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THE ALGEBRA OF TOPOLOGY

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There are various connections between modern algebra and topology. In both these branches of mathematics, in the first place, a peculiarly strong tendency obtains, to define the object of investigation by means of abstract postulates. In the domain of combinatorial topology, moreover, methods and results of algebra are invariably applied. Such applications have occurred much less frequently in the field of point-set topology. But on the other hand, various fragments and arguments of point-set topology have themselves an algebraic character; and, in view of the simplicity and elegance of an algebraic presentation, several topologists have attempted to present in this way a sizeable portion of their subject.[1]

The idea therefore suggests itself, of creating an algebraic apparatus adequate for the treatment of portions of point-set topology. In the present paper we attempt to make a contribution to such a development. For this purpose we shall set up the foundation of a new algebraic calculus, which could be regarded as a sort of algebra of topology; and we shall study both the internal algebraic properties of this calculus and its relation to topology as ordinarily conceived. In particular our methods will enable us to settle a problem regarding the axiomatic foundations of topology which has remained open for a rather long time.

In §1 we shall present postulates for the sort of algebra of topology under consideration. This algebra, which we shall call *closure algebra*, is arrived at by adding to the postulates for Boolean algebra some additional postulates which express the properties of the closure operation[2] usually assumed in topology.

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[1] See, for instance, the following: F. Riesz, *Stetigkeitsbegriff und abstrakte Mengenlehre*, Atti del 4 Congresso International dei Mathematici, Rome, 1910, vol. 2, p. 18; C. Kuratowski, *L'opèration $\bar{A}$ de l'analysis situs*, Fundamenta Mathematicae, vol. 3 (1922), pp. 182-199; C. Kuratowski, *Topologie* I, Warsaw, 1933; R. L. Moore, *On the foundations of plane analysis situs*, Transactions of the American Mathematical Society, vol. 17 (1916), pp. 131–164; E. W. Chittenden, *On general topology and the relation of the properties of the class of all continuous functions to the properties of the space*, Transactions of the American Mathematical Society, vol. 31 (1929), pp. 290–321; S. T. Sanders, Jr., *Derived sets and their complements*, Bulletin of the American Mathematical Society, vol. 42 (1936), pp. 577–584; E. C. Stopher, *Cyclic relations in point set theory*, Bulletin of the American Mathematical Society, vol. 43 (1937), pp. 686–694; E. C. Stopher, *Point set operators and their interrelations*, Bulletin of the American Mathematical Society, vol. 46 (1939), pp. 758–762; M. Ward, *The closure operations of a lattice*, these Annals, vol. 43 (1942), pp. 191–196. [Further references are to be found in P. Alexandroff and H. Hopf, *Topologie* I, Leipzig, 1935. One of the first mathematicians to emphasize the importance in topology of an algebraic algorithm was S. Janiszewski.

[2] Similar methods can be applied to other topological notions which cannot be algebraically defined in terms of closure. Thus we could develop analogously an algebra of derivatives. We shall make some further remarks in this connection in Part I of the Appendix.

This section also contains definitions of numerous notions which will be used later, and we establish here some of the elementary properties of these notions.

In §2 we concern ourselves with the relations between closure algebras and topological spaces. It is clear that the family of all sets of a topological space constitutes a closure algebra (called the *closure algebra over* the given space), and the same is true of certain subfamilies of this family. We shall show here that in a certain sense the converse also holds: that every closure algebra is isomorphic with a family of sets situated in a topological space. Thus we shall establish here a representation theorem for closure algebras (making an essential use of the known results on the representation of Boolean algebras). This result is of importance to our investigations because it shows that the notion of a closure algebra is not too broad a generalization, so far as concerns our present purposes, of the notion of a topological space.

In §3 we define a *universal* closure algebra for a class $\mathfrak{A}$ of algebras, to be one which contains a subalgebra isomorphic with each algebra of the class $\mathfrak{A}$. We are interested here especially in universal algebras for all finite algebras. We shall see that the closure algebra over any null-dimensional dense-in-itself subspace of Euclidean space (e.g., over Cantor's discontinuum, or the space of all points with rational coordinates) is a universal algebra of this kind. Moreover we shall show that the closure algebra over Euclidean space itself is also a universal algebra for all finite algebras in a certain generalized sense; strictly speaking, for every finite closure algebra K we can find an open subset S of the given Euclidean space, such that K is isomorphic with a subalgebra of the algebra over S.

In §4 we introduce the notion of a *closure-algebraic function*. Roughly speaking, a closure-algebraic function is one which corresponds to an expression built up from variables by means of the various operations of closure algebra. Thus such functions play somewhat the same role in our investigations as is played by polynomials in classical algebra. In this section we establish several properties of closure-algebraic functions, some of which are of interest in themselves, and some of which are useful as lemmas for later developments.

The results of §§3 and 4 pave the way for the discussion in §5, which constitutes the central part of our paper. In §5 we define a *functionally free* algebra to be one which satisfies no topological equations except those which hold in every closure algebra; by topological equations we understand here those whose terms are expressions involving only the fundamental operations of closure algebra. We show that every closure algebra of a rather wide class is functionally free; this class comprehends in particular the closure algebras over Euclidean space of any number of dimensions, and more generally over an arbitrary dense-in-itself, separable metric space. Hence it follows that every topological equation which holds in Euclidean space of a given number of dimensions holds also in every other Euclidean space, and in fact in every topological space. (This is the solution of the problem which was mentioned before[3].) We are also concerned in this section with free closure algebras with given generators. We

[3] See the paper by Kuratowski cited in Footnote 1. Also page 18 of his *Topologie*.

shall see that various functionally free closure algebras—among them all closure algebras over Euclidean spaces—contain free closure algebras generated by n elements (for n an arbitrary positive integer). On the other hand we shall show that no free closure algebra generated by a finite number of elements is functionally free.

The appendix at the end of the paper contains some remarks on logical questions connected with our investigations, as well as some indications of possible ways of extending these results.

§1. Fundamental Notions

Due to the nature of our subject, we shall have to use notions from several domains: from general set theory, from topology, from Boolean algebra, and finally from general abstract algebra. In this section we shall define some of these notions which are of an elementary and fundamental nature, or which will be used throughout the rest of our paper. The more special notions, and all those of general algebra, will be introduced subsequently as the need for them arises.

We shall use lower-case Latin letters to denote integers, points of a space, or elements of an algebra; and lower-case Greek letters to denote real numbers, and also sometimes ordinal numbers. Greek capitals will be used to denote algebras (in those cases when they are treated as systems of sets and operations). We use Latin capitals for sets; however sets of sets (or *families* of sets, as we shall call them) and sets (classes) of algebras will usually be denoted by German capitals.

We shall use from the general theory of sets only rather elementary notions, which belong mostly to what is usually called the calculus of sets. The formulas

$$x \in A \quad \text{and} \quad A \subseteq B$$

will express as usual, that the element x belongs to A, or that the set A is contained in B, respectively. The expressions

$$A \cup B \quad \text{and} \quad \bigcup_{X \in \mathfrak{A}} X$$

will denote the union of two sets A and B, or of all sets of the family $\mathfrak{A}$; similarly

$$A \cap B \quad \text{and} \quad \bigcap_{X \in \mathfrak{A}} X$$

will denote the intersection of A and B, or of all sets of the family $\mathfrak{A}$. The difference of two sets A and B (i.e., the set of all elements that belong to A but not to B) will be denoted by

$$A - B$$

and the empty set by

$$\Lambda.$$

By

$$\{x\} \quad \text{and} \quad \{x_1, \cdots, x_n\}$$

we mean the set whose only member is x, and the set whose only members are $x_1, \cdots, x_n$. By

$$\langle x, y\rangle$$

we shall mean the ordered couple whose first element is x, and whose second element is y. In most applications we are concerned with sets all of which are contained in a given set S; in this case, X being an arbitrary subset of S, we shall refer to the difference $S - X$ as the complement of X (to S); in symbols $-X$. A non-empty family $\mathfrak{A}$ of subsets of a set S is called a *field* of sets if together with any two sets X and Y it contains $X \cup Y$ and $X \cap Y$, and together with any set X it contains its complement with respect to the set S (i.e., $S - X = -X$).

It is well-known that the study of fields of sets can be developed in an abstract way, as an independent algebraic system which is referred to as *Boolean algebra.* In developing this algebra we speak, not of sets of a given family $\mathfrak{A}$, but of elements $x, y, z, \cdots$ of an arbitrary class K, for which certain operations are defined: two binary operations, addition (formation of union) and multiplication (formation of intersection), and one unary operation, complementation. Thus strictly speaking a Boolean algebra is a system constituted by a class K (containing at least two different elements), and the three operations listed above, which are supposed to satisfy certain postulates.[4]

We shall still denote the sum and product of two elements x and y of a Boolean algebra by

$$x \cup y \quad \text{and} \quad x \cap y,$$

respectively. It will be clear from the context in each case whether the symbols "$\cup$" and "$\cap$" are intended in their set-theoretical or algebraic meaning.

