iff  $a \Rightarrow b = 1$ .
2.  $\langle A, *, 1 \rangle$  is a commutative monoid.
3. The pair  $(*, \Rightarrow)$  is residuated with respect to  $\leq$ , i.e.  $a * b \leq c$  iff  $a \leq b \Rightarrow c$ .
4.  $a \Rightarrow b = \sim b \Rightarrow \sim a$ .
5.  $a \Rightarrow 0 = \sim a$  and  $\sim \sim a = a$ .
6.  $a^2 \leq a^3$ .
7.  $(a \vee b)^2 \leq a^2 \vee b^2$ .

### 3.2 $\mathcal{S}$ -algebras as residuated lattices

In this section, we introduce an equivalent presentation of  $\mathcal{S}$ -algebras that takes precisely the properties in Proposition 3.4 as postulates. We begin by recalling the following well-known definitions (see e.g. [15, p. 185]):

DEFINITION 3.5

A *commutative integral residuated lattice (CIRL)* is an algebra  $\mathbf{A} := \langle A, \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$  of type  $\langle 2, 2, 2, 2, 0, 0 \rangle$  such that:

1.  $\langle A, \wedge, \vee \rangle$  is a lattice (with ordering  $\leq$ ) with maximum element 1.
2.  $\langle A, *, 1 \rangle$  is a commutative monoid.
3.  $(*, \Rightarrow)$  forms a residuated pair with respect to  $\leq$ , that is:  $a * b \leq c$  iff  $b \leq a \Rightarrow c$  for all  $a, b, c \in A$ .

We say that a CIRL is *three-potent*<sup>1</sup> when  $a^2 \leq a^3$  for all  $a \in \mathbf{A}$  (in which case it follows that  $a^2 = a^3$ ). If the lattice ordering of  $\mathbf{A}$  also has 0 as a minimum element, then  $\mathbf{A}$  is a *commutative integral bounded residuated lattice (CIBRL)*. Setting  $\sim a := a \Rightarrow 0$ , we then say that a CIBRL is *involutive* when it satisfies the equation  $\sim \sim x \approx x$  [16, p. 186]. The latter last equation implies that  $x \Rightarrow y \approx \sim y \Rightarrow \sim x$  [25, Lemma 3.1].

The property of integrality mentioned in the above definition corresponds to the requirement that 1 be at the same time the neutral element of the monoid and the top element of the lattice order. One easily sees that integrality entails that the operation  $*$  is  $\leq$ -decreasing ( $a * b \leq a$ ) and that the term  $x \Rightarrow x$  defines thus an algebraic constant in the lattice that is interpreted as 1.

DEFINITION 3.6

An  *$\mathcal{S}'$ -algebra* is a three-potent involutive CIBRL.

<sup>1</sup>The reader should be advised, however, that for some authors, e.g. [15, p. 96], three-potency corresponds to the equation  $x^3 \approx x^4$  and two-potency to  $x^2 \approx x^3$ .



Since CIBRLs form an equational class [15, Theorem 2.7], it is clear that  $\mathcal{S}'$ -algebras are also an equational class. By contrast, from Definition 3.3 it is far from obvious whether  $\mathcal{S}$ -algebras are equationally axiomatizable or not. By Proposition 3.4, though, we immediately obtain the following result:

PROPOSITION 3.7

Let  $\mathbf{A} := \langle A, \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$  be an  $\mathcal{S}$ -algebra. Setting  $x * y := \sim(x \Rightarrow \sim y)$ , we have that  $\mathbf{A}' := \langle A, \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$  is an  $\mathcal{S}'$ -algebra.

The next lemma will allow us to verify that, conversely, every  $\mathcal{S}'$ -algebra has a term-definable  $\mathcal{S}$ -algebra structure. Thus, as anticipated,  $\mathcal{S}'$ -algebras and  $\mathcal{S}$ -algebras can be viewed as two presentations (in slightly different languages) of the same class of abstract structures. To establish this, we shall check that every  $\mathcal{S}'$ -algebra satisfies all (quasi)equations introduced in Definition 3.3.

LEMMA 3.8 ([21], Lemma 1).

1. Any CIBRL satisfies the equation  $(x \vee y) * z \approx (x * z) \vee (y * z)$ .
2. Any CIBRL satisfies  $x^2 \vee y^2 \approx (x^2 \vee y^2)^2$ .
3. Any three-potent CIBRL satisfies  $(x \vee y^2)^2 \approx (x \vee y)^2$ .
4. Any three-potent CIBRL satisfies  $(x \vee y)^2 \approx x^2 \vee y^2$ .

PROPOSITION 3.9

Let  $\mathbf{A}' := \langle A, \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$  be an  $\mathcal{S}'$ -algebra. Setting  $\sim x := x \Rightarrow 0$ , we have that  $\mathbf{A} := \langle A, \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$  is an  $\mathcal{S}$ -algebra.

PROOF. Let  $\mathbf{A}'$  be an  $\mathcal{S}'$ -algebra. We first consider the equations obtained from Definition 3.3.1. To check E(A1) (namely, the equation  $x \Rightarrow x \approx 1$ ), one may use residuation and the facts that  $1 * a \leq a$  and that 1 is the maximum element. E(A2) follows from the fact that 0 is the minimum element of  $\mathbf{A}'$ . E(A3) follows from the definition of  $\sim$  in  $\mathcal{S}'$  and from E(A1). E(A4) follows from the fact that  $1 \approx 1 \Rightarrow 1$ . E(A5) follows from the fact that  $\mathbf{A}'$  is involutive. We look next at the quasiequations obtained from Definition 3.3.3:

Q(P) follows from the commutativity of  $*$  and from the equation  $(a * b) \Rightarrow c \approx a \Rightarrow (b \Rightarrow c)$ .

Q(C) follows from 3-potency: since  $a^2 \leq a^3$ , we have that  $a^3 \Rightarrow b \approx 1$  implies  $a^2 \Rightarrow b \approx 1$ .

Q(E) follows from the fact that  $\mathbf{A}'$  carries a partial order  $\leq$  that is determined by the implication  $\Rightarrow$ .

To prove Q( $\Rightarrow$  1), suppose  $a \leq b$  and  $c \leq d$ . From  $c \leq d$ , as  $b \Rightarrow c \leq b \Rightarrow c$ , using residuation we have that  $b * (b \Rightarrow c) \leq c$ , thus  $b * (b \Rightarrow c) \leq d$  and therefore  $b \Rightarrow c \leq b \Rightarrow d$ . As  $a \leq b$ , using residuation and the  $\leq$ -monotonicity of  $*$ , we have that  $a * (b \Rightarrow d) \leq b * (b \Rightarrow d) \leq d$ , therefore  $b \Rightarrow d \leq a \Rightarrow d$  and thus  $b \Rightarrow c \leq a \Rightarrow d$ . Now, since  $b \Rightarrow c \leq a \Rightarrow d$  iff  $a * (b \Rightarrow c) \leq d$  iff  $a \leq (b \Rightarrow c) \Rightarrow d$ , we obtain the desired result.

For Q( $\Rightarrow$  r), we need to prove that if  $d \approx 1$ , then  $b \Rightarrow d \approx 1$ . By residuation, recall that  $1 \leq b \Rightarrow d$  iff  $1 * b \leq d$ . The latter equation is however obviously true, given that  $d \approx 1$ .

The quasiequations Q( $\wedge$ 11), Q( $\wedge$ 12), Q( $\wedge$ r), Q( $\vee$ 11), Q( $\vee$ r1) and Q( $\vee$ r2) follow straightforwardly from the fact that  $\mathbf{A}'$  is partially ordered and the order is determined by the implication.

To prove Q( $\vee$ 12), notice that  $(b \vee c)^2 \leq b^2 \vee c^2$  by Lemma 3.8.4. Suppose  $b^2 \leq d$  and  $c^2 \leq d$ , then since  $\mathbf{A}'$  is a lattice, we have  $b^2 \vee c^2 \leq d$  and we conclude that  $(b \vee c)^2 \leq d$  and thus  $(b \vee c)^2 \Rightarrow d \approx 1$ .

As to Q( $\sim \Rightarrow$  1), by E(A1) we know that  $b \Rightarrow b \approx 1$ , therefore we have  $b * c \leq b$  and  $b * c \leq c$ . Thus  $b * c \leq b \wedge c$ . Now, if  $b \wedge c \leq d$ , then  $b * c \leq d$ .

## 10 Nelson's logic

To prove  $\mathbf{Q}(\sim \Rightarrow r)$ , suppose  $d^2 \leq b \wedge c$ . Using the  $\leq$ -monotonicity of  $*$ , we have  $d^2 * d^2 \leq (b \wedge c) * (b \wedge c)$ , i.e.  $d^4 \leq (b \wedge c)^2$ . Using 3-potency, we have  $d^4 \approx d^2$ , therefore  $d^2 \leq (b \wedge c)^2$ . Since  $(b \wedge c)^2 \leq b * c$ , we have  $d^2 \leq (b * c)$ .

