An Exact Algorithm for an Art Gallery Problem

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Abstract. *In this master thesis, a geometrical* NP-HARD *problem, the* Art Gallery problem*, is studied. The problem goal is to minimize the number of guards sufficient to cover the interior of an art gallery whose boundary is represented by a simple polygon. Among the many variants, we focus on one where the guards are stationary and restricted to vertices of the polygon, which can be either orthogonal or general simple, without holes. An exact algorithm is proposed in which the original continuous problem is discretized. Proofs of correctness and convergence of the algorithm to an optimal solution are given. Extensive experimentation with the algorithm show that it solves to optimality instances with more than ten times the size of the largest ones reported earlier in the literature.*

Keywords. *combinatorial geometry, combinatorial optimization, art gallery problem, exact algorithm, set covering, visibility.*

1. Introduction

Nowadays, security is one of the main concerns in our society. Stories concerning robbed banks, museums, homes and other buildings are well known all over the world. As the demand for low cost and effective security systems increases, it is important to fully understand the problem in order to improve efficiency. The research on the Art Gallery problems (AGP) plays an important role in the decisions involved.

In 1973, Victor Klee posed the original AGP problem, which consists in determining the minimum number of guards sufficient to cover the interior of an n-wall art gallery. This work focus on the minimization problem of the specific variation where guards have a $360°$ field of vision and their placement is restricted to the vertices of the simple polygon, either orthogonal or general simple, that represents the outer boundary of a given art gallery. Both minimization problems have been proven NP-HARD and, while being one of the most studied problems in computational geometry, almost no attempts to develop an exact algorithm have been made, and none are complete, efficient and practical. In the literature there is a number of works on approximation algorithms, as well as a few on the placement of a sub-optimal number of guards.

This master thesis successfully addresses this problem and is well aligned within the main context. We propose an exact and efficient algorithm, perform a theoretical analysis on the convergence and exactness of the algorithm, and carry out an extensive experimental study on its practicality.

The proposed approach is iterative and consists of building an initial discretization of the simple polygon that represents the floor plan of the gallery. Once that phase is complete, the problem is modeled as an instance of the classical *Set Cover problem* (SCP), which is solved and then refined until an optimal solution for the original problem is attained.

Results from our work have been published on four international conferences and one international journal. Additionally, a technical video about our work was also published accompanying one of those papers. Lastly, a benchmark comprised of all our instances and results was made publicly available, for future comparative research. Some of these papers are included as chapters of the thesis, as sanctioned by the university's graduate program, along with detailed comments scattered within those chapters to help the reader to completely understand the details of the work.

In the first three chapters, we introduce the problem and its motivation, explain the theoretical concepts and definitions required for a full understanding of the details, and present the design of the algorithm, the proofs of correctness and convergence, the reduction between the problems, the discretization strategies used in the experimental evaluation and the process of building the instances. Finally, there is an appendix that describes the development and implementation of the algorithm, the issues solved, the design and project, and two interfaces built to perform the experimental and the human evaluation and visualization of the algorithm.

2. Inception of the Algorithm

The geometrical perspective of the Art Gallery problem represents an n-wall art gallery as a planar region whose boundary consists of a simple polygon P with a set V of n vertices. A vertex $v \in V$ is denoted *convex* if the internal angle at v is smaller than $180°$. Whenever no confusion arises, ***a point in*** P will mean a point either in the interior or on the boundary of P.

A visibility region $\text{Vis}(v)$ of a vertex $v \in V$ is the set of all points in P visible from v. Any point $y \in P$ is visible from any other point $x \in P$ if and only if the closed segment joining x and y does not intersect the exterior of P.

A set of points S is called a ***guard set*** for P if and only if every point $p \in P$ is visible from at least a point $s \in S$. Thus, a ***vertex guard set*** G is any subset of vertices such that $\bigcup_{g \in G} \text{Vis}(g) = P$. As a vertex guard set guarantees that the entire gallery is overseen by the guards in the set, it is easy to see that an AGP amounts to finding the smallest subset $G \subset V$ that is a vertex guard set for P.

The problem of finding the smallest vertex guard set for P can be regarded as a specific Set Cover problem (SCP), where we wish to find a smallest cardinality set of visibility regions of the vertices of P whose union cover P, see Figure 1.