By the zero-, or empty, element Λ and the universe-element V of a Boolean algebra we understand the unique elements satisfying the formulas

$$\Lambda = x \cap -x \quad \text{and} \quad \mathrm{V} = x \cup -x,$$

where x is an arbitrary element of the algebra. If x and y are any elements such that

$$x \cup y = y,$$

we say that x is included in y, or that y *contains* x, in symbols

$$x \subseteq y.$$

If in addition $x \neq y$, we shall say that x is a *proper* part of y. If $x \cap y = \Lambda$, the elements x and y are called *mutually exclusive*, or *disjoint.*

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[4] For postulates for Boolean algebra, see, for example, E. V. Huntington, *Sets of inde-independent postulates for the algebra of logic,* Transactions of the American Mathematical Society, vol. 5 (1904), pp. 288–309.

An element a of a Boolean algebra is called an *atom* if $a \neq \Lambda$, and if Λ and a are the only elements contained in a. A Boolean algebra is called *atomistic* if every non-empty element contains an atom.

Let A be any set of elements of a Boolean algebra. If there is an element z, which contains every element of A, and is included in every other element y which contains every element of A, then we call z the *sum* of the elements of A, in symbols

$$\bigcup_{x \in A} x.$$

Analogously we define the *product* of the elements of A, in symbols

$$\bigcap_{x \in A} x.$$

If the sum exists for every countable set A, we say the Boolean algebra is *countably additive*; and if it exists for every A we say the algebra is *completely additive*. (A countably, or completely, additive Boolean algebra could as well be called countably, or completely, multiplicative. For, as is known, a Boolean algebra is, e.g., countably additive, if and only if for every countable set of its elements the product exists.)

By saying that a set S is a *topological space*[5] *with respect to a closure operation* $\mathbf{C}$ one means that $\mathbf{C}$ is a unary operation which satisfies the following conditions:

(1) If $A \subseteq S$, then $A \subseteq \mathbf{C}A = \mathbf{CC}A \subseteq S$;
(2) If $A \subseteq S$ and $B \subseteq S$, then $\mathbf{C}(A \cup B) = \mathbf{C}A \cup \mathbf{C}B$;
(3) If $A \subseteq S$, and A contains at most one point, then $\mathbf{C}A = A$.

From conditions (2) and (3) one can of course draw the immediate inference that any finite set is equal to its own closure.

When condition (3) in the definition of topological space is replaced by the weaker condition:

(3′) $\mathbf{C}\Lambda = \Lambda$

then we say that S is a *topological space in the wider sense*.

The family of all sets of a topological space is clearly a Boolean algebra with respect to the operations $\cup$, $\cap$, and $-$. For this algebra however, in addition to the normal Boolean operations we have also a new operation—the operation of closure. Hence, by means of the same process of abstraction which took us from fields of sets to Boolean algebra, we go from topological spaces to closure algebras.

DEFINITION 1.1. *We say that a set K is a* closure algebra *with respect to the operations* $\cup$, $\cap$, $-$, *and* $\mathbf{C}$, *when*:

[5] See Kuratowski's paper referred to above, or the book by Alexandroff and Hopf. Alexandroff and Hopf call a space which satisfies (1), (2), and (3) a "T_1-space", and a space which satisfies (1), (2), and (3′) simply a "topological space".

1.11 *K is a Boolean algebra with respect to* $\cup$, $\cap$, *and* $-$,
1.12 *If x is in K, then $\mathbf{C}x$ is in K*,
1.13 *If x is in K, then $x \subseteq \mathbf{C}x$*,
1.14 *If x is in K, then $\mathbf{CC}x = \mathbf{C}x$*,
1.15 *If x and y are in K, then $\mathbf{C}(x \cup y) = \mathbf{C}x \cup \mathbf{C}y$*,
1.16 $\mathbf{C}\Lambda = \Lambda$.

When it is necessary in the interests of clarity, we shall use Greek capitals for closure algebras, and shall speak of the closure algebra

$$\Gamma = (K, \cup, \cap, -, \mathbf{C}).$$

Oftentimes, however, we shall use the more usual way of speaking, making such statements, for example, as that K is a closure algebra with respect to $\cup$, $\cap$, $-$, and $\mathbf{C}$.

As an immediate consequence of this definition[6] we have:

COROLLARY 1.2. *In any closure algebra*
(i) $\mathbf{C}\mathrm{V} = \mathrm{V}$,
(ii) *If $x \subseteq y$, then $\mathbf{C}x \subseteq \mathbf{C}y$.*

We shall use in connection with closure algebras certain terms with which the reader will be familiar from topology.

DEFINITION 1.3. *By the interior of an element x, is meant the element $\mathbf{I}x = -\mathbf{C}-x$.*

COROLLARY 1.4. *In any closure algebra*
(i) $\mathbf{I}x \subseteq x$,
(ii) $\mathbf{II}x = \mathbf{I}x$.
(iii) $\mathbf{I}(x \cap y) = \mathbf{I}x \cap \mathbf{I}y$,
(iv) If $x \subseteq y$, *then* $\mathbf{I}x \subseteq \mathbf{I}y$.

DEFINITION 1.5. *An element x is called closed, if $\mathbf{C}x = x$.*

DEFINITION 1.6. *An element x is called open, if $\mathbf{I}x = x$.*

COROLLARY 1.7. *In any closure algebra*
(i) *The complement of an open element is closed, and the complement of a closed element is open,*
(ii) *The sum of any finite number of closed elements is closed, and the sum of any number of open elements is open,*
(iii) *The product of any number of closed elements is closed, and the product of any finite number of open elements is open,*
(iv) *$\mathbf{C}x$ is closed and $\mathbf{I}x$ is open,*
(v) *Λ and V are both open and closed,*
(vi) $\mathbf{C}(\mathbf{C}x \cap \mathbf{C}y) = \mathbf{C}x \cap \mathbf{C}y$.

COROLLARY 1.8. *If x is any open element of a closure algebra and y is an arbitrary element, then*

[6] Proofs are to be found in Kuratowski's *Topologie* of the analogues for topological spaces of our 1.2, 1.4, 1.7, and 1.8. Since the arguments given by Kuratowski are algebraic in character (i.e., do not involve points, but only sets of points) they can be immediately carried over to closure algebras.

(i) $x \cap \mathbf{C}(x \cap y) = x \cap \mathbf{C}y$,

(ii) $x \cap y = \Lambda$ *implies* $x \cap \mathbf{C}y = \Lambda$.

We shall be concerned in this paper with certain special types of closure algebras which we are now going to define:

DEFINITION 1.9. *A closure algebra is called* <u>*connected*</u> *if* $\mathbf{C}x \cap \mathbf{C}-x = \Lambda$ *implies either* $x = \Lambda$ *or* $x = \mathrm{V}$

DEFINITION 1.10. *A closure algebra is called* <u>*well-connected*</u> *if* $\mathbf{C}x \cap \mathbf{C}y = \Lambda$ *implies either* $x = \Lambda$ *or* $y = \Lambda$.

DEFINITION 1.11. *A closure algebra is called* <u>*totally disconnected*</u> *if every non-empty open element is expressible as the sum of two mutually exclusive non-empty open elements.*

The notions of a connected and of a totally disconnected closure algebra are essentially known from topology (where of course they are applied not to closure algebras but to topological spaces). On the contrary that of a well-connected algebra is a new notion, which will prove very useful in the subsequent discussion. It can be illustrated by the following example: Let A_1, A_2, and A_3 be, respectively, the set of points interior to a certain circle of the Euclidean plane, the set of points exterior to the circle, and the points on the perimeter of the circle. Let $\mathfrak{K}$ be the family of sets consisting of the 8 sets, Λ, A_1, A_2, A_3, $A_1 \cup A_2$, $A_1 \cup A_3$, $A_2 \cup A_3$, and $A_1 \cup A_2 \cup A_3 = \mathrm{V}$. Then it is easily seen that $\mathfrak{K}$ is a closure algebra with respect to union, intersection, complementation, and closure. Moreover if X is any element of $\mathfrak{K}$ except Λ, we notice that $A_3 \subseteq \mathbf{C}X$; and hence $\mathfrak{K}$ is well-connected.

§2. Relation between Topological Spaces and Closure Algebras

The following theorem is an immediate consequence of the definitions of topological spaces and closure algebras.

THEOREM 2.1. *If S is a topological space (in either sense), then the family $\mathfrak{F}$ of all subsets of S is a closure algebra with respect to the set-theoretical operations of union, intersection, and complementation, and the topological operation of closure. The same is true of any field of subsets of S which is closed under the closure operation.*

DEFINITION 2.2. *For every topological space S, the family $\mathfrak{F}$ of Theorem 2.1 is called the closure algebra* <u>*over*</u> *S; we shall also sometimes say that $\mathfrak{F}$ is the closure algebra* <u>*determined*</u> *by S.*

We shall apply to closure algebras the general algebraic notion of subalgebras; thus a subalgebra of a closure algebra K is a subclass K_1 which is a closure algebra with respect to the same operations. Clearly it is sufficient that K_1 be closed under all the operations involved.

As an illustration of this notion, suppose that K is a closure algebra, and let K_1 be the class of those elements of K which are expressible as sums of finitely many products of an open element by a closed element: i.e., K_1 consists of all elements z such that

$$z = (x_1 \cap y_1) \cup \cdots \cup (x_n \cap y_n),$$

where each x_i is open and each y_i is closed. Making use of Corollary 1.7, it is easily shown that K_1 is a subalgebra of K. K_1 is the smallest subalgebra of K which contains all the closed elements of K.