$\mathbf{Q}(\sim \wedge 1)$ ,  $\mathbf{Q}(\sim \wedge r)$ ,  $\mathbf{Q}(\sim \vee 1)$  and  $\mathbf{Q}(\sim \vee r)$  follow from the De Morgan's Laws [15, Lemma 3.17]. Finally, we have  $\mathbf{Q}(\sim \sim 1)$  and  $\mathbf{Q}(\sim \sim r)$  because  $\mathbf{A}'$  is involutive.

It remains to prove the quasiequation according to which  $(a \Rightarrow b) \approx 1$  and  $(b \Rightarrow a) \approx 1$  imply  $a \approx b$ . We have that  $a \leq b$  and  $b \leq a$ , since  $\leq$  is anti-symmetric it follows that  $a \approx b$ .  $\square$

From Propositions 3.7 and 3.9 above, we obtain the desired result:

### THEOREM 3.10

The classes of  $\mathcal{S}$ -algebras and of  $\mathcal{S}'$ -algebras are term-equivalent.<sup>2</sup>

In the next section, we are going to use the algebraic insight gained through Theorem 3.10 to provide an alternative and more perspicuous axiomatization of  $\mathcal{S}$ .

## 4 A finite Hilbert-style calculus for $\mathcal{S}$

We are now going to introduce a finite Hilbert-style calculus and prove that it is algebraizable with respect to the class of  $\mathcal{S}'$ -algebras (hence, with respect to  $\mathcal{S}$ -algebras). This will give us a finite presentation of  $\mathcal{S}$  that is equivalent to Nelson's calculus of Section 2 but with the added advantage of involving only a finite number of axiom schemata.

Our calculus is an axiomatic strengthening of the *full Lambek calculus with exchange and weakening* ( $\mathcal{FL}_{ew}$ ; see e.g. [26]), which will allow us to obtain the algebraizability of  $\mathcal{S}$  as an easy extension of the corresponding result about  $\mathcal{FL}_{ew}$ .

### The calculus $\mathcal{S}'$

#### DEFINITION 4.1

The logic  $\mathcal{S}' := \langle \mathbf{Fm}, \vdash_{\mathcal{S}'} \rangle$  is the sentential logic in the language  $\langle \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$  of type  $\langle 2, 2, 2, 2, 0, 0 \rangle$  defined by the Hilbert-style calculus with the following axiom schemata and *modus ponens* as its only rule schema:

- (S1)  $(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \gamma) \Rightarrow (\varphi \Rightarrow \gamma))$
- (S2)  $(\varphi \Rightarrow (\psi \Rightarrow \gamma)) \Rightarrow (\psi \Rightarrow (\varphi \Rightarrow \gamma))$
- (S3)  $\varphi \Rightarrow (\psi \Rightarrow \varphi)$
- (S4)  $\varphi \Rightarrow (\psi \Rightarrow (\varphi * \psi))$
- (S5)  $(\varphi \Rightarrow (\psi \Rightarrow \gamma)) \Rightarrow ((\varphi * \psi) \Rightarrow \gamma)$
- (S6)  $(\varphi \wedge \psi) \Rightarrow \varphi$
- (S7)  $(\varphi \wedge \psi) \Rightarrow \psi$
- (S8)  $(\varphi \Rightarrow \psi) \Rightarrow ((\varphi \Rightarrow \gamma) \Rightarrow (\varphi \Rightarrow (\psi \wedge \gamma)))$
- (S9)  $\varphi \Rightarrow (\varphi \vee \psi)$
- (S10)  $\psi \Rightarrow (\varphi \vee \psi)$

<sup>2</sup>We refer the reader to [36, p. 329] for a formal definition of term-equivalence. Informally, Theorem 3.10 is saying that  $\mathcal{S}$ -algebras and  $\mathcal{S}'$ -algebras may be seen as two equivalent presentations of the 'same' class of algebras in different algebraic languages, analogous to the well-known presentation of Boolean algebras as Boolean rings or to that of  $MV$ -algebras as certain lattice-ordered groups.

- (S11)  $(\varphi \Rightarrow \gamma) \Rightarrow ((\psi \Rightarrow \gamma) \Rightarrow ((\varphi \vee \psi) \Rightarrow \gamma))$   
 (S12) 1  
 (S13)  $0 \Rightarrow \varphi$   
 (S14)  $((\varphi \Rightarrow 0) \Rightarrow 0) \Rightarrow \varphi$   
 (S15)  $(\varphi \Rightarrow (\varphi \Rightarrow (\varphi \Rightarrow \psi))) \Rightarrow (\varphi \Rightarrow (\varphi \Rightarrow \psi))$ .

Axioms from (S1) to (S13) are those that axiomatize  $\mathcal{FL}_{ew}$  as presented in [37, Section 5], where  $\mathcal{FL}_{ew}$  is proven to be algebraizable. From that result, we can immediately obtain the following:

**THEOREM 4.2**

The calculus  $\mathcal{S}'$  is algebraizable with the same defining equation and equivalence formulas as  $\mathcal{S}$  (cf. Theorem 3.2). Its equivalent algebraic semantics is the class of  $\mathcal{S}'$ -algebras.

**PROOF.** We know from [37, Lemmas 5.2 and 5.3] that  $\mathcal{FL}_{ew}$  is algebraizable with respect to the class of CIBRLs. Given that  $\mathcal{S}'$  is an axiomatic strengthening of  $\mathcal{FL}_{ew}$ , by [14, Proposition 3.31], it is also algebraizable with the same defining equation and equivalence formulas. The corresponding class of algebras is a subvariety of the class of CIBRLs, and it can be axiomatized by adding equations corresponding to the new axioms, as described in Def. 3.3.1. It is easy to check that these imply precisely that the equivalent algebraic semantics of  $\mathcal{S}'$  is the class of all involutive (S14) and three-potent (S15) CIBRLs, i.e. the class of  $\mathcal{S}'$ -algebras.  $\square$

Although the logics  $\mathcal{S}$  and  $\mathcal{S}'$  were initially defined over different propositional languages (namely  $\langle \wedge, \vee, \Rightarrow, \sim, 0 \rangle$  for  $\mathcal{S}$  and  $\langle \wedge, \vee, \Rightarrow, *, \sim, 0, 1 \rangle$  for  $\mathcal{S}'$ ), we can obviously expand the language of  $\mathcal{S}$  to include the connectives 1 and  $*$  defined by  $1 := \sim 0$  and  $\phi * \psi := \sim(\phi \Rightarrow \sim\psi)$ . This allows us to state the following:

**COROLLARY 4.3**

The calculi  $\mathcal{S}$  (in the above-defined expanded language) and  $\mathcal{S}'$  define the same consequence relation.

**PROOF.** The result follows straightforwardly from the fact that  $\mathcal{S}$  and  $\mathcal{S}'$  are algebraizable (with the same defining equation and equivalence formulas) with respect to the same class of algebras. To add some detail one can invoke the algorithm of [14, Proposition 3.47], which allows one to obtain an axiomatization of an algebraizable logic from a presentation of the corresponding class  $K$  that comprises its equivalent algebraic semantics; notice that the algorithm uses only the (quasi)equations that axiomatize  $K$  and the defining equations and equivalence formulas witnessing algebraizability.  $\square$

We close the section with a non-trivial result about  $\mathcal{S}$  that would also not have been easily established if one had to work with Nelson's original presentation. It is well known that substructural logics enjoy a generalized version of the Deduction-Detachment Theorem [15, Theorem 2.14]. Combining this result with the algebraic insight obtained in Subsection 3.2 allows us to obtain a 'global' deduction theorem for  $\mathcal{S}$ :

**THEOREM 4.4 (Deduction-Detachment Theorem).**

For all  $\Gamma \cup \{\phi\} \subseteq Fm$ ,  $\Gamma \cup \{\phi\} \vdash_{\mathcal{S}} \psi$  if and only if  $\Gamma \vdash_{\mathcal{S}} \phi^2 \Rightarrow \psi$ .

**PROOF.** From [15, Corollary 2.15], we have that  $\Gamma \cup \{\phi\} \vdash \psi$  iff  $\Gamma \vdash \phi^n \Rightarrow \psi$  for some  $n$ . Now it is easy to see that in  $\mathcal{S}$ , thanks to (S15), we can always choose  $n = 2$ .  $\square$

Theorem 4.4 suggests that, upon defining  $\varphi \rightarrow \psi := \varphi^2 \Rightarrow \psi$ , one may obtain in  $\mathcal{S}$  a new implication-type connective  $\rightarrow$  that enjoys the standard formulation of the Deduction-Detachment Theorem (for which  $n = 1$ ). This is precisely what happens in Nelson's logic  $\mathcal{N}3$ , where in fact  $\rightarrow$  is usually taken as the primitive implication and  $\Rightarrow$  as defined (see Subsection 6.2). Whether a similar interdefinability result holds for  $\mathcal{S}$  as well is actually an interesting open question, to which we shall return in Section 8. For now, what we can say is that the above-defined term  $\rightarrow$  does indeed behave on  $\mathcal{S}$ -algebras like an implication operation, at least in the abstract sense introduced by Blok, Köhler and Pigozzi [5]. The latter paper is the second of a series devoted to classes of algebras of non-classical logics [9, Definition II.9.3], focusing in particular on varieties that enjoy the property of having *equationally definable principal congruences* or EDPC for short [5, p. 338]. This is quite a strong property. In particular, it implies congruence-distributivity and the congruence extension property [6, Theorem 1.2]. It is well known that a logic that is algebraizable with respect to some variety of algebras enjoys a (generalized) Deduction-Detachment Theorem<sup>3</sup> if and only if its associated variety has EDPC [14, Corollary 3.86]. This applies, in particular, to our logic  $\mathcal{S}$  and to  $\mathcal{S}$ -algebras.