It is important to notice that this is a *continuous* SCP since there are infinitely many points in the interior of P to be covered. Notwithstanding, the original problem can be discretized in a finite number of representative points of P, $D(P)$, so that the formulation becomes manageable. This leads to the following IP formulation of the SCP, where A is a matrix and A_{ij} is set to 1 if and only if point $i \in D(P)$ is visible from vertex j, or 0 otherwise; x_j represents whether vertex j is in the solution set or not. We aim to minimize $z = \min \sum_{j \in V} x_j$ subject to $\sum_{j \in V} A_{ij} x_j \geq 1, \forall p_i \in D(P)$.

Given a feasible solution x to the IP above, let $Z(x) = \{j \in V \mid x_j = 1\}$. The constraints on the formulation states that every point in $D(P)$ is visible from at least one selected guard in the solution and the objective function minimizes the cardinality z of $Z(x)$. As the set $D(P)$ is finite, the solution set for the discretized problem $Z(x)$ may not be a viable solution for the original problem, i.e., a vertex guard set for P. In this scenario, the algorithm must iterate, picking a new discretization point inside each uncovered region and solving an IP formulation for the new SCP instance created.

One of the important results we proved is that once the algorithm finds a solution for the discretized instance $I(P, D(P))$ which is viable for the original problem, that solution is also optimal, i.e., a minimum vertex guard cover for P.

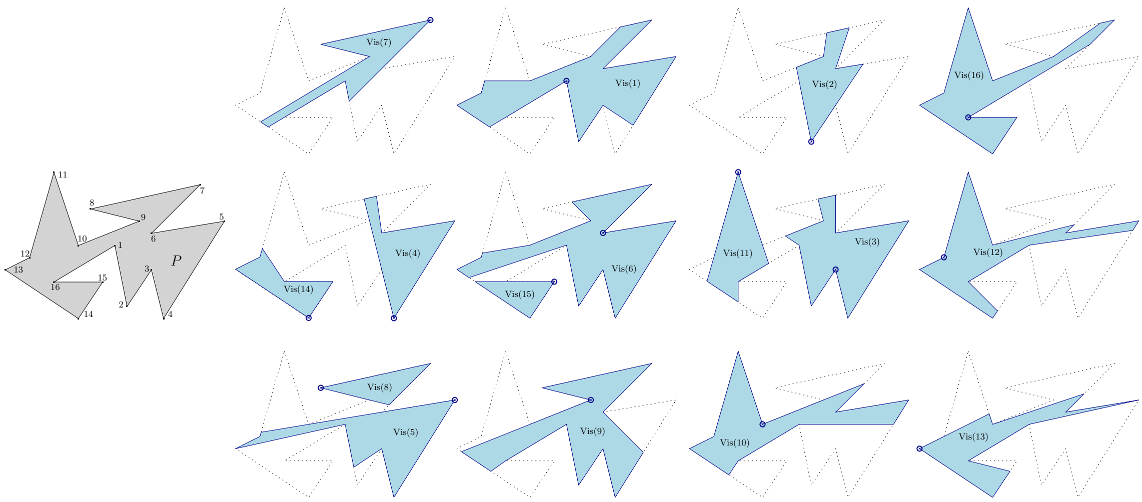

Figure 1. AGP **as a specific** SCP

In order to prove that the algorithm will always find a viable, and thus optimal, solution for the original problem, it suffices to determine the worst case for the number of iterations. Consider the set of all visibility regions of the vertices in V. The edges of these regions induce an arrangement of line segments within P, whose faces are called *atomic visibility polygons* (AVP). It has been shown that there are at most $O(n^3)$ AVPs.

It follows from the definition of AVPs that if any point in the interior of an atomic visibility polygon $\mathcal{V}$ is visible from a vertex guard, the entirety of $\mathcal{V}$ must also be. Since any uncovered region is formed by the union of neighboring AVPs, and since after each iteration one discretization point inside each of those regions is included into the formulation, an upper bound on the maximum number of iterations performed by the algorithm is $O(n^3)$ and this establishes its convergence, see Figure 2.

Moreover, the above algorithm is actually a rough description of a Turing's reduction from the Art Gallery problem to the Set Cover problem. The reader must also consider that if the initial discretization is comprised of the centroids of all atomic visibility polygons, the algorithm needs no more than the first iteration to obtain an optimal solution and this establishes a Karp's reduction.