As a second illustration, let K_2 consist of all elements x of K such that $\mathbf{C}x \cap \mathbf{C}-x = \Lambda$; K_2 can be shown to be a subalgebra of K by making use of Corollary 1.2(ii).

We can obviously apply to closure algebras the well-known algebraic notions of isomorphism and homomorphism. We are going to prove that every closure algebra is isomorphic with a subalgebra of the closure algebra over some topological space.

LEMMA 2.3. *Let K be a completely additive Boolean algebra; let $\mathbf{C}_1$ be a unary operation defined over a certain subclass K_1 of K in such a way that*

(i) $\Lambda \in K_1$, *and* $\mathbf{C}_1\Lambda = \Lambda$,

(ii) If $x \in K_1$, *then* $\mathbf{C}_1x \in K_1$, *and* $x \subseteq \mathbf{C}_1x = \mathbf{C}_1\mathbf{C}_1x$,

(iii) If $x \in K_1$ *and* $y \in K_1$, *then* $x \cup y \in K_1$, *and* $\mathbf{C}_1(x \cup y) = \mathbf{C}_1x \cup \mathbf{C}_1y$.

Then there is a closure operation $\mathbf{C}$ *such that*

(I) *K is a closure algebra with respect to $\mathbf{C}$ (and the original Boolean operations),*

(II) *If $x \in K_1$, then* $\mathbf{C}x = \mathbf{C}_1X$.

PROOF. We shall say in this proof that an element x of K is covered by an element y of K_1 if $x \subseteq y$ and $\mathbf{C}_1y = y$. We set $\mathbf{C}x$ equal to the product of all the elements which cover x. Since our Boolean algebra is completely additive, this product always exists. (In case no element covers x, we have $\mathbf{C}x = V$.) We shall show that K is a closure algebra with respect to $\mathbf{C}$, and that whenever x is in K_1 then $\mathbf{C}x = \mathbf{C}_1x$.

Since x is contained in every y which covers x, it is seen that $x \subseteq \mathbf{C}x$.

In particular, $\mathbf{C}x \subseteq \mathbf{C}\mathbf{C}x$. On the other hand, it is easily seen that every element which covers x also covers $\mathbf{C}x$; and hence $\mathbf{C}\mathbf{C}x \subseteq \mathbf{C}x$. From the two inclusions, we have $\mathbf{C}x = \mathbf{C}\mathbf{C}x$.

To see that $\mathbf{C}(x \cup y) = \mathbf{C}x \cup \mathbf{C}y$, let A_1, A_2, and A_3 be the sets of elements which cover x, y, and $x \cup y$ respectively. Then A_3 consists of just those elements which can be expressed as the sum of two elements y_1 and y_2 such that $y_1 \in A_1$ and $y_2 \in A_2$; and hence[7]

$$\mathbf{C}x \cup \mathbf{C}y = \bigcap_{y_1 \in A_1} y_1 \cup \bigcap_{y_2 \in A_2} y_2 = \bigcap_{y_1 \in A_1} \Big[\bigcap_{y_2 \in A_2} (y_1 \cup y_2) \Big] = \bigcap_{y \in A_3} y = \mathbf{C}(x \cup y),$$

as was to be shown.

Since Λ is covered by Λ, it is clear that $\mathbf{C}\Lambda = \Lambda$.

To prove (II), suppose that $x \in K_1$. Then x is covered by $\mathbf{C}_1x$, since $x \subseteq \mathbf{C}_1x$ and $\mathbf{C}_1\mathbf{C}_1x = \mathbf{C}_1x$; and hence $\mathbf{C}x \subseteq \mathbf{C}_1x$. On the other hand, let y be any ele-

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[7] It is well known that in every Boolean algebra not only the finite but also the infinite distributive laws hold, under the assumption that the sums and products involved exist; and hence they hold identically in every completely additive Boolean algebra. Cf. A. Tarski, *Grundzüge des Systemenkalküls*, First Part, Fundamenta Mathematicae, vol. 25 (1935), p. 510, footnote.

ment which covers x; then we have $x \subseteq y$ and $\mathbf{C}_1 y = y$; hence $\mathbf{C}_1 x \subseteq \mathbf{C}_1 y$, and therefore $\mathbf{C}_1 x \subseteq y$. Thus $\mathbf{C}_1 x$ is contained in every element which covers x, and hence $\mathbf{C}_1 x \subseteq \mathbf{C} x$. From the two inclusions we have $\mathbf{C} x = \mathbf{C}_1 x$, as was to be shown.

THEOREM 2.4. *Every closure algebra is isomorphic with a subalgebra of the closure algebra over a topological space in the wider sense.*

PROOF. Let K_2 be a closure algebra with respect to the operations $\cup_2$, $\cap_2$, $-_2$, and $\mathbf{C}_2$. By the well-known representation theorem for Boolean algebra,[8] K_2 is isomorphic as a Boolean algebra with a field of sets $\mathfrak{K}_2$, where $\cup_2$, $\cap_2$, and $-_2$ correspond to the set-theoretical operations $\cup$, $\cap$, and $-$ in $\mathfrak{K}_2$. We define a unary operation $\mathbf{C}_1$ over the members of $\mathfrak{K}_2$ as follows: if X is any member of $\mathfrak{K}_2$, if X coresponds to the element x of K_2, if $\mathbf{C}_2 x = y$, and if y corresponds to Y in $\mathfrak{K}_2$, then we put $\mathbf{C}_1 X = Y$. Then clearly the closure algebra K_2 is isomorphic with the closure algebra $\mathfrak{K}_2$. Hence our theorem will be proved if we can show that $\mathfrak{K}_2$ is isomorphic with a subalgebra of the closure algebra over a topological space in the wider sense.

Suppose all the sets of the family $\mathfrak{K}_2$ are subsets of a set S, and let $\mathfrak{K}$ be the family consisting of all subsets of S. Then $\mathfrak{K}$ is a completely additive Boolean algebra with respect to $\cup$, $\cap$, and $-$; and we see that the family $\mathfrak{K}_2$ and the operation $\mathbf{C}_1$ satisfy the hypothesis of Lemma 2.3. Hence there is a closure operation $\mathbf{C}$ such that $\mathfrak{K}$ is a closure algebra with respect to $\mathbf{C}$; and such that $\mathbf{C}X = \mathbf{C}_1 X$ for X in $\mathfrak{K}_2$. We see then that S is a topological space in the wider sense with respect to $\mathbf{C}$, and that our original algebra is isomorphic with a subalgebra of the closure algebra over this space.

THEOREM 2.5. *If S is any topological space in the wider sense, then there is a topological space S_1 (in the strict sense) such that the closure algebra over S is isomorphic with a subalgebra of the closure algebra over S_1.*[9]

PROOF. Let S be a topological space in the wider sense, with the closure operation $\mathbf{C}$. Let h be a function which is defined for every point of S, and which assumes as values infinite sets, in such a way that if x and y are distinct points of S then $h(x) \cap h(y) = \Lambda$. If X is any subset of S, we set

$$h(X) = \bigcup_{x \in X} h(x). \tag{1}$$

We set

$$S_1 = h(S). \tag{2}$$

We notice that

$$h(X \cup Y) = h(X) \cup h(Y). \tag{3}$$

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[8] See M. H. Stone, *The theory of representations for Boolean algebras*, Transactions of the American Mathematical Society, vol. 40 (1936), pp. 37–111.

[9] A similar theorem to this (but weaker, since S was supposed finite) was proved in McKinsey's paper, *A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology*, Journal of Symbolic Logic, vol. 6 (1941), pp. 117–134; see Theorem 20.

If X is any subset of S_1, then by $g(X)$ we shall mean the set of all points y of S such that $X \cap h(y)$ contains infinitely many elements. If X and Y are any subsets of S_1 and if z is any point of S, then to say that $(X \cup Y) \cap h(z)$ is infinite, is equivalent to saying that either $X \cap h(z)$ is infinite or $Y \cap h(z)$ is infinite; hence

$$g(X \cup Y) = g(X) \cup g(Y). \tag{4}$$

We notice that if X_1 is any finite subset of S_1 then

$$g(X_1) = \Lambda. \tag{5}$$

It is also easily seen, that if X is any subset of S, then

$$gh(X) = X. \tag{6}$$

We now set, for any subset X of S_1,

$$\mathbf{C}_1(X) = X \cup h\mathbf{C}g(X). \tag{7}$$

We are to show that S_1 is a topological space (in the strict sense) with respect to $\mathbf{C}_1$, and that the closure algebra over S is isomorphic with a subalgebra of the closure algebra over S_1.

It is clear from (7) that we have $X \subseteq \mathbf{C}_1(X)$.

From (7), (4), and (3) we have $\mathbf{C}_1(X \cup Y) = X \cup Y \cup h\mathbf{C}g(X \cup Y) = X \cup Y \cup h\mathbf{C}(g(X) \cup g(Y)) = X \cup Y \cup h(\mathbf{C}g(X) \cup \mathbf{C}g(Y)) = X \cup Y \cup h\mathbf{C}g(X) \cup h\mathbf{C}g(Y) = (X \cup h\mathbf{C}g(X)) \cup (Y \cup h\mathbf{C}g(Y)) = \mathbf{C}_1(X) \cup \mathbf{C}_1(Y)$.