In the context of varieties of non-classical logic having EDPC, the authors of [5] single out those that possess term-definable operations that can be viewed as generalizations of intuitionistic conjunction, implication and bi-implication. These operations are called, respectively, *weak meet*, *weak relative pseudo-complementation* and *Gödel equivalence*. Algebras containing such operations are called *weak Brouwerian semilattices with filter-preserving operations*, or WBSO for short [5, Definition 2.1].

According to [1], a variety having a constant 1 is called *subtractive* if the congruences of any algebra in the variety permute at 1. Subtractive WBSO varieties are particularly interesting because the lattice of congruences of any algebra  $\mathbf{A}$  belonging to a subtractive WBSO variety is isomorphic to the ideal lattice of  $\mathbf{A}$  for a certain uniformly-defined notion of ideal. As observed in [5, p. 358], the algebraic counterpart of Nelson's logic  $\mathcal{N}3$  is a WBSO variety. The same is true for  $\mathcal{S}$ -algebras; in fact, we can here prove a slightly stronger result:

#### THEOREM 4.5

$\mathcal{S}$ -algebras form a WBSO variety in which a *weak meet* is given by  $\wedge$  (or, equivalently, by  $*$ ), *weak relative pseudo-complementation* is given by the term  $x^2 \Rightarrow y$  and *Gödel equivalence* is  $x \Leftrightarrow y$ . In fact,  $\mathcal{S}$ -algebras form a *subtractive WBSO variety*.

PROOF. One could directly check that, with the above choice of terms,  $\mathcal{S}$ -algebras satisfy all properties of [5, Definition 2.1]. But we can provide a more compact proof as follows. As mentioned earlier, since the logic  $\mathcal{S}$  has a form of Deduction-Detachment Theorem (our Theorem 4.4), we know that the variety of  $\mathcal{S}$ -algebras has EDPC [14, Corollary 3.86]. We can then apply [35, Theorem 3.3] (note that  $\mathcal{S}$ -algebras satisfy the premisses of the theorem thanks to [35, Lemma 3.2]) to conclude that  $(x \Leftrightarrow y)^2 * z$  is a *ternary deductive term* for  $\mathcal{S}$ -algebras in the sense of [8, Definition 2.1] that is moreover *regular with respect to 1* [8, Definition 4.1]. Then, by [8, Theorem 4.4], we have that  $\mathcal{S}$ -algebras form a WBSO variety. Finally, to check that  $\mathcal{S}$ -algebras are subtractive [1, p. 214], it is sufficient to note that they satisfy the equation  $1^2 \Rightarrow x \approx x$  [1, p. 215].  $\square$

<sup>3</sup>See e.g. [14, Definition 3.76] for a precise definition of generalized deduction-detachment theorem.

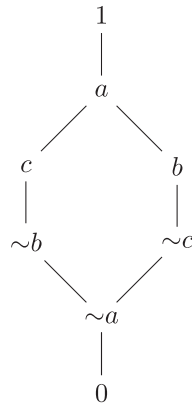


FIGURE 1  $\mathbf{A}_8$

Regarding the proof of the preceding theorem, it may be interesting to note that applying [8, Theorem 4.4] to the ternary deductive term  $(x \Leftrightarrow y)^2 * z$  would give us different witnessing terms: namely, we would obtain  $x^2 * y$  as weak meet,  $(x^2 \Rightarrow (x^2 * y))^2$  as weak relative pseudo-complementation and  $(x \Leftrightarrow y)^2$  as Gödel equivalence. This is not surprising, for such terms need not be unique.

## 5 More on $\mathcal{S}$ -algebras

### 5.1 A non-distributive $\mathcal{S}$ -algebra

We are now going to look at a particular  $\mathcal{S}$ -algebra that provides a counter-example for several formulas of  $\mathcal{S}$  that are not valid, including the formulas which Nelson claims (without proof) not to be provable in his calculus [23, p. 213].

#### EXAMPLE 5.1

The algebra  $\mathbf{A}_8$  shown in Figure 1 is an  $\mathcal{S}$ -algebra whose lattice reduct is obviously not distributive. The table for the implication  $\Rightarrow$  of  $\mathbf{A}_8$  is shown below.

$\Rightarrow$	0	1	c	$\sim c$	$\sim b$	b	a	$\sim a$
0	1	1	1	1	1	1	1	1
1	0	1	c	$\sim c$	$\sim b$	b	a	$\sim a$
c	$\sim c$	1	1	b	a	b	1	b
$\sim c$	c	1	c	1	c	1	1	c
$\sim b$	b	1	1	b	1	b	1	b
b	$\sim b$	1	c	a	c	1	1	c
a	$\sim a$	1	c	b	c	b	1	$\sim a$
$\sim a$	a	1	1	1	1	1	1	1

In Figure 1, we consider that  $\sim 1 = 0$  and that  $\sim \sim x = x$ , for each  $x \in A_8$ . The above sound model of  $\mathcal{S}'$  was found by using the Mace4 Model Searcher [19]. We are going to check that  $\mathbf{A}_8$  witnesses the failure of all the formulas listed below:

## PROPOSITION 5.2

The following formulas cannot be proved in  $\mathcal{S}$ .

1.  $p \vee \sim p$  (Excluded Middle)
2.  $\sim(p \wedge \sim p)$
3.  $(p \wedge \sim p) \Rightarrow q$  (Ex Contradictione)
4.  $(p \Rightarrow (p \Rightarrow q)) \Rightarrow (p \Rightarrow q)$  (Contraction)
5.  $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \wedge q) \Rightarrow r)$
6.  $(p \wedge \sim q) \Rightarrow \sim(p \Rightarrow q)$
7.  $((p \Rightarrow q) \Rightarrow q) \Rightarrow ((q \Rightarrow p) \Rightarrow p)$  (Łukasiewicz)
8.  $(p \wedge (q \vee r)) \Rightarrow ((p \wedge q) \vee (p \wedge r))$  (Distributivity)
9.  $((p^2 \Rightarrow q) \wedge ((\sim q)^2 \Rightarrow \sim p)) \Rightarrow (p \Rightarrow q)$  (Nelson)

PROOF. Thanks to [14, Lemma 2.6], if  $\phi$  can be proved in  $\mathcal{S}$ , then  $v(\phi) = v(\phi \rightarrow \phi)$ . As  $v(\phi \rightarrow \phi) = 1$ , it suffices to find, for each of the above formulas, some valuation  $v : \mathbf{Fm} \rightarrow \mathbf{A}_8$  such that  $v(\phi) \neq 1$ .

1. Setting  $v(p) = c$ , we have  $v(p \vee \sim p) = v(p) \vee \sim v(p) = c \vee \sim c = a$ .
2. Setting  $v(p) = c$ , we have  $v(\sim(p \wedge \sim p)) = \sim(c \wedge \sim c) = \sim(\sim a) = a$ .
3. Let  $v(p) = c$  and  $v(q) = 0$ . Then  $v((p \wedge \sim p) \Rightarrow q) = \sim a \Rightarrow 0 = a$ .
4. Let  $v(p) = c$  and  $v(q) = 0$ . Then  $v((p \Rightarrow (p \Rightarrow q)) \Rightarrow (p \Rightarrow q)) = (c \Rightarrow (c \Rightarrow 0)) \Rightarrow (c \Rightarrow 0) = (c \Rightarrow \sim c) \Rightarrow \sim c = b \Rightarrow \sim c = a$ .
5. Let  $v(p) = v(q) = c$  and  $v(r) = 0$ , then  $v((p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \wedge q) \Rightarrow r)) = (c \Rightarrow (c \Rightarrow 0)) \Rightarrow ((c \wedge c) \Rightarrow 0) = b \Rightarrow (c \Rightarrow 0) = b \Rightarrow \sim c = a$ .
6. Let  $v(p) = v(q) = c$ , then  $v((p \wedge \sim q) \Rightarrow \sim(p \Rightarrow q)) = (c \wedge \sim c) \Rightarrow \sim(c \Rightarrow c) = \sim a \Rightarrow \sim 1 = a$ .
7. Let  $v(p) = \sim c$  and  $v(q) = c$ . We have  $v(((p \Rightarrow q) \Rightarrow q) \Rightarrow ((q \Rightarrow p) \Rightarrow p)) = ((\sim c \Rightarrow c) \Rightarrow c) \Rightarrow ((c \Rightarrow \sim c) \Rightarrow \sim c) = (c \Rightarrow c) \Rightarrow (b \Rightarrow \sim c) = 1 \Rightarrow a = a$ .
8. Let  $v(p) = c$ ,  $v(q) = \sim c$  and  $v(r) = \sim b$ . We have  $v((p \wedge (q \vee r)) \Rightarrow ((p \wedge q) \vee (p \wedge r))) = ((c \wedge (\sim c \vee \sim b)) \Rightarrow ((c \wedge \sim c) \vee (c \wedge \sim b))) = (c \wedge a) \Rightarrow (\sim a \vee \sim b) = c \Rightarrow \sim b = a$ .
9. Let  $v(p) = c$  and  $v(q) = \sim b$ . We have  $v(((p^2 \Rightarrow q) \wedge ((\sim q)^2 \Rightarrow \sim p)) \Rightarrow (p \Rightarrow q)) = (\sim(c \Rightarrow \sim c)) \Rightarrow \sim b \wedge ((\sim(b \Rightarrow \sim b)) \Rightarrow \sim c) \Rightarrow (c \Rightarrow \sim b) = (\sim b \Rightarrow \sim b) \wedge (\sim c \Rightarrow \sim c) \Rightarrow a = 1 \Rightarrow a = a$ .  $\square$