Notice that each iteration of the algorithm solves an NP-HARD problem, the SCP, and its convergence is closely connected to the number of uncovers regions founded. Furthermore, as the uncovered regions depends on the choice of the initial discretization, there is a trade off between speed and precision that one must take into account when designing a good discretization strategy. It must ideally be light enough to set up instances of SCP that can rapidly be solved while minimizing the number of iterations required to attain an optimal solution. At the same time, it is important to start off with a discretization that represents the polygon well.

3. Evaluation of the Algorithm

In order to evaluate the algorithm and test its practical usability, several discretization strategies were built, each one with its own purpose. Thereafter, an instance building scheme was developed and implemented, with the ability to generate thousands of random orthogonal and general simple polygons, as well as of a few other classes of polygons.

Thus, an extensive experimental evaluation was conducted, and the results, summarized in Figure 3, show that the overall best performance was achieved by the *convex*

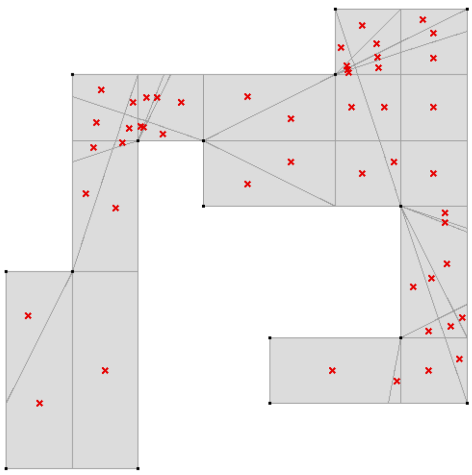

Figure 2. Visibility arrangement and a discretization with the centroids of all AVPs.

vertices strategy, where only the convex vertices of the polygon are included in the initial discretization set.

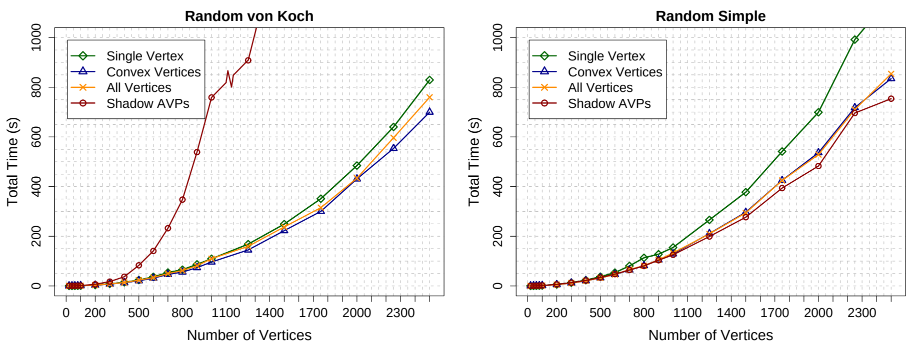

Figure 3. Total processing time for different discretization strategies.

The Convex Vertices strategy yields a sparse discretization and, as a consequence, small SCP instances, and yet dense enough to represent the polygon in a fairly broad sense. As it can be seen in Figure 3, this leads to a very fast implementation for instances of up to 2500 vertices.

4. Conclusion and Key Contributions

In this master thesis, we describe our extensive research of an Art Gallery problem, where the goal is to minimize the number of stationary vertex guards sufficient to cover the interior of an art gallery whose boundary is represented by a simple polygon.

We designed the first known exact algorithm able to solve to optimally (within excellent overall computational times) instances of up to ten times the size of the ones found in prior literature. Moreover, the algorithm proved to be robust, in the sense it was able to tackle instances from a broad range of polygon classes, including very degenerate ones.

A theoretical analysis on the subject is presented, including a full chapter with all basic definitions and concepts one needs to fully understand the problem, and a detailed explanation of the reductions, both Turing's and Karp's, from the Art Gallery problem to the Set Cover problem, which is the basis for our algorithm, and finally the proof from its correctness and convergence.

In order to validate the algorithm, an implementation was fully coded, as well as a generator of multiple class instances. Besides generating instances from a few well known polygon classes, we developed a new benchmark class of random von Koch polygons – an extremely complex and hard to solve class of instances derived from a modified von Koch curve. In the last part of the thesis, we share insights about implementation issues, degenerate instances, exact arithmetic which allowed our implementation to be robust and exact.