From this last result, together with (7) and (6), we have $\mathbf{C}_1\mathbf{C}_1(X) = \mathbf{C}_1(X \cup h\mathbf{C}g(X)) = \mathbf{C}_1(X) \cup \mathbf{C}_1h\mathbf{C}g(X) = \mathbf{C}_1(X) \cup h\mathbf{C}g(X) \cup h\mathbf{C}gh\mathbf{C}g(X) = \mathbf{C}_1(X) \cup h\mathbf{C}gh\mathbf{C}g(X) = \mathbf{C}_1(X) \cup h\mathbf{C}\mathbf{C}g(X) = \mathbf{C}_1(X) \cup h\mathbf{C}g(X) = \mathbf{C}_1(X)$.

In order to complete the proof that S_1 is a topological space, we notice from (5) that, if X is a finite set, then $\mathbf{C}_1(X) = X \cup h\mathbf{C}g(X) = X \cup h\mathbf{C}\Lambda = X \cup h\Lambda = X \cup \Lambda = X$.

The function h establishes an isomorphism between the closure algebra over S and a certain subalgebra of the closure algebra over S_1. This is obvious so far as concerns the Boolean operations. And on the other hand we have $\mathbf{C}_1h(X) = h(X) \cup h\mathbf{C}gh(X) = h(X) \cup h\mathbf{C}(X) = h(X \cup \mathbf{C}(X)) = h\mathbf{C}(X)$. This completes the proof.

THEOREM 2.6. *Every closure algebra is isomorphic with a subalgebra of the closure algebra over a topological space* (*in the strict sense*).

PROOF. From Theorem 2.4 and Theorem 2.5.

From Theorems 2.1, 2.4, and 2.6 we see that a topological equation is true in every closure algebra if and only if it is true in every topological space.

§3. Universal Algebras

If in a given algebra K we can construct models for all closure algebras of a given class $\mathfrak{A}$, then we call K a *universal* algebra with respect to $\mathfrak{A}$. Using more technical terminology, we say that K is a universal algebra for all algebras of the class $\mathfrak{A}$ of algebras, if every algebra of the class $\mathfrak{A}$ is isomorphic with a subalgebra of K. As applied specifically to closure algebras, this general algebraic notion can be subjected to a certain extension. In order to make this extension, we first introduce the notion of a relativized subalgebra, which corresponds to the familiar notion of a relativized topological space.[10]

DEFINITION 3.1. *If* $\Gamma = (K, \cup, \cap, -, \mathbf{C})$ *is any closure algebra and* a *is any element of* K *such that* $a \neq \Lambda$, *then by the* <u>*relativized subalgebra of* Γ *with respect to* a</u>, *in symbols* Γ_a, *we understand the algebra constituted by the class* K_a *of all elements* x *of* K *such that* $x \subseteq a$, *by the original operations* $\cup$ *and* $\cap$, *and by the unary operations* $-_a$ *and* $\mathbf{C}_a$ *determined by the formulas*: $-_a x = a \cap -x$ *and* $\mathbf{C}_a x = a \cap \mathbf{C}x$.

As an immediate consequence we have:

COROLLARY 3.2. *If* $\Gamma = (K, \cup, \cap, -, \mathbf{C})$ *is a closure algebra, and* a *is any non-empty element of* K, *then* Γ_a *is a closure algebra. If* a *is open, then the open elements of* Γ_a *are at the same time open elements of* Γ.

PROOF. The proof of the first part is almost immediate. To prove the second part, we make use of Corollary 1.8(i).

Now we define:

DEFINITION 3.3. *By saying that an algebra* Γ *is a* <u>*generalized universal algebra*</u> *for a class* $\mathfrak{A}$ *of closure algebras, we mean that for each algebra* Δ *of the class* $\mathfrak{A}$ *there is an open element* a *of* Γ *such that* Δ *is isomorphic with a subalgebra of* Γ_a.

In this section we shall be concerned primarily with universal algebras for all finite algebras. The main result in this section will be Theorem 3.12, which implies that the closure algebra over Euclidean space is an algebra of this sort in the generalized sense just defined. In establishing this theorem we use only a few rather special algebraic properties of Euclidean space, and these same properties will be necessary for a further algebraic study of Euclidean space. However these properties are rather involved, and it is by no means obvious that they apply to Euclidean space. We shall denote a closure algebra having these properties as a *dissectable* algebra. We shall first show that the closure algebra over Euclidean space is a dissectable closure algebra (though the converse does not hold), and then that every dissectable algebra is a generalized universal algebra for all finite algebras. If the definition of dissectable algebras which we are now going to give seems rather strange, it should be kept in mind that this notion is merely an instrument which facilitates the study of Euclidean space.

DEFINITION 3.4. *A closure algebra* K *is said to be* <u>*dissectable*</u> *if, for every non-empty open element* a *of* K, *and for every pair of integers* r *and* s, *where* $r \geqq 0$ *and* $s > 0$, *there are* $r + s$ *non-empty, mutually exclusive elements* $a_1, \cdots, a_r, b_1, \cdots, b_s$ *of* K *such that*

(i) *The elements* $a_1, \cdots, a_r$ *are all open,*

(ii) $\mathbf{C}b_1 = \cdots = \mathbf{C}b_s$,

(iii) $a_1 \cup \cdots \cup a_r \cup b_1 \cup \cdots \cup b_s = a$,

(iv) $\mathbf{C}a \cap -a \subseteq \mathbf{C}b_i \subseteq \mathbf{C}a_j$ *for* $i \leqq s$ *and* $j \leqq r$.

---

[10] See Kuratowski, *Topologie*, p. 17.

It should be especially noticed that in the above definition we do not exclude the case $r = 0$; thus any non-empty open element a of a dissectable closure algebra can be represented as the sum of s elements $b_1, \cdots, b_s$ such that $\mathbf{C}b_1 = \cdots = \mathbf{C}b_s$, and such that $\mathbf{C}a \cap -a \subseteq \mathbf{C}b_i$ for each $i \leqq s$.

Making use of the second part of Corollary 3.2, we see as an almost immediate consequence of this definition that if Γ is a dissectable closure algebra and a is any non-empty open element of Γ then Γ_a is also dissectable.

In order to formulate our next theorem, it is necessary to recall a few special notions of topology.

If x is a point of a topological space, then we say that x is a *limit-point* of a subset A of the space, if $x \in \mathbf{C}(A - \{x\})$. A topological space S is called *dense-in-itself* if every point of S is a limit-point of S.

A topological space is said to be *normal* if, for every two subsets X_1 and X_2 of S such that $\mathbf{C}X_1 \cap \mathbf{C}X_2 = \Lambda$, there are two mutually exclusive open subsets Y_1 and Y_2 of S such that $\mathbf{C}X_1 \subseteq Y_1$ and $\mathbf{C}X_2 \subseteq Y_2$.

A topological space is said to have a *countable basis*, if there exists an infinite sequence $X_1, X_2, \cdots, X_n, \cdots$ of non-empty open subsets of S, such that every non-empty open subset X of S can be represented in the form

$$X = X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_n} \cup \cdots$$

where $i_1, i_2, \cdots, i_n, \cdots$ is a sequence of positive integers.

(The above notions could also be applied to closure algebras, but we do not need them in that connection.)

Theorem 3.5. *The closure algebra over every normal, dense-in-itself topological space with a countable basis is dissectable.*[11]

Proof. Let S be a normal, dense-in-itself topological space with a countable basis (with the closure operation $\mathbf{C}$), let A be any non-empty open subset of S, and let r and s be two integers with $r \geqq 0$ and $s > 0$. Then we are to show that there are $r + s$ non-empty mutually exclusive subsets $A_1, \cdots, A_r, B_1, \cdots, B_s$ of A which satisfy the four conditions of Definition 3.4.

By a well-known topological theorem, the conditions imposed on S in the hypothesis of the theorem imply that the space S is metrizable. That is to say, there is a *distance-function* $d(x, y)$, defined for all points x and y of S, assuming non-negative real numbers as values, and satisfying the following conditions (for all x, y, and z in S):

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) + d(y, z) \geqq d(x, z)$,

(iv) If $X \subseteq S$, then $x \in \mathbf{C}X$ if and only if for every real number $\varepsilon > 0$ there exists a $y \in X$ such that $d(x, y) < \varepsilon$.

[11] This theorem is closely analogous to a theorem to be found in Tarski's *Der Aussagenkalkül und die Topologie*, Fundamenta Mathematicae, vol. 31 (1938), pp. 103–134; see Satz 3.10. The present theorem is somewhat stronger, however, than the theorem given there, the proof of which, in its most general form, was due to S. Eilenberg.

It can also be concluded that S contains a countable subset $E = \{e_1, e_2, \cdots\}$ such that every point of S is a limit-point of E.