If we were to add the (Łukasiewicz) formula from Proposition 5.2 as a new axiom schema to  $\mathcal{S}$  (or  $\mathcal{S}'$ ), we would obtain precisely the *three-valued Łukasiewicz logic* [12, Chapter 4.1]. No other non-classical logic in the Łukasiewicz family is comparable with  $\mathcal{S}$  because, on the one hand, they all lack three-potency, and, on the other,  $\mathcal{S}$  does not satisfy (Łukasiewicz), which is valid in all of them.

## 5.2 The Galatos–Raftery doubling construction

We present here an adaptation of the construction introduced in [16, Section 6] to embed a commutative integral residuated lattice into one having an involutive negation. This will provide us with a simple recipe for constructing  $\mathcal{S}$ -algebras and will also prove useful in studying the relation between subclasses of residuated lattices and subclasses of  $\mathcal{S}$ -algebras (see Section 7).

## DEFINITION 5.3

Given a CIRL  $\mathbf{A} := \langle A, \wedge, \vee, *, \Rightarrow, 1 \rangle$ , let  $\sim A := \{\sim a : a \in A\}$  be a disjoint copy of  $A$  and let  $A^* := A \cup \sim A$ . We extend the lattice order of  $\mathbf{A}$  to  $A^*$  as follows. For all  $a, b \in A$ :

1.  $a \leq_{A^*} b$  iff  $a \leq_A b$ .
2.  $\sim a \leq_{A^*} b$ .

3.  $\sim a \leq_{A^*} \sim b$  iff  $b \leq_A a$ .

For each  $a \in A$ , we define  $\sim(\sim a) := a$ . The behavior of the lattice operations is fixed according to De Morgan's laws:  $\sim a \wedge \sim b := \sim(a \vee b)$  and  $\sim a \vee \sim b := \sim(a \wedge b)$ . The operations  $*$  and  $\Rightarrow$  are extended to  $A^*$  as follows:

$$\begin{aligned}
 a * \sim b &:= \sim(a \Rightarrow b) & \sim a * \sim b &:= \sim 1 \\
 a \Rightarrow \sim b &:= \sim(a * b) & \sim a \Rightarrow \sim b &:= b \Rightarrow a & \sim a \Rightarrow b &:= 1.
 \end{aligned}$$

It is shown in [16, Section 6] that, if  $\mathbf{A}$  is a CIRL, then  $\mathbf{A}^*$  is an involutive CIBRL into which  $\mathbf{A}$  is embedded in the obvious way. Moreover, we have the following:

PROPOSITION 5.4

$\mathbf{A}^*$  is an  $\mathcal{S}$ -algebra if and only if  $\mathbf{A}$  is a three-potent CIRL.

PROOF. One direction is immediate: if  $\mathbf{A}^*$  is an  $\mathcal{S}$ -algebra, then it is three-potent, hence so is  $\mathbf{A}$  as a  $\{\wedge, \vee, *, \Rightarrow, 1\}$ -subalgebra of  $\mathbf{A}^*$ . Conversely, if  $\mathbf{A}$  is a three-potent CIRL, since we already know that  $\mathbf{A}^*$  is a CIBRL, it remains to show that  $a^2 \leq a^3$  for all  $a \in \mathbf{A}^*$ . For  $a \in A$ , the result follows from 3-potency of  $\mathbf{A}$ . If  $a \in \sim A$ , then by Definition 5.3, we have  $a^2 = \sim 1 = a^3$ .  $\square$

The following corollary concerns *implicative lattices*, namely CIRLs where  $(\wedge, \Rightarrow)$  forms a residuated pair and hence  $\wedge$  and  $*$  coincide. Implicative lattices are precisely the 0-free subreducts of Heyting algebras.

COROLLARY 5.5

If  $\mathbf{A}$  is either an implicative lattice or an  $\mathcal{S}$ -algebra, then  $\mathbf{A}^*$  is an  $\mathcal{S}$ -algebra.

In fact, it is not difficult to check that if  $\mathbf{A}$  is an implicative lattice, then  $\mathbf{A}^*$  is a special  $\mathcal{S}$ -algebra known as an  $\mathcal{N}3$ -lattice (we shall deal with these structures in Section 6.2).

EXAMPLE 5.6

Consider the three-element linearly ordered *MV*-algebra [12, Definition 1.1.1], that we shall call  $\mathbb{L}_3$  (for Łukasiewicz three-valued logic), defined as follows. The universe is  $\{0, \frac{1}{2}, 1\}$  with the obvious lattice ordering. We consider  $\mathbb{L}_3$  in the algebraic language  $\langle \wedge, \vee, *, \Rightarrow, \sim, 0, 1 \rangle$  with the (non-lattice) operations being given by the following tables:

$\Rightarrow$	0	$\frac{1}{2}$	1	$*$	0	$\frac{1}{2}$	1	$\sim$	0	1
0	1	1	1	0	0	0	0	0	0	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1	1	0	$\frac{1}{2}$	1	1	1	0

We note that  $\mathbb{L}_3$  is an involutive CIBRL (see, e.g. [12, Lemma 1.1.4 and Proposition 1.1.5]). It is also easy to check that  $\mathbb{L}_3$  is three-potent, and so it is an  $\mathcal{S}$ -algebra.

Applying the doubling construction to  $\mathbb{L}_3$ , we obtain the six-element linearly ordered  $\mathcal{S}$ -algebra  $(\mathbb{L}_3)^*$  with universe  $\{\sim 1, \sim \frac{1}{2}, \sim 0, 0, \frac{1}{2}, 1\}$ . The lattice operations are determined in the obvious way

and the implication is given by the table below.

$\Rightarrow$	$\sim 1$	$\sim \frac{1}{2}$	$\sim 0$	$0$	$\frac{1}{2}$	$1$
$\sim 1$	1	1	1	1	1	1
$\sim \frac{1}{2}$	$\frac{1}{2}$	1	1	1	1	1
$\sim 0$	0	$\frac{1}{2}$	1	1	1	1
$0$	$\sim 0$	$\sim 0$	$\sim 0$	1	1	1
$\frac{1}{2}$	$\sim \frac{1}{2}$	$\sim 0$	$\sim 0$	$\frac{1}{2}$	1	1
$1$	$\sim 1$	$\sim \frac{1}{2}$	$\sim 0$	0	$\frac{1}{2}$	1

$(\mathcal{L}_3)^*$  is an example of an  $\mathcal{S}$ -algebra that is distributive but fails to satisfy the equation corresponding to Nelson axiom from Proposition 5.2.9. Setting  $v(p) := \sim 0$  and  $v(q) := \sim \frac{1}{2}$ , we have  $((\sim 0)^2 \Rightarrow \sim \frac{1}{2}) \wedge ((\sim \frac{1}{2})^2 \Rightarrow \sim \sim 0) \Rightarrow (\sim 0 \Rightarrow \sim \frac{1}{2}) = ((\sim 1 \Rightarrow \sim \frac{1}{2}) \wedge (0 \Rightarrow 0)) \Rightarrow \frac{1}{2} = (1 \wedge 1) \Rightarrow \frac{1}{2} = \frac{1}{2}$ .