With all that, we carried out a broad experimental evaluation of the algorithm implementation and analyzed the behavior and impact of several discretization strategies on the many classes of instances at our disposal. In the end, the best overall performance strategy was found to be the one built with only the convex vertices of the polygon. It was shown to provide a perfect balance between speed and accuracy, performing only a few iterations in a short computational time.

Besides, we generated and made available for future public reference the first and only known benchmark of instances for the Art Gallery problem, comprised of more than 10,000 instances with up to 2,500 vertices as well as all the detailed results presented in our work.

In conclusion we successfully built, implemented and tested an exact algorithm that is able to solve extremely large instances (of up to 2,500 vertices) in a number of iterations several orders smaller than the theoretical upper bound, within 800 seconds when the best overall discretization strategy we found was employed.

5. Final Remarks and Relevance

This master thesis presents a broad multidisciplinary study on two variants of an important geometrical NP-HARD problem, the Art Gallery problem. This problem has been subject of study by many of the best known researchers in Computational Geometry for the past 40 years, including J. O'Rourke, V. Chvátal, D. T. Lee, A. K. Lin, A. Aggarwal, J. Urrutia, S. K. Ghosh, among others. See [O'Rourke 1987, Lee and Lin 1986, Urrutia 2000, Ghosh 2010].

The fact that our published papers have already begun to be cited attests to the relevance of our work: we have had 3 citations according to Scopus and 8 according to Google Scholar (excluding self-citations). Also, our paper *"An IP solution to the art gallery problem"*, which appeared in the Proceedings of the 25th ACM Annual Symposium on Computational Geometry, has been downloaded from the ACM Digital Library 85 times in the last 12 months and six times in the last six weeks alone.

We attribute the visibility our work is having to the fact that it is the first exact and efficient algorithm to optimally solve the problem in question, as well as for the extensive experimental analysis of the practical viability of the algorithm, and lastly, for having the first known benchmark of instances to the problem, available for public reference and use at *www.ic.unicamp.br/∼cid/Problem-instances/Art-Gallery*. Hopefully this will allow other researchers to compare their results with our exact ones.

Moreover, it is undoubtedly a recognition of excellence the fact that four refereed international conferences and one international journal published the papers that came out of this Master's thesis work. One of said conferences happens to be one of the most prestigious ones in experimental evaluation of algorithms — the Symposium on Experimental Algorithms, SEA, formerly know as Workshop on Experimental Algorithms, WEA.

Here is the full list of publication this work has generated.

1. M. C. Couto, P. J. de Rezende, and C. C. de Souza, *An exact algorithm for minimizing vertex guards on art galleries.* In **International Transactions in Operational Research**, available online since March 1st, 2011, no. doi: 10.1111/j.1475-3995.2011.00804.x., 27 pages.
2. M. C. Couto, P. J. de Rezende, and C. C. de Souza. *An IP solution to the art gallery problem.* In **SCG '09: Proceedings of the 25th annual symposium on Computational geometry**, pages 88–89, New York, NY, USA, 2009. ACM.
3. M. C. Couto, C. C. de Souza, and P. J. de Rezende. *Strategies for optimal placement of surveillance cameras in art galleries.* In **GraphiCon 2008: XI International Conference on Computer Graphics & Vision**, vol. 1, page *http://www.graphicon.ru/2008/proceedings/technical.html.* Lomonosov Moscow State University, 2008.
4. M. C. Couto, C. C. de Souza, and P. J. de Rezende. *Experimental evaluation of an exact algorithm for the orthogonal art gallery problem.* In **WEA**, **Lecture Notes in Computer Science**, volume 5038, pages 101–113. Springer, 2008.
5. M. C. Couto, C. C. de Souza, and P. J. de Rezende. *An exact and efficient algorithm for the orthogonal art gallery problem.* In **Proc. of the XX Brazilian Symp. on Comp. Graphics and Image Processing**, pages 87–94. IEEE Computer Society, 2007.

This master thesis opens plenty opportunities for future research which are detailed both in the publications, and in the thesis itself.

Acknowledgments. This research was partially supported by CNPq – Conselho Nacional de Desenvolvimento Científico e Tecnológico – Grant #301732/2007-8, #472504/2007-0, #483177/2009-1, #473867/2010-9 and FAPESP – Fundação de Amparo à Pesquisa do Estado de São Paulo – Grant #07/52015-0 and a Grant from FAEPEX/UNICAMP.

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