If x is any point of S, and if X is any non-empty subset of S, then by $d(x, X)$ we shall mean the greatest lower bound of the set of numbers $d(x, y)$ where $y \in X$; we set

$$d(x, \Lambda) = 1.$$

If X and Y are non-empty subsets of S, then by $d(X, Y)$ we shall mean the greatest lower bound of the set of numbers $d(x, y)$, where $x \in X$ and $y \in Y$; we set

$$d(X, \Lambda) = d(\Lambda, X) = d(\Lambda, \Lambda) = 1.$$

If X is any non-empty subset of S, then by $m(X)$ we shall mean the greatest lower bound of the set of numbers $d(x, X - \{x\})$, where $x \in X$; we set $m(\Lambda) = 1$.

If x is any point of S, and λ is any positive real number, then by $Sp(x, \lambda)$ we shall mean the set of all points y of S such that $d(x, y) \leqq \lambda$. It is clear that $Sp(x, \lambda)$ is a closed set, and that $x \in Sp(x, \lambda)$.

We shall now define, for each non-negative integer n, certain positive real numbers ε_n, δ_n, as well as certain sets U_n, V_n, $H_n^{(1)}, \cdots, H_n^{(r)}, K_n^{(1)}, \cdots, K_n^{(s)}$.

We put $\varepsilon_0 = \delta_0 = 1$, $V_0 = H_0^{(1)} = \cdots = H_0^{(r)} = K_0^{(1)} = \cdots = K_0^{(s)} = \Lambda$, and $U_0 = A$, where A is our given non-empty open set.

Supposing now that these numbers and sets have all been defined for $n = k$ (and supposing that U_k is a non-empty open set) we shall define them for $n = k + 1$. Let u and v be the first two elements of the countable set E which are in U_k. Then we denote by ε_{k+1} one-third the smallest of the three numbers

$$\frac{1}{k+1}, \qquad d(u, -U_k), \qquad \text{and } d(u, v);$$

i.e., we set

$$\varepsilon_{k+1} = 1/3 \min \left[\frac{1}{k+1}, d(u, -U_k), d(u,v)\right].$$

Let $x_1, \cdots, x_{r+s}$ be the first $r + s$ elements of the sequence E which are in $Sp(u, \varepsilon_{k+1})$. Then we set

$$\delta_{k+1} = 1/3 \min [d(\{x_1, \cdots, x_{r+s}\}, -Sp(u, \varepsilon_{k+1})), m(\{x_1, \cdots, x_{r+s}\})\Lambda].$$

We put

$$H_{k+1}^{(1)} = Sp(x_1, \delta_{k+1}), \cdots, H_{k+1}^{(r)} = Sp(x_r, \delta_{k+1}),$$

and

$$K_{k+1}^{(1)} = \{x_{r+1}\}, \cdots, K_{k+1}^{(s)} = \{x_{r+s}\}.$$

Finally we set

$$V_{k+1} = H_{k+1}^{(1)} \cup \cdots \cup H_{k+1}^{(r)} \cup K_{k+1}^{(1)} \cup \cdots \cup K_{k+1}^{(s)},$$

and

$$U_{k+1} = U_k - V_{k+1}.$$

Now, making use of the fact that $C_0 = A$ is by hypothesis a non-empty open set, it is easily shown that, for every n, U_n is a non-empty open set, and that all the numbers and sets ε_n, δ_n, U_n, V_n, $H_n^{(1)}, \cdots, H_n^{(r)}, K_n^{(1)}, \cdots, K_n^{(s)}$ are actually defined for each n. Then the following can easily be proved by mathematical induction:

(1) If $x \in V_n$ and $y \in V_n$, then $d(x, y) \leqq 1/n$,

(2) $V_{n+1} \subseteq U_n \subseteq A$, and hence $V_n \cap \mathbf{C}A \cap -\bar{A} = \Lambda$,

(3) The sets $H_n^{(1)}, \cdots, H_n^{(r)}, K_n^{(1)}, \cdots, K_n^{(s)}$ are mutually exclusive,

(4) If $n \neq 0$, then $\mathbf{I}H_n^{(m)} \neq \Lambda$, and $K_n^{(m)} \neq \Lambda$.

We now define sets $A_1, \cdots, A_r$ and $B_1, \cdots, B_s$ as follows:

(5)
$$
\begin{aligned}
A_1 &= \mathbf{I}H_1^{(1)} \cup \cdots \cup \mathbf{I}H_n^{(1)} \cup \cdots \\
&\ \ \vdots \\
A_r &= \mathbf{I}H_1^{(r)} \cup \cdots \cup \mathbf{I}H_n^{(r)} \cup \cdots \\
B_1 &= K_1^{(1)} \cup \cdots \cup K_n^{(1)} \cup \cdots \\
&\ \ \vdots \\
B_{s-1} &= K_1^{(s-1)} \cup \cdots \cup K_n^{(s-1)} \cup \cdots \\
B_s &= A - (A_1 \cup \cdots \cup A_r \cup B_1 \cup \cdots \cup B_{s-1}).
\end{aligned}
$$

From (3), (4), and (5) we immediately see that the sets $A_1, \cdots, A_r, B_1, \cdots, B_s$ are non-empty and mutually exclusive. We see by Theorem 1.10 and Theorem 1.5 that each A_i is open. It is clear from the definition of B_s that

$$A_1 \cup \cdots \cup A_r \cup B_1 \cup \cdots \cup B_s = A.$$

In order to complete our proof, it remains therefore only to show that conditions (ii) and (iv) of Definition 3.4 are satisfied by the sets $A_1, \cdots, A_r, B_1, \cdots, B_s$.

By means of a familiar kind of argument involving limits of sequences of points, it can be shown that:

(6) $$H_n^{(m)} \cap \mathbf{C}-H_n^{(m)} \subseteq \mathbf{C}A_i \quad \text{and} \quad H_n^{(m)} \cap \mathbf{C}-H_n^{(m)} \subseteq \mathbf{C}B_j,$$

for every m and n, and for $1 \leqq i \leqq r$ and $1 \leqq j \leqq s$.

From (6) we derive, by a similar argument:

(7) $$A - (A_1 \cup \cdots \cup A_r) \subseteq \mathbf{C}A_i \quad \text{and} \quad A - (A_1 \cup \cdots \cup A_r) \subseteq \mathbf{C}B_j$$

for $1 \leqq i \leqq r$ and $1 \leqq j \leqq s$;

and moreover

(8) $$\mathbf{C}A \cap -A \subseteq \mathbf{C}A_i$$

for $1 \leqq i \leqq r$.

From (7) we easily see that condition (ii) of Definition 3.4 is satisfied. For since $B_i \subseteq A - (A_1 \cup \cdots \cup A_r)$, we see by (7) that $B_i \subseteq \mathbf{C}B_j$, and hence that $\mathbf{C}B_i \subseteq \mathbf{C}B_j$, as was to be shown.

In a similar way we see from the first part of (7) that $\mathbf{C}B_i \subseteq \mathbf{C}A_j$. From this, together with (8), we see that condition (iv) of Definition 3.4 is satisfied, which completes the proof of our theorem.

Since it is well-known that Euclidean space is dense-in-itself, normal, and possesses a countable basis, we have the following corollary:

COROLLARY 3.6. *The closure algebra over Euclidean space is dissectable.*

THEOREM 3.7. *Every dissectable closure algebra is a universal algebra for the class of all well-connected finite closure algebras.*

PROOF. It is convenient to prove the following slightly more general theorem (from which our desired theorem results immediately by setting $a = V$):

If $\Gamma = (K, \cup, \cap, -, \mathbf{C})$ is a dissectable closure algebra, and a is an open non-empty element of Γ, and if $\Phi = (K', \cup', \cap', -', \mathbf{C}')$ is any finite well-connected closure algebra, then there is a subalgebra Δ of Γ_a, such that

(i) Δ is isomorphic with Φ,

(ii) $\mathbf{C}a \cap - a \subseteq \mathbf{C}x$ for every non-empty x of Δ.

We shall prove this latter theorem by an induction with respect to the number of atoms of Φ. If there is just one atom, then the theorem is obvious. Hence we shall suppose the theorem is true for every finite algebra with less than p atoms, and that Φ contains just p atoms.

Since Φ is well-connected, there is an atom b_1 of K which is contained in every non-empty closed element of Φ. Let $b_2, \cdots, b_k$ be the other atoms (if any) such that $\mathbf{C}'b_1 = \mathbf{C}'b_2 = \cdots = \mathbf{C}'b_k$.

It is then easily seen that $\mathbf{C}'b_1 = \cdots = \mathbf{C}'b_k = b_1 \cup' \cdots \cup' b_k$. For first it is clear that $b_1 \cup' \cdots \cup' b_k \subseteq \mathbf{C}'(b_1 \cup' \cdots \cup' b_k) = \mathbf{C}'b_1 \cup' \cdots \cup' \mathbf{C}'b_k = \mathbf{C}'b_1$. And moreover, if x is any atom contained in $\mathbf{C}'b_1$ then $\mathbf{C}'x \subseteq \mathbf{C}'\mathbf{C}'b_1 = \mathbf{C}'b_1$; and since by hypothesis $b_1 \subseteq x$, we have also $\mathbf{C}'b_1 \subseteq \mathbf{C}'x$, and hence $\mathbf{C}'x = \mathbf{C}'b_1$; thus for some i we have $x = b_i$.

Let $c_1, \cdots, c_q$ be the other atoms (if any) of Φ besides the atoms $b_1, \cdots, b_k$. Thus $k + q = p$, and hence (since $k \neq 0$) we have $q < p$.