## 6 $\mathcal{N}3$ and $\mathcal{N}4$

As mentioned earlier, David Nelson is remembered for having introduced, besides  $\mathcal{S}$ , two better-known logics:  $\mathcal{N}3$ , which is usually called just *Nelson logic* [22], and  $\mathcal{N}4$  which is known as *paraconsistent Nelson logic* [4]. Both logics are algebraizable with respect to classes of residuated structures (called, respectively,  $\mathcal{N}3$ -lattices or *Nelson algebras*, and  $\mathcal{N}4$ -lattices). The question then arises of what is precisely the relation between  $\mathcal{S}$  and these other logics, or equivalently between  $\mathcal{S}$ -algebras,  $\mathcal{N}3$ -lattices and  $\mathcal{N}4$ -lattices. Can we meaningfully say that one is stronger than the other? By looking at their algebraic models, it will not be difficult to show that  $\mathcal{N}3$  (which is known to be an axiomatic strengthening of  $\mathcal{N}4$ ) can also be viewed as an axiomatic strengthening of  $\mathcal{S}$ , while  $\mathcal{N}4$  and  $\mathcal{S}$  do not seem to be comparable in any meaningful way. Just to fix terminology for what follows, we shall say that a logic  $\mathcal{L}'$  is a *conservative expansion* of a logic  $\mathcal{L}$  when the language of  $\mathcal{L}'$  expands that of  $\mathcal{L}$  and yet the consequence relations of both logics coincide on the common formulas.

### 6.1 $\mathcal{N}4$

#### DEFINITION 6.1

$\mathcal{N}4 := \langle \mathbf{Fm}, \vdash_{\mathcal{N}4} \rangle$  is the sentential logic in the language  $\langle \wedge, \vee, \rightarrow, \sim \rangle$  of type  $\langle 2, 2, 2, 1 \rangle$  defined by the Hilbert-style calculus with the following axiom schemata and *modus ponens* as its only rule schema:

- (N1)  $\phi \rightarrow (\psi \rightarrow \phi)$
- (N2)  $(\phi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \gamma))$
- (N3)  $(\phi \wedge \psi) \rightarrow \phi$
- (N4)  $(\phi \wedge \psi) \rightarrow \psi$
- (N5)  $(\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \gamma) \rightarrow (\phi \rightarrow (\psi \wedge \gamma)))$
- (N6)  $\phi \rightarrow (\phi \vee \psi)$
- (N7)  $\psi \rightarrow (\phi \vee \psi)$
- (N8)  $(\phi \rightarrow \gamma) \rightarrow ((\psi \rightarrow \gamma) \rightarrow ((\phi \vee \psi) \rightarrow \gamma))$
- (N9)  $\sim \sim \phi \leftrightarrow \phi$
- (N10)  $\sim(\phi \vee \psi) \leftrightarrow (\sim \phi \wedge \sim \psi)$
- (N11)  $\sim(\phi \wedge \psi) \leftrightarrow (\sim \phi \vee \sim \psi)$
- (N12)  $\sim(\phi \rightarrow \psi) \leftrightarrow (\phi \wedge \sim \psi)$



Here  $\phi \leftrightarrow \psi$  abbreviates  $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ . The implication  $\rightarrow$  in  $\mathcal{N}4$  is usually called *weak implication*, in contrast to the *strong implication*  $\Rightarrow$  that is defined by the following term:

$$\phi \Rightarrow \psi := (\phi \rightarrow \psi) \wedge (\sim\psi \rightarrow \sim\phi).$$

As the notation suggests, it is the strong implication, not the weak one, that we shall compare with the implication of  $\mathcal{S}$ . This appears indeed to be the more meaningful choice, as explained below.

A remarkable feature of the weak implication of  $\mathcal{N}4$  is that, on the one hand (unlike the implication of  $\mathcal{S}$ ), it enjoys the Deduction-Detachment Theorem in its standard formulation:  $\Gamma \cup \{\varphi\} \vdash_{\mathcal{N}4} \psi$  if and only if  $\Gamma \vdash_{\mathcal{N}4} \varphi \rightarrow \psi$ . On the other hand, contraposition fails ( $\varphi \rightarrow \psi \not\vdash_{\mathcal{N}4} \sim\psi \rightarrow \sim\varphi$ ), and the corresponding ‘weak bi-implication’ (again unlike  $\mathcal{S}$ , as axiom (A5) in Definition 2.1 makes clear) does not satisfy the following congruence property:  $\vdash_{\mathcal{N}4} \varphi \leftrightarrow \psi$  need not imply  $\vdash_{\mathcal{N}4} \sim\varphi \leftrightarrow \sim\psi$ . By contrast, the strong implication of  $\mathcal{N}4$  does not enjoy the Deduction-Detachment Theorem but (like the implication of  $\mathcal{S}$ ) it satisfies contraposition, and the associated bi-implication  $(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$  enjoys the congruence property. The same considerations apply to the logic  $\mathcal{N}3$  considered in the next subsection.

It is well known [30, Theorem 2.6] that  $\mathcal{N}4$  is algebraizable (though not implicative) with defining equation  $\mathbf{E}(\phi) := \{\phi \approx \phi \rightarrow \phi\}$  and equivalence formulas  $\Delta(\phi, \psi) := \{\phi \Rightarrow \psi, \psi \Rightarrow \phi\}$ . The implication in  $\mathbf{E}(\phi)$  could as well be taken to be the strong one, so  $\mathbf{E}(\phi) := \{\phi \approx \phi \Rightarrow \phi\}$  would work too. By contrast, letting  $\Delta(\phi, \psi) := \{\phi \rightarrow \psi, \psi \rightarrow \phi\}$  or the equivalent  $\Delta(\phi, \psi) := \{\phi \leftrightarrow \psi\}$  would not work precisely because of the failure of the above-mentioned congruence property.

The equivalent algebraic semantics of  $\mathcal{N}4$  is the class of  $\mathcal{N}4$ -lattices defined below [24, Definition 8.4.1]:

DEFINITION 6.2

An algebra  $\mathbf{A} := \langle A, \vee, \wedge, \rightarrow, \sim \rangle$  of type of type  $\langle 2, 2, 2, 1 \rangle$  is an  $\mathcal{N}4$ -lattice if it satisfies the following properties:

1.  $\langle A, \vee, \wedge, \sim \rangle$  is a De Morgan lattice.
2. The relation  $\preceq$  defined for all  $a, b \in A$  by  $a \preceq b$  iff  $(a \rightarrow b) \rightarrow (a \rightarrow b) = a \rightarrow b$  is a pre-order on  $\mathbf{A}$ .
3. The relation  $\equiv$  defined for all  $a, b \in A$  as  $a \equiv b$  iff  $a \preceq b$  and  $b \preceq a$  is compatible with  $\wedge, \vee, \rightarrow$  and the quotient  $\mathbf{A}_{\equiv} := \langle A, \vee, \wedge, \rightarrow \rangle / \equiv$  is an implicative lattice.
4. For all  $a, b \in A$ ,  $\sim(a \rightarrow b) \equiv a \wedge \sim b$ .
5. For all  $a, b \in A$ ,  $a \preceq b$  iff  $a \preceq b$  and  $\sim b \preceq \sim a$ , where  $\preceq$  is the lattice order of  $\mathbf{A}$ .

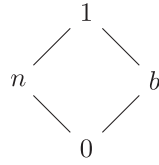
Despite this somewhat exotic definition, the class of  $\mathcal{N}4$ -lattices can actually be axiomatized by equations only [24, Definition 8.5.1].

A simple example of an  $\mathcal{N}4$ -lattice is  $\mathbf{A}_4$ , shown in Figure 2, whose lattice reduct is the four-element De Morgan algebra.

The tables for weak implication and negation in  $\mathbf{A}_4$  are as follows:

$\rightarrow$	0	$n$	$b$	1
0	1	1	1	1
$n$	1	1	1	1
$b$	0	$n$	$b$	1
1	0	$n$	$b$	1

$\sim$	0	1
0	1	0
$n$	$n$	$n$
$b$	$b$	$b$
1	1	0

FIGURE 2  $\mathbf{A}_4$ 

One can check that  $\mathbf{A}_4$  satisfies all properties of Definition 6.2, the quotient  $\mathbf{A}_4/\equiv$  mentioned in Definition 6.2.3 being the two-element Boolean algebra. It is also not difficult to see that no constant term is definable in  $\mathbf{A}_4$ . In fact, since the singleton  $\{b\}$  is a subuniverse of  $\mathbf{A}_4$ , this element would be the only possible interpretation for an algebraic constant. But  $\{0, 1\}$  is also a subuniverse of  $\mathbf{A}_4$ , so  $b$  cannot be the algebraic constant. This implies that  $\mathbf{A}_4$  has no term-definable  $\mathcal{S}$ -algebra structure and that no constant term exists in the whole class of  $\mathcal{N}4$ -lattices. In particular, neither the equation  $x \rightarrow x \approx y \rightarrow y$  nor  $x \Rightarrow x \approx y \Rightarrow y$  hold in all  $\mathcal{N}4$ -lattices.

In order to compare  $\mathcal{N}4$  and  $\mathcal{S}$ , we must fix a common propositional language, an obvious choice being  $\langle \wedge, \vee, \rightarrow, \sim \rangle$ , which is the primitive language of  $\mathcal{N}4$  as introduced above. That is, we interpret the implication of  $\mathcal{S}$  (up to now denoted  $\Rightarrow$ ) as the weak implication  $\rightarrow$  of  $\mathcal{N}4$ . Under this interpretation, it is easy to check that for instance the  $\mathcal{N}4$  axiom (N12) is not provable in  $\mathcal{S}$  (Proposition 5.2.6). On the other hand, it is well known that the weak implication of  $\mathcal{N}4$  does not satisfy the contraposition axiom (A5) of our Definition 2.1:  $(\phi \rightarrow \psi) \leftrightarrow (\sim\psi \rightarrow \sim\phi)$ . Thus we must conclude that, over this language,  $\mathcal{N}4$  and  $\mathcal{S}$  are incomparable.