We now choose from among the atoms $c_1, \cdots, c_q$ a set $d_1, \cdots, d_n$ of atoms in the following way. Let d_1 be the first atom in the sequence $c_1, \cdots, c_q$ whose closure does not contain as proper part the closure of any atom c_i. Let d_2 be the first atom in the sequence $c_1, \cdots, c_q$ whose closure is different from the closure of d_1, and whose closure does not contain as proper part the closure of any atom c_i. And so on.

It will be seen that for every $j \leqq q$ there is an $i \leqq n$ such that $d_i \subseteq \mathbf{C}'d_i \subseteq \mathbf{C}'c_j$. Let e_i be the sum of all atoms c_j such that $d_i \subseteq \mathbf{C}'c_j$; i.e.,

$$(1) \qquad e_i = \bigcup_{d_i \subseteq \mathbf{C}'c_j} c_j \quad \text{for} \quad i = 1, \cdots, u.$$

Thus

$$(2) \qquad e_1 \cup' \cdots \cup' e_n = c_1 \cup' \cdots \cup' c_q.$$

Moreover we shall put

$$(3)\qquad e_0 = b_1 \cup' \cdots \cup' b_k ;$$

so that

$$(4)\qquad e_0 \cup' e_1 \cup' \cdots \cup' e_n = V.$$

It can easily be shown that e_i is open for $i \geqq 1$.

Since Γ is a dissectable algebra, there are non-empty, mutually exclusive elements $a_1, \cdots, a_n, f_1, \cdots, f_k$ of Γ which satisfy the conditions:

$$(5)\qquad \text{the elements } a_1, \cdots, a_n \text{ are open,}$$

$$(6)\qquad \mathbf{C}f_1 = \cdots = \mathbf{C}f_k,$$

$$(7)\qquad a_1 \cup \cdots \cup a_n \cup f_1 \cup \cdots \cup f_k = a,$$

$$(8)\qquad \mathbf{C}a \cap -a \subseteq \mathbf{C}f_i \subseteq \mathbf{C}a_j \quad \text{for} \quad i \leqq k \quad \text{and} \quad j \leqq n.$$

We shall set

$$(9)\qquad a_0 = f_1 \cup \cdots \cup f_k .$$

It is then easily shown that

$$(10)\qquad \mathbf{C}f_1 = \cdots = \mathbf{C}f_k = \mathbf{C}a.$$

Let Δ_0 be the subalgebra of Γ_{a_0} which consists of all sums formed from the elements $f_1, \cdots, f_k$ (including also the null-element). Thus $\Delta_0 = (K_0, \cup, \cap, -_0, \mathbf{C}_0)$, where K is the set of all elements expressible as sums of the elements $f_1, \cdots, f_k$, and where $-_0 x = a_0 \cap -x$ and $\mathbf{C}_0 x = a_0 \cap \mathbf{C}x$, for all x in K_0.

Moreover, we define a function h in the following way. If

$$x = b_{i_1} \cup' \cdots \cup' b_{i_q}$$

is any element of Φ_{e_0}, then we set

$$(11)\qquad h_0(x) = f_{i_1} \cup \cdots \cup f_{i_q} ;$$

we set $h(\Lambda) = \Lambda$. It can now be shown that the function h establishes an isomorphism between Φ_{e_0} and Δ_0; that is to say, if x and y are any elements of Φ_{e_0}, then

$$(12)\qquad h_0(x \cup' y) = h_0(x) \cup h_0(y),$$

$$(13)\qquad h_0(x \cap' y) = h_0(x) \cap h_0(y),$$

$$(14)\qquad h_0(-'_{e_0} x) = -_0 h_0(x),$$

$$(15)\qquad h_0(\mathbf{C}'_{e_0} x) = \mathbf{C}_0 h_0(x).$$

The first three of these equations are clearly true from Boolean algebra. To establish the last one, we need only observe that if $x = \Lambda$, then both sides are equal to Λ; and if $x \neq \Lambda$, then both sides are equal to a_0.

Let i be an arbitrary one of the integers from 1 to n. We notice that the relativized algebra Φ_{e_i} is well-connected; for if x is any element of Φ_{e_i} except Λ, we see from (1) that $d_i \subseteq \mathbf{C}'_{e_i}x$. Moreover the number of atoms in Φ_{e_i} is at most equal to q and so is less than p. Hence our induction hypothesis applies to Φ_{e_i}. Since a_i is an open element of Γ, we therefore see that there is a subalgebra $\Delta_i = (K_i, \cup, \cap, -_i, \mathbf{C}_i)$ of Γ_{a_i} such that

(16) Φ_{e_i} is ismorphic with Δ_i

(17) $\mathbf{C}a \cap -a \subseteq \mathbf{C}x$ for every non-empty x of Δ_i.

Thus there is a function h_i, which has a single-valued inverse and satisfies the following conditions, for every x and y in Φ_{e_i}:

(18) $$h_i(x \cup' y) = h_i(x) \cup h_i(y),$$

(19) $$h_i(x \cap' y) = h_i(x) \cap h_i(y),$$

(20) $$h_i(-'_{e_i}x) = -_ih_i(x),$$

(21) $$h_i(\mathbf{C}'_{e_i}x) = \mathbf{C}_ih_i(x).$$

We now define a function h as follows (where x is an arbitrary element of Φ):

(22) $$h(x) = h_0(x \cap' e_0) \cup \cdots \cup h_n(x \cap' e_n).$$

It is seen that $h(x)$ is always an element of Γ_a. It is easily shown that if $h(x) = h(y)$ then $x = y$. Moreover, remembering that $a_i \cap a_j = \Lambda$ for $i \neq j$, and that a_i is open for $i \geqq 1$, we can prove the following (making use also of equations (12)–(15) and (18)–(21)):

(23) $$h(x \cup' y) = h(x) \cup h(y),$$

(24) $$h(x \cap' y) = h(x) \cap h(y),$$

(25) $$h(-'x) = -_ah(x),$$

(26) $$h(\mathbf{C}'x) = \mathbf{C}_ah(x).$$

Thus if we set $\Delta = (K'', \cup, \cap, -_a, \mathbf{C}_a)$, where K'' is the set of values assumed by $h(x)$, then we see that Δ is ismorphic with Φ. Since Δ is isomorphic with Φ, it also follows that Δ is a closure algebra, and hence a subalgebra of Γ. Making use of equations (8) and (17) we can also show that

(27) $\mathbf{C}a \cap -a \subseteq \mathbf{C}x$ for every non-empty x in Δ.

Thus our theorem is also true for finite algebras which contain p atoms, and hence by mathematical induction is true generally.

THEOREM 3.8. *Every totally disconnected dissectable closure algebra is a universal algebra for the class of all finite closure algebras.*

PROOF. Let $\Gamma = (K, \cup, \cap, -, \mathbf{C})$ be a totally disconnected dissectable closure algebra; we shall show that every closure algebra is isomorphic with a subalgebra of Γ. We shall carry out the proof by an induction on the number of atoms in the finite algebra. If the finite algebra contains just one atom, then it is obviously isomorphic with a subalgebra of Γ. Hence we suppose that every finite closure algebra with less than p atoms is isomorphic with a subalgebra of Γ, and we let $\Phi = (K', \cup', \cap', -', \mathbf{C}')$ be a closure algebra with just p atoms.

If Φ is well-connected, then Φ is isomorphic with a subalgebra of Γ by Theorem 3.7. Hence we can suppose that Φ is not well-connected.

Let $c_1, \cdots, c_p$ be the atoms of Φ. We choose from among these atoms, as in the proof of Theorem 3.7, a set $d_1, \cdots, d_n$ such that: $\mathbf{C}'d_i \neq \mathbf{C}'d_j$ for $i \neq j$; $\mathbf{C}'d_i$ does not contain as proper part any $\mathbf{C}'c_j$; and for every j there exists an i such that $d_i \subseteq \mathbf{C}'c_j$. As in the proof of Theorem 3.7, we set

$$(1) \qquad e_i = \bigcup_{d_i \subseteq \mathbf{C}'c_j} c_j \quad \text{for} \quad i = 1, \cdots, n.$$

We see that

$$(2) \qquad e_1 \cup' \cdots \cup' e_n = c_1 \cup' \cdots \cup' c_p = \mathrm{V}.$$

Moreover, since Φ is not well-connected, we see that, for every $i \leqq n$, $e_i \neq \mathrm{V}$, and hence that the number of atoms contained in e_i is less than p. Thus our induction hypothesis applies to each of the finite algebras $\Phi_{e_1}, \cdots, \Phi_{e_n}$

Since Γ is totally disconnected, it is easily seen that there are n non-empty mutually exclusive open elements $a_1, \cdots, a_n$ in Γ such that

$$(3) \qquad a_1 \cup \cdots \cup a_n = \mathrm{V}.$$

Since a_i is open, we see that Γ_{a_i} is a totally disconnected closure algebra. Hence by the induction hypothesis we see that there is a subalgebra Δ_i of Γ_{a_i} such that Δ_i is isomorphic with Φ_{l_i}. Let h_i be the function which establishes this isomorphism, so that we have:

$$(4) \qquad h_i(x \cup' y) = h_i(x) \cup h_i(y),$$

$$(5) \qquad h_i(x \cap' y) = h_i(x) \cap h_i(y),$$

$$(6) \qquad h_i(-'_{e_i}x) = -_{a_i}h_i(x),$$

$$(7) \qquad h_i(\mathbf{C}'_{e_i}x) = \mathbf{C}_{a_i}h_i(x).$$

We now define a function h as follows (for x any element of Φ):

$$(8) \qquad h(x) = h_1(x \cap' e_1) \cup \cdots \cup h_n(x \cap' e_n).$$

As in the proof of Theorem 3.7, we can show that this function establishes an isomorphism between Φ and a subalgebra Δ of Γ.

From the above theorem we see that the closure algebra determined by any dense-in-itself and totally disconnected subspace of Euclidean space (for example, by Cantor's discontinuum, or by the set of all points with rational coordinates) is a universal algebra for all finite algebras. Finally we shall consider the problem whether every dissectable algebra is a universal algebra in the generalized sense for all finite algebras; and we shall show that the answer to this question is positive. To obtain this result we need some lemmas.

LEMMA 3.9. *Let K be a closure algebra (with respect to $\cup$, $\cap$, $-$, and $\mathbf{C}$), and let K^* be the set of all couples $\langle x, y\rangle$, where x is an arbitrary element of K, and y is either Λ or V. Let the operations $\cup^*$, $\cap^*$, $-^*$, and $\mathbf{C}^*$ be defined as follows:*

(i) $\langle x, y\rangle \cup^* \langle u, v\rangle = \langle x \cup u, y \cup v\rangle,$

(ii) $\langle x, y\rangle \cap^* \langle u, v\rangle = \langle x \cap u, y \cap v\rangle,$

(iii) $-^*\langle x, y\rangle = \langle -x, -y\rangle,$

(iv) $\mathbf{C}^*\langle x, y\rangle = \langle \mathbf{C}x, V\rangle$, *unless* $x = \Lambda$ *and* $y = \Lambda$,

(v) $\mathbf{C}^*\langle \Lambda, \Lambda\rangle = \langle \Lambda, \Lambda\rangle.$

Then K^* is a well-connected closure algebra with respect to $\cup^*$, $\cap^*$, $-^*$, and $\mathbf{C}^*$.

PROOF. To see that K^* is a closure algebra, it is only necessary to verify that conditions 1.11–1.16 are satisfied. It will be noticed that the null-element of this closure algebra is $\langle \Lambda, \Lambda\rangle$, and the universe-element is $\langle V, V\rangle$.

By (iv) it is seen that the closure of every element except $\langle \Lambda, \Lambda\rangle$ contains the element $\langle \Lambda, V\rangle$. Hence, by Definition 1.10, K^* is well-connected.

LEMMA 3.10. *If K and K^* are related as in Lemma 3.9, then K is isomorphic with K^* relativized to a certain open element; in fact, K is isomorphic with $(K^*)_{\langle V, \Lambda\rangle}$.*

PROOF. It is clear that $\langle V, \Lambda\rangle$ is open, since $\langle V, \Lambda\rangle$ is closed. To establish the isomorphism, let the element x of K correspond to the element $\langle x, \Lambda\rangle$ of $(K^*)_{\langle V, \Lambda\rangle}$.

LEMMA 3.11. *If K is a dissectable closure algebra, and if K and K^* are related as in Lemma 3.9, then K^* is also dissectable.*

PROOF. The closed elements of K^* consist of the $\langle \Lambda, \Lambda\rangle$, together with all couples $\langle x, V\rangle$, where x is a closed element of K. Hence the open elements of K^* consist of the couple $\langle V, V\rangle$, together with all couples $\langle x, \Lambda\rangle$, where x is an open element of K.

Hence we can see that K^* is dissectable if K is dissectable, as follows. If $\langle a, \Lambda\rangle$ is an open element of K^* then a is an open element of K. Hence for $r \geqq 0$ and $s > 0$, there are elements $a_1, \cdots, a_r, b_1, \cdots, b_s$ of K whose sum is a, and which satisfy the other conditions of Definition 3.4. Then we can easily show that the elements $\langle a_1, \Lambda\rangle, \cdots, \langle a_r, \Lambda\rangle, \langle b_1, \Lambda\rangle, \cdots, \langle b_s, \Lambda\rangle$ of K^* have $\langle a, \Lambda\rangle$ for their sum, and satisfy the other conditions of Definition 3.4. If we consider the open element $\langle V, V\rangle$ of K^*, on the other hand, we need to modify the construction only slightly; namely, we now consider the elements

$$\langle a_1, \Lambda\rangle, \cdots, \langle a_r, \Lambda\rangle, \langle b_1, \Lambda\rangle, \cdots, \langle b_{s-1}, \Lambda\rangle, \langle b_s, V\rangle.$$

THEOREM 3.12. *Every dissectable closure algebra is a generalized universal algebra for the class of all finite closure algebras.*

PROOF. Let K be any dissectable closure algebra, and let H be any finite algebra; we are to show that there is an open element a of K such that H is isomorphic with a sublagebra of K_a. Let H^* be the closure algebra related to H as described in Lemma 3.9. It is clear that H^* is also finite; and from Lemma 3.9 we see that H^* is well-connected. Hence, by Theorem 3.7, H* is isomorphic with a subalgebra G of K. Since the couple $\langle V, \Lambda \rangle$ is an open element of H^*, we see that under this isomorphism $\langle V, \Lambda \rangle$ must correspond to an open element a of G. Then $(H^*)_{\langle V,\Lambda\rangle}$ is isomorphic with G_a; and hence with a subalgebra of K_a. Since, by Lemma 3.10, H is isomorphic with $(H^*)_{\langle V,\Lambda\rangle}$, we see that H is isomorphic with a subalgebra of K_a, as was to be shown.

The above lemmas also enable us to answer negatively a certain question which arises naturally regarding universal algebras—the question wnether Theorem 3.7 could be strengthened to assert that every dissectable closure algebra is a universal algebra for all connected finite algebras. To see that this is not the case, we need only observe that Theorem 3.5 and Lemmas 3.9 and 3.11 imply the existence of well-connected dissectable algebras. Now if K is a well-connected dissectable algebra, it is seen that every finite subalgebra of K must also be well-connected; hence K cannot be a universal algebra for all connected finite algebras, since there exist connected finite algebras which are not well-connected. (The question remains open, however, whether the closure algebra over Euclidean space is a universal algebra for all connected finite algebras.)

§4. Closure-Algebraic Functions

We shall concern ourselves in this section with functions f of n variables (n-ary operations) which correlate with every sequence of n elements $a_1, \cdots, a_n$ of each closure algebra an element $f(a_1, \cdots, a_n)$ of that algebra; we shall call these functions simply *closure* functions. Each such function will be conceived as defined over all possible closure algebras, so that strictly speaking it is a function of $n + 1$ variables—the first variable denoting the whole algebra, and the n remaining ones denoting elements of this algebra. Consequently if for instance Γ is a closure algebra, then the function value b of Γ which corresponds to the argument values $a_1, \cdots, a_n$ could be symbolized by

$$b = f_\Gamma(a_1, \cdots, a_n).$$

In practice, however, the symbol "Γ" will often be omitted, and f will be referred to as a function of n arguments. As examples we can take the n *identity functions* of n variables:

$$f_\Gamma^{(i)}(x_1, \cdots, x_n) = x_i \quad \text{for} \quad 1 \leqq i \leqq n.$$

Furthermore the functions of one variable

$$f_\Gamma(x) = \mathbf{C}_\Gamma(x) \quad \text{and} \quad g_\Gamma(x) = -_\Gamma(x),$$

and the functions of two variables

$$h_\Gamma(x, y) = x \cup_\Gamma y \quad \text{and} \quad k_\Gamma(x, y) = x \cap_\Gamma y.$$

In this whole context a closure algebra Γ is considered not merely as a set of elements, but rather as a system consisting of this set together with the four fundamental operations, which are denoted here by:

$$\mathsf{C}_\Gamma, \ -_\Gamma, \ \cup_\Gamma, \text{ and } \cap_\Gamma .$$

Two functions f and g are of course said to be (*identically*) *equal*,

$$f = g,$$

when

$$f_\Gamma(x_1, \cdots, x_n) = g_\Gamma(x_1, \cdots, x_n)$$

for every algebra Γ and for all elements $x_1, \cdots, x_n$ of Γ. To express the fact that the latter equation holds for all elements $x_1, \cdots, x_n$ of a given algebra Γ, we shall say that f and g are (*identically*) *equal in* Γ, or that f and g are (*identically*) *equal in* K, where K is the class of elements of the algebra Γ.

Among all closure functions, we can easily single out a special category which will be referred to as *inner algebraic functions of closure algebras*, or briefly as *closure-algebraic functions*. With this in view, we first define certain operations on functions corresponding to the fundamental operations of closure algebra.