As mentioned earlier, another possible choice for a common language would be one that replaces  $\rightarrow$  by  $\Rightarrow$ , interpreting the original implication of  $\mathcal{S}$  as the strong implication  $\Rightarrow$  of  $\mathcal{N}4$ . This is also a sensible option, for it has been recently shown [38] that the whole logic  $\mathcal{N}4$  can be equivalently presented in this language: the weak implication is term-definable in the  $\langle \wedge, \vee, \Rightarrow, \sim \rangle$ -fragment of  $\mathcal{N}4$  (namely, by setting  $\phi \rightarrow \psi := \phi \wedge (((\phi \wedge (\psi \Rightarrow \psi)) \Rightarrow \psi) \Rightarrow ((\phi \wedge (\psi \Rightarrow \psi)) \Rightarrow \psi)) \Rightarrow ((\phi \wedge (\psi \Rightarrow \psi)) \Rightarrow \psi)$ , see [38, Theorem 2.1]).

Under the latter interpretation, the above-mentioned contraposition axiom turns out to be valid in both logics. However, the fact that the equation  $x \Rightarrow x \approx y \Rightarrow y$  does not hold in all  $\mathcal{N}4$ -lattices implies (via the algebraizability of  $\mathcal{N}4$ ) that the formula  $(\phi \Rightarrow \phi) \Rightarrow (\psi \Rightarrow \psi)$ , which is valid in  $\mathcal{S}$ , is not provable in  $\mathcal{N}4$ . On the other hand, the (Distributivity) axiom is valid in  $\mathcal{N}4$  but not in  $\mathcal{S}$ , as we have seen (Proposition 5.2.8). All the above arguments continue to hold also if we were to consider conservative expansions of  $\mathcal{N}4$  such as the logic  $\mathcal{N}4^\perp$  of [24].

Taking into account the above observations, we conclude the following.

**PROPOSITION 6.3**

$\mathcal{N}4$  (together with all of its conservative expansions) and  $\mathcal{S}$  are incomparable over either language  $\langle \wedge, \vee, \rightarrow, \sim \rangle$  or  $\langle \wedge, \vee, \Rightarrow, \sim \rangle$ .

In the next section, we are going to see that at least in the case of the logic  $\mathcal{N}3$  the second choice of language allows us to show that the two logics are indeed comparable, with  $\mathcal{S}$  being the deductively weaker among them.

## 6.2 $\mathcal{N}3$

Nelson's logic  $\mathcal{N}3 := \langle \mathbf{Fm}, \vdash_{\mathcal{N}3} \rangle$  is the axiomatic strengthening of  $\mathcal{N}4$  obtained by adding the following axiom:

$$(N13) \quad \sim\phi \rightarrow (\phi \rightarrow \psi)$$

As an axiomatic strengthening of  $\mathcal{N}4$ , we have that  $\mathcal{N}3$  is also algebraizable with the same defining equation and equivalence formulas.  $\mathcal{N}3$  is in fact implicative, and its equivalent algebraic semantics is the variety of  $\mathcal{N}3$ -lattices, which are just  $\mathcal{N}4$ -lattices satisfying the equation corresponding to the above axiom (namely,  $\sim x \rightarrow (x \rightarrow y) \approx x \rightarrow x$ ) or, equivalently,  $x \rightarrow x \approx y \rightarrow y$  (which forces integrality). The latter equation implies that each  $\mathcal{N}3$ -lattice has two algebraic constants, given by  $1 := x \rightarrow x$  and  $0 := \sim 1$ .

In his 1959 paper [23, p. 215], Nelson mentioned that a calculus for  $\mathcal{N}3$  (there denoted by  $N$ ) could be obtained from his calculus for  $\mathcal{S}$  by removing certain rules and adding others, thus leaving it unclear whether one logic could be viewed as a strengthening of the other. Our algebraizability result for  $\mathcal{S}$  gives us a way to settle this issue.

As in the preceding subsection, we may compare  $\mathcal{S}$  and  $\mathcal{N}3$  over the languages  $\langle \wedge, \vee, \rightarrow, \sim, 0, 1 \rangle$  and  $\langle \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$ , this time including the propositional constants that are term-definable in both logics. The first option yields no new results, for the arguments of the preceding subsection continue to hold for  $\mathcal{N}3$  too. Thus,  $\mathcal{S}$  and  $\mathcal{N}3$  are also incomparable over  $\langle \wedge, \vee, \rightarrow, \sim, 0, 1 \rangle$ . The second option instead gives us the following.

**PROPOSITION 6.4**

$\mathcal{N}3$  is a (proper) strengthening of  $\mathcal{S}$  over the language  $\langle \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$ .

**PROOF.** It follows from [36, Theorem 3.12] that (the  $\langle \wedge, \vee, *, \Rightarrow, \sim, 0, 1 \rangle$ -reduct of) every  $\mathcal{N}3$ -lattice satisfies all properties of our Definition 3.6, and thus is an  $\mathcal{S}$ -algebra. On the other hand,  $\mathcal{N}3$ -lattices (like  $\mathcal{N}4$ -lattices) are distributive while  $\mathcal{S}$ -algebras need not be. Thus, invoking the algebraizability of  $\mathcal{N}3$  and of  $\mathcal{S}$  once more, we have that  $\mathcal{N}3$  is a *proper* strengthening of  $\mathcal{S}$ .  $\square$

Taking into account the axiomatization of  $\mathcal{N}3$  given in [36, p. 326], we can add further information to the preceding proposition by saying that  $\mathcal{N}3$  can be viewed as the *axiomatic* strengthening of  $\mathcal{S}$  obtained by adding the (Distributivity) and the (Nelson) axioms from Proposition 5.2. One can, in fact, do even better, showing that an  $\mathcal{S}$ -algebra satisfying the equation corresponding to the (Nelson) axiom must satisfy (Distributivity) as well [11, Remark 3.7]. Thus we obtain the following.

**PROPOSITION 6.5**

$\mathcal{N}3$  over the language  $\langle \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$  is the axiomatic strengthening of  $\mathcal{S}$  by the (Nelson) axiom.

It is not difficult to verify (see Example 5.6) that adding (Distributivity) to  $\mathcal{S}$  does not allow us to prove (Nelson). Thus, if we do so, we obtain a distinct logic that is intermediate between  $\mathcal{S}$  and  $\mathcal{N}3$ . On the other hand, the weakest strengthening of both  $\mathcal{S}$  and  $\mathcal{N}4$  is the logic  $\mathcal{N}3$  itself. To see this, recall that  $\mathcal{S}$ -algebras are integral residuated lattices, and therefore satisfy the equation  $x \Rightarrow x \approx y \Rightarrow y$ . Now, an  $\mathcal{N}4$ -lattice satisfying such equation (i.e. an algebra that is at the same time an  $\mathcal{N}4$ -lattice and an  $\mathcal{S}$ -algebra) must actually be an  $\mathcal{N}3$ -lattice. This can be easily checked using Odintsov's *twist-structure* representation of  $\mathcal{N}4$ -lattices [24, Proposition 8.4.3]. Thus,  $\mathcal{N}3$  is the join of  $\mathcal{S}$  and  $\mathcal{N}4$  in the lattice of all strengthenings of  $\mathcal{S}$ .

**PROPOSITION 6.6**

$\mathcal{N}3$ -lattices =  $\mathcal{S}$ -algebras  $\cap$   $\mathcal{N}4$ -lattices.

The following information on  $\mathcal{S}$ -algebras is also obtained as a straightforward consequence of the previous results.

**PROPOSITION 6.7**

The variety of  $\mathcal{S}$ -algebras is not finitely generated.

PROOF. Suppose, by contradiction, that the variety of  $\mathcal{S}$ -algebras was  $V(K)$ , with  $K$  a finite set of finite algebras (see [9, Definitions II.9.1 and II.9.4] for the meaning of the  $V, H$  and  $S$  operators). Since  $\mathcal{S}$ -algebras are congruence-distributive (this follows from [9, Theorem II.12.3] and also from our Theorem 4.5), we could apply Jónsson's lemma [9, Corollary IV.6.10] to conclude that the subdirectly irreducible algebras of  $V(K)$  are in  $HS(K)$ . Thus, there would be only finitely many subdirectly irreducible  $\mathcal{S}$ -algebras. However, it is well known that there are infinitely many subdirectly irreducible  $\mathcal{N}3$ -lattices (see, e.g., [24, Corollary 9.2.11]), and so there are infinitely many subdirectly irreducible  $\mathcal{S}$ -algebras.  $\square$

Concerning the relation between  $\mathcal{S}$ -algebras and  $\mathcal{N}3$ -lattices, we can employ the construction introduced in Definition 5.3 to state an analogue of Proposition 5.4.

PROPOSITION 6.8

Let  $\mathbf{A}^*$  be an  $\mathcal{S}$ -algebra constructed according to Definition 5.3. Then:

1.  $\mathbf{A}^*$  is an  $\mathcal{N}3$ -lattice if and only if  $\mathbf{A}$  is an implicative lattice.
2.  $\mathbf{A}^*$  is an  $MV$ -algebra ([12, Definition 1.1.1]) if and only if  $\mathbf{A}$  is trivial (and hence, if and only if  $\mathbf{A}^*$  is the two-element Boolean algebra).

PROOF. 1. Assume  $\mathbf{A}$  is an implicative lattice, and let us check that the Nelson equation  $((x^2 \Rightarrow y) \wedge ((\sim y)^2 \Rightarrow \sim x)) \Rightarrow (x \Rightarrow y) \approx 1$  holds in  $\mathbf{A}^*$ . Let  $a, b \in A$ . Clearly, the only non-trivial cases are when either  $a$  or  $b$  are in  $\sim A$ . Using the fact that  $a^2 = a$  and  $b^2 = b$ , we have  $((a^2 \Rightarrow \sim b) \wedge ((\sim \sim b)^2 \Rightarrow \sim a)) \Rightarrow (a \Rightarrow \sim b) = ((a \Rightarrow \sim b) \wedge (b \Rightarrow \sim a)) \Rightarrow (a \Rightarrow \sim b) = 1$ . In addition, given that  $\sim a \leq b$ , we have  $((\sim a)^2 \Rightarrow b) \wedge ((\sim b)^2 \Rightarrow \sim \sim a) \Rightarrow (\sim a \Rightarrow b) = (((\sim a)^2 \Rightarrow b) \wedge ((\sim b)^2 \Rightarrow a)) \Rightarrow 1 = 1$ . Lastly, we have  $((\sim a)^2 \Rightarrow \sim b) \wedge ((\sim \sim b)^2 \Rightarrow \sim \sim a) \Rightarrow (\sim a \Rightarrow \sim b) = (((\sim a)^2 \Rightarrow \sim b) \wedge (b^2 \Rightarrow a)) \Rightarrow (b \Rightarrow a) = (((\sim a)^2 \Rightarrow \sim b) \wedge (b \Rightarrow a)) \Rightarrow (b \Rightarrow a) = 1$ .

Conversely, assume  $\mathbf{A}^*$  is an  $\mathcal{N}3$ -lattice. Then, for all  $a \in A$ , we can show that  $a \Rightarrow a^2 = 1$  by instantiating the equation corresponding to the Nelson axiom as follows:  $((a^2 \Rightarrow a^2) \wedge ((\sim (a^2))^2 \Rightarrow \sim a)) \Rightarrow (a \Rightarrow a^2) = 1$ . Since  $b^2 = \sim 1$  for all  $b \in \sim A$ ,  $((a^2 \Rightarrow a^2) \wedge ((\sim (a^2))^2 \Rightarrow \sim a)) \Rightarrow (a \Rightarrow a^2) = (1 \wedge (\sim 1 \Rightarrow \sim a)) \Rightarrow (a \Rightarrow a^2) = 1 \Rightarrow (a \Rightarrow a^2) = a \Rightarrow a^2$ , which implies the desired result. Thus  $a = a^2$ , which implies that  $\mathbf{A}$  is an implicative lattice. 2. Assume  $\mathbf{A}^*$  is an  $MV$ -algebra. Then  $\mathbf{A}^*$  satisfies the so-called divisibility equation  $x * (x \Rightarrow y) = x \wedge y$  [12, Proposition 1.1.5]. Now let  $a \in A$ , so that  $\sim a <_{A^*} a$ . We have  $\sim a = a \wedge \sim a = a * (a \Rightarrow \sim a) = a * \sim(a * a)$ . But  $a * \sim a = \sim 1$  and so, by  $\leq$ -monotonicity,  $a * \sim(a * a) = \sim 1$ . Thus,  $\sim a = \sim 1$ , which implies that  $A = \{1\}$ .  $\square$

The preceding proposition, besides characterizing the algebras of the form  $\mathbf{A}^*$  that happen to be  $\mathcal{N}3$ -lattices, highlights the fact that  $\mathbf{A}^*$  turns out to be an  $MV$ -algebra (or Heyting, or Boolean algebra) only in the degenerate case. For the reader familiar with the twist-structure representation of  $\mathcal{N}3$ -lattices [24, Proposition 8.4.3], it may be worth mentioning that (as one can easily verify) when  $\mathbf{A}^*$  is an  $\mathcal{N}3$ -lattice, the quotient algebra  $\mathbf{A}^*_{\simeq}$  given by the twist-structure representation (cf. Definition 6.2) is isomorphic to the ordinal sum of  $\mathbf{A}$  with a one-element algebra (which constitutes the bottom element of  $\mathbf{A}^*_{\simeq}$ ).

## 7 Strengthenings of $\mathcal{S}$

In this section, we take a brief look at the finitary<sup>4</sup> strengthenings of  $\mathcal{S}$ . As is usual in algebraic logic, we shall in fact consider the equivalent question about sub quasivarieties of  $\mathcal{S}$ -algebras (on

<sup>4</sup>For the sake of simplicity, we restrict our attention to *finitary* strengthenings, though all the considerations of the present section generalize straightforwardly to arbitrary strengthenings (i.e. sub-generalized-quasivarieties).

quasivarieties and quasiequations, see e.g. [9, Definition V.2.24]). It is well known that finitary strengthenings of an algebraizable logic (in our case  $\mathcal{S}$ ) form a lattice that is dually isomorphic to the lattice of sub quasivarieties of its equivalent algebraic semantics—in our case,  $\mathcal{S}$ -algebras (see e.g. [14, Theorem 3.33]). Similarly, axiomatic strengthenings correspond to subvarieties [14, Corollary 3.40].

By combining, for instance, [11, Corollary 5.3] with [15, Theorem 1.59], we know that there are continuum many sub(quasi)varieties of  $\mathcal{N}3$ -lattices, and from this it follows that there are at least continuum many sub quasivarieties of  $\mathcal{S}$ -algebras. An interesting question is how many of these sub quasivarieties are included between  $\mathcal{S}$ -algebras and  $\mathcal{N}3$ -lattices. Using the doubling construction of Subsection 5.2, we can obtain some partial results in this direction.

Let  $\{e_i : i \in I\} \cup \{e\} \subseteq Fm \times Fm$  be equations in the language of residuated lattices (which does not include the 0 constant). Let

$$q(\vec{x}) := \&\{e_i : i \in I\} \implies e$$

be a quasiequation where  $\vec{x}$  are all the variables appearing in  $\{e_i : i \in I\} \cup \{e\}$ . Define the quasiequation

$$q^*(\vec{x}) := \&(\{e_i : i \in I\} \cup \{\sim x \sqsubseteq x : x \in \vec{x}\}) \implies e$$

that is built with formulas  $Fm^*$  in the language of  $\mathcal{S}$ -algebras (which includes 0 and therefore the negation), where  $x \sqsubseteq y$  is a shorthand for the equation  $y \vee x \approx y$ . Notice that if  $q(\vec{x})$  is an equation (i.e. the set  $\{e_i : i \in I\}$  is empty), then  $q^*(\vec{x})$  is a quasiequation of the form:

$$\&\{\sim x \sqsubseteq x : x \in \vec{x}\} \implies e.$$

It is not difficult to see that such a quasiequation is equivalent, in the context of  $\mathcal{S}$ -algebras, to the equation  $e^*$  that is obtained from  $e$  by substituting every variable  $x$  in  $e$  with the term  $x \vee \sim x$ . In other words, if  $q(\vec{x})$  is an equation in the language of residuated lattices, then  $q^*(\vec{x})$  is (equivalent to) an equation in the language of  $\mathcal{S}$ -algebras.

Let  $\mathbf{A}$  be a three-potent CIRL, and let  $\mathbf{A}^*$  be the bounded CIRL obtained as in Definition 5.3 (which is an  $\mathcal{S}$ -algebra by Proposition 5.4). Notice that within  $\mathbf{A}^*$  the elements of  $A$  are precisely the solutions to the equation  $x \approx x \vee \sim x$  (abbreviated  $\sim x \sqsubseteq x$ ), i.e.  $A = \{a \in \mathbf{A}^* : \sim a \leq a\}$ .

**PROPOSITION 7.1**

For any CIRL  $\mathbf{A}$  and any quasiequation  $q(\vec{x})$  in the language of residuated lattices,

$$\mathbf{A} \models q(\vec{x}) \quad \text{if and only if} \quad \mathbf{A}^* \models q^*(\vec{x}).$$

**PROOF.** For the rightward direction, it is sufficient to notice that all elements in  $\mathbf{A}^*$  satisfying the premisses of  $q^*(\vec{x})$  must belong to  $A$ , so we can use  $q(\vec{x})$  to obtain the desired result. As to the leftward direction, since any element  $a \in A$  satisfies  $\sim a \leq a$ , we can use  $q^*(\vec{x})$  to show that  $q(\vec{x})$  holds in  $\mathbf{A}$ .  $\square$

Let  $\mathcal{Q}$  be a quasivariety of commutative, integral, 3-potent residuated lattices. Then  $\{\mathbf{A}^* : \mathbf{A} \in \mathcal{Q}\}$  is a class of  $\mathcal{S}$ -algebras by Proposition 5.4, and we can consider the quasivariety  $\mathcal{Q}^* := \mathcal{Q}(\{\mathbf{A}^* : \mathbf{A} \in \mathcal{Q}\})$  generated by this class (see [14, Definition 1.72] for a definition of the  $\mathcal{Q}$  operator).  $\mathcal{Q}^*$  is then a quasivariety of  $\mathcal{S}$ -algebras, and from our previous considerations we also know that if  $\mathcal{Q}$  is a variety, then  $\mathcal{Q}^*$  is also a variety. Moreover, from Proposition 7.1 we have the following result:

**PROPOSITION 7.2**

For any quasivariety  $\mathcal{Q}$  of CIRLs and any quasiequation  $q$  in the language of  $\mathcal{Q}$ , we have  $\mathcal{Q} \models q$  if and only if  $\mathcal{Q}^* \models q^*$ .

Denote by  $\mathcal{RL}_3$  the variety of three-potent CIRLs, by  $\mathcal{T}$  the trivial variety (in the language of residuated lattices) and by  $[\mathcal{RL}_3, \mathcal{T}]$  the lattice of all sub quasivarieties of  $\mathcal{RL}_3$ . Similarly, we denote by  $[(\mathcal{RL}_3)^*, \mathcal{BA}]$  the interval (in the lattice of all sub quasivarieties of  $\mathcal{S}$ -algebras) between  $(\mathcal{RL}_3)^* := Q(\{\mathbf{A}^* : \mathbf{A} \in \mathcal{RL}_3\})$  and the variety of Boolean algebras  $\mathcal{BA}$ . Notice that  $\mathcal{BA} = (\mathcal{T})^*$  by Proposition 6.8.2.

**PROPOSITION 7.3**

The map  $(\cdot)^*$  is a lattice embedding of  $[\mathcal{RL}_3, \mathcal{T}]$  into  $[(\mathcal{RL}_3)^*, \mathcal{BA}]$ .

**PROOF.** It is obvious that the map  $(\cdot)^*$  is order-preserving. We show that it is also order-reflecting, which implies that it is injective. Let  $Q_1, Q_2$  be quasivarieties of three-potent CIRLs. Assume  $(Q_1)^* \subseteq (Q_2)^*$ , let  $\mathbf{A} \in Q_1$  and suppose  $q$  is any quasiequation such that  $Q_2 \models q$ . Then, by Proposition 7.1, we have  $(Q_2)^* \models q^*$ . By definition, we have  $\mathbf{A}^* \in (Q_1)^*$  and therefore  $\mathbf{A}^* \in (Q_2)^*$ , which means that  $\mathbf{A}^* \models q^*$ . Then again by Proposition 7.1, we have  $\mathbf{A} \models q$ , which means that  $\mathbf{A} \in Q_2$ . Hence,  $Q_1 \subseteq Q_2$  as required. Thus the map  $(\cdot)^*$  is an order embedding and therefore a (complete) lattice embedding.  $\square$

By our previous considerations,  $(\mathcal{RL}_3)^*$  is a variety of  $\mathcal{S}$ -algebras, and in fact it is not difficult to show that it is a proper subvariety of  $\mathcal{S}$ -algebras; it is proper because, e.g. the equation  $(x \Rightarrow \sim x) \vee (\sim x \Rightarrow x) \approx 1$  is valid in  $(\mathcal{RL}_3)^*$  but not in all  $\mathcal{S}$ -algebras (the algebra  $\mathbf{A}_8$  shown earlier being a witness). Similarly, denoting by  $\mathcal{IL}$  the variety of implicative lattices, we have by Proposition 6.8.1 that  $(\mathcal{IL})^*$  is a proper subvariety of  $\mathcal{N}3$ -lattices.

By [15, Theorem 9.54], the cardinality of  $[\mathcal{RL}_3, \mathcal{T}]$  is greater than or equal to the continuum. By Proposition 7.3, this implies that there are at least continuum many quasivarieties in  $[(\mathcal{RL}_3)^*, \mathcal{BA}] \subseteq [\mathcal{S}, \mathcal{BA}]$ . It is not difficult to see (e.g. by observing again that the equation  $(x \Rightarrow \sim x) \vee (\sim x \Rightarrow x) \approx 1$  need not be satisfied in all  $\mathcal{N}3$ -lattices) that  $[\mathcal{N}3, \mathcal{BA}]$  is not a sublattice of  $[(\mathcal{RL}_3)^*, \mathcal{BA}]$ , and this entails that we actually have now some more information than when we started off. Similarly, denoting by  $\kappa$  the cardinality of  $[\mathcal{RL}_3, \mathcal{IL}]$ , we now know that the cardinality of  $[(\mathcal{RL}_3)^*, (\mathcal{IL})^*]$ , and therefore that of  $[\mathcal{S}, (\mathcal{IL})^*]$ , must be at least as large as  $\kappa$ . Unfortunately, as far as we know, the cardinality of  $[\mathcal{RL}_3, \mathcal{IL}]$  is at present unknown. This leaves us with an interesting open problem, namely the study of the cardinality (and the structure) of the lattice of logics/algebras  $[\mathcal{S}, \mathcal{N}3]$ . We mention a few more open problems in the next section.

## 8 Future work

To the best of the authors' knowledge, the present paper—together with its precursor [21]—is the first devoted to a semantical study of Nelson's logic  $\mathcal{S}$ . We have though but scratched the surface of what may turn out to be an interesting topic for future research. We mention here but three directions.

The first is to study other types of calculi for  $\mathcal{S}$ , for example sequent-style or display-style calculi, in particular one would be interested in calculi that enjoy certain desirable properties (e.g. analytic, cut-free ones) and that fit well within the general proof-theoretic framework of substructural logics. Encouraging results in this direction have been obtained about  $\mathcal{N}3$  but at this point it seems far from obvious whether (or how) these may be extended to our  $\mathcal{S}$ .

The second issue may be cast in purely algebraic terms. Thanks to recent work of Spinks and Veroff, we know that  $\mathcal{N}4$ -lattices as well as  $\mathcal{N}3$ -lattices can be equivalently presented taking either the strong implication ( $\Rightarrow$ ) or the weak one ( $\rightarrow$ ) as primitive. In the case of  $\mathcal{N}4$ -lattices, this result turns out to be surprisingly hard to prove, not so hard for  $\mathcal{N}3$ -lattices, where the term defining the weak implication from the strong one is also simpler [36, Theorem 1.1.3], *viz.*

$\phi \rightarrow \psi := \phi \Rightarrow (\phi \Rightarrow \psi)$ . The same question can now be asked about  $\mathcal{S}$ -algebras: is it possible to axiomatize them taking the weak implication as primitive? For this, one might start by checking which theorems of  $\mathcal{N}3$  and  $\mathcal{N}4$  regarding the weak implication are valid in all  $\mathcal{S}$ -algebras, once we translate them according to the preceding term. The answer seems at the moment far from obvious and might provide us with further logical insight into  $\mathcal{S}$ . It is well known, for example, that the  $\{\wedge, \vee, \rightarrow\}$ -fragment of  $\mathcal{N}4$  (recall that here the implication  $\rightarrow$  is the weak one) coincides with the corresponding fragment of intuitionistic logic which means that  $\mathcal{N}4$  (and  $\mathcal{N}3$ ) may be regarded as strengthenings of intuitionistic logic by an involutive De Morgan negation. A solution to the above-mentioned problem would then tell us whether an analogous result can be stated for the logic  $\mathcal{S}$  as well.

Lastly, a promising line of research may be opened by the study of the Nelson axiom/equation (Proposition 5.2.9) in a more abstract algebraic setting. It is not difficult to see that the Nelson equation is equivalent, in the context of  $\mathcal{S}$ -algebras, to the following condition:

$$a^2 \Rightarrow b = 1 \text{ and } (\sim b)^2 \Rightarrow \sim a = 1 \text{ imply } a \leq b.$$

One can also (less immediately) show that this is in turn equivalent to the following:

$$\vartheta(b, 1) \subseteq \vartheta(a, 1) \text{ and } \vartheta(a, 0) \subseteq \vartheta(b, 0) \text{ imply } a \leq b \quad (1)$$

where  $\vartheta(a, 1)$  denotes the congruence generated by the set  $\{a, 1\}$  and so on. It is interesting to notice that the latter condition is almost purely algebraic, for it only relies on the presence of two distinguished elements in the algebra and (inessentially) of a partial order. Moreover, it closely reminds the properties of *congruence orderability* and *congruence quasi-orderability* studied in [2, 3]. This suggests that a purely algebraic investigation of the Nelson equation, restated as (1), along the lines of Aglianò's work may be a fruitful one. The first results in this direction have by now been published as [31, 34]. Further results are to be found in [32].

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