DEFINITION 4.1. *Let f and g be two closure functions; then by $f \cup g$ and $f \cap g$ we understand the functions h' and h'' determined by the formulas*

$$h'_\Gamma(x_1, \cdots, x_n) = f_\Gamma(x_1, \cdots, x_n) \cup g_\Gamma(x_1, \cdots, x_n)$$

$$h''_\Gamma(x_1, \cdots, x_n) = f_\Gamma(x_1, \cdots, x_n) \cap g_\Gamma(x_1, \cdots, x_n)$$

(*for every closure algebra Γ and for all elements $x_1, \cdots, x_n$ of Γ*). *Similarly by $-f$ and $\mathsf{C}f$ we understand the functions k' and k'' such that*

$$k'_\Gamma(x_1, \cdots, x_n) = -f_\Gamma(x_1, \cdots, x_n)$$

$$k''_\Gamma(x_1, \cdots, x_n) = \mathsf{C}f_\Gamma(x_1, \cdots, x_n).$$

From this definition we easily infer the following:

COROLLARY 4.2. *The set K of all closure functions is a closure algebra with respect to the operations $\cup$, $\cap$, $-$, and C of Definition* 4.1.

If K is any closure algebra, and S is any subset of K, then there exists clearly a smallest subalgebra L of K which contains S. L is called, as usual, the subalgebra *generated* by S. In particular, if $\mathfrak{K}$ is the set of closure functions, and if $\mathfrak{S}$ is the set of identity functions of n variables, then there is a smallest subalgebra $\mathfrak{L}$ of $\mathfrak{K}$ which contains $\mathfrak{S}$; we call the elements of this smallest subalgebra $\mathfrak{L}$ *closure-algebraic* functions.

It is clear that the closure-algebraic functions are just those closure functions which can be obtained from identity functions by a finite number of applications of the operations $\cup$, $\cap$, $-$, and C. Thus we see that closure-algebraic functions are those for which there exist *chains* in the sense of the following definition:

Definition 4.3. *A finite sequence* $f_1, \cdots, f_r$ *of closure functions (all of the same number of variables) is said to be a chain for the function* f *if,*

(i) $f_r = f$,

(ii) *every function* $f_i (i = 1, \cdots, r)$ *is either an identity function, or is the sum, product, complement, or closure of functions preceeding it in the sequence.*

The number r is called the *length* of the chain. The length of a shortest chain for f, is called the *order* of f.

The above construction involves nothing specific for closure algebras, and can easily be extended to any other type of algebra. Thus we are dealing here with notions belonging to the domain of general algebra. In Part II of the Appendix we deal with some logical difficulties involved in the notions of closure functions and of closure-algebraic functions.

The real importance of these notions will appear in the next section. In order to prepare the ground for some later developments however, we shall state here a few theorems on closure-algebraic functions of a rather isolated character.

Theorem 4.4. *If* a *is any open element of a closure algebra* Γ, *and* $x_1, \cdots, x_n$ *are arbitrary elements of* Γ, *then for any closure-algebraic function* f *of* n *variables, and for any* $i \leqq n$, *we have*

$$a \cap f(x_1, \cdots, x_n) = a \cap f(x_1, \cdots, x_{i-1}, a \cap x_i, x_{i+1}, \cdots, x_n).$$

Proof. From the definition of closure-algebraic functions, it is seen that in order to show all closure-algebraic functions have a certain property, it suffices to show that : (i) the identity functions have the property; (ii) if two functions f and g have the property, then $f \cup g$ has it; (iii) if two functions f and g have the property, then $f \cap g$ has it; (iv) if a function f has the property, then $-f$ has it; and (v) if a function f has the property, then $\mathbf{C}f$ has it. By Boolean algebra, moreover, it is seen that (ii) is a consequence of (iii) and (iv), and that (iii) is a consequence of (ii) and (iv); hence it suffices to show (i), (iv), (v), and either (ii) or (iii).

It is immediately evident that the identity functions have the property asserted in our theorem.

Suppose that f and g are any two functions satisfying the theorem, so that

$$(1) \qquad a \cap f(x_1, \cdots, x_n) = a \cap f(x_1, \cdots, x_{i-1}, a \cap x_i, x_{i+1}, \cdots, x_n)$$

$$(2) \qquad a \cap g(x_1, \cdots, x_n) = a \cap g(x_1, \cdots, x_{i-1}, a \cap x_i, x_{i+1}, \cdots, x_n).$$

Adding the corresponding sides of (1) and (2), and applying the distributive law of multiplication with respect to addition, we see that the function $f \cup g$ also satisfies the theorem.

Suppose that f is any function satisfying (1). Taking the complements of the two sides of (1), and multiplying the resulting equation through by a, we have

$$(3) \quad a \cap [-a \cup -f(x_1, \cdots, x_n)] = a \cap [-a \cup -f(x_1, \cdots, x_{i-1}, a \cap x_i, x_{i+1}, \cdots, x_n)]$$

or

(4) $a \cap -f(x_1, \cdots, x_n) = a \cap -f(x_1, \cdots, x_{i-1}, a \cap x_i, x_{i+1}, \cdots, x_n)$.

From (4) we see that the function $-f$ also has the property asserted in the theorem.

Finally, if f is any function satisfying (1), and if we first form the closures of the two sides of (1), and then multiply through by a we obtain

(5) $a \cap \mathsf{C}[a \cap f(x_1, \cdots, x_n)] = a \cap \mathsf{C}[a \cap f(x_1, \cdots, x_{i-1}, a \cap x_i, x_{i+1}, \cdots, x_n)$.

Remembering that by hypothesis a is open, and applying Corollary 1.8, we conclude from (5) that

(6) $a \cap \mathsf{C}f(x_1, \cdots, x_n) = a \cap \mathsf{C}f(x_1, \cdots, x_{i-1}, a \cap x_i, x_{i+1}, \cdots, x_n)$.

From (6) we see that the function $\mathsf{C}f$ also satisfies our theorem.

This theorem has various different consequences. We shall state here some of them which seem interesting in themselves, even though they will not be used in the future discussion.

COROLLARY 4.5. *If a is open element of a closure algebra Γ, and y an arbitrary element such that $a \cap y = \Lambda$, then for any closure-algebraic function f we have*

$$a \cap f(x \cup y) = a \cap f(x).$$

If in addition b is open and $b \cap x = \Lambda$, then

$$(a \cup b) \cap f(x \cup y) = [a \cap f(x)] \cup [b \cap f(y)].$$

PROOF. Applying the theorem, we have

$$\begin{aligned} a \cap f(x \cup y) &= a \cap f[(a \cap x) \cup (a \cap y)] \\ &= a \cap f[(a \cap x) \cup \Lambda] \\ &= a \cap f(a \cap x) \\ &= a \cap f(x) \end{aligned}$$

as was to be shown.

If in addition b is open and $b \cap x = \Lambda$ we also have

$$b \cap f(x \cup y) = b \cap f(y).$$

Adding this equation to the one first proved, and applying the distributive law, we have

$$(a \cup b) \cap f(x \cup y) = [a \cap f(x)] \cup [b \cap f(y)].$$

COROLLARY 4.6. *If x and y are elements of a closure algebra such that $x \cap \mathsf{C}y = \Lambda$, then*

$$x \cap f(x \cup y) = x \cap f(x).$$

If in addition $y \cap \mathbf{C}x = \Lambda$, *then*

$$(x \cup y) \cap f(x \cup y) = [x \cap f(x)] \cup [y \cap f(y)].$$

PROOF. Since $-\mathbf{C}y$ is open, and $-\mathbf{C}y \cap y = \Lambda$, we see from Corollary 4.5 that

$$-\mathbf{C}y \cap f(x \cup y) = -\mathbf{C}y \cap f(x).$$

Multiplying this equation through by x we have

$$x \cap -\mathbf{C}y \cap f(x \cup y) = x \cap -\mathbf{C}y \cap f(x),$$

or (since, from the hypothesis, $x \cap -\mathbf{C}y = x$)

$$x \cap f(x \cup y) = x \cap f(x),$$

as was to be shown.

The second of these corollaries implies directly

COROLLARY 4.7. *Let* Γ *be a closure algebra, and let* f *be a closure-algebraic function such that, for every* z *in* Γ, $f(z) \subseteq z$; *then for every pair of elements* x *and* y *of* Γ *where* $x \cap \mathbf{C}y = \Lambda = y \cap \mathbf{C}x$, *we have*

$$f(x \cup y) = f(x) \cup f(y).$$

THEOREM 4.8. *If* a *is an open element of a closure algebra* Γ, *and* $x_1, \cdots, x_n$ *are elements include in* a, *then for every closure-algebraic function* f *of* n *variables, we have*

$$f_{\Gamma_a}(x_1, \cdots, x_n) = a \cap f_\Gamma(x_1, \cdots, x_n).$$

PROOF. If f is an identity function, so that for some $i \leq n$ we have

$$f_\Gamma(x_1, \cdots, x_n) = x_i,$$

then, since $x_i \subseteq a$, we have

$$a \cap f_\Gamma(x_1, \cdots, x_n) = a \cap x_i = x_i = f_{\Gamma_a}(x_1, \cdots, x_n).$$

If the theorem is true for functions f and g, then we have

$$\begin{aligned} a \cap (f \cup g)_\Gamma(x_1, \cdots, x_n) &= [a \cap f_\Gamma(x_1, \cdots, x_n)] \cup [a \cap g_\Gamma(x_1, \cdots, x_n)] \\ &= f_{\Gamma_a}(x_1, \cdots, x_n) \cup g_{\Gamma_a}(x_1, \cdots, x_n) \\ &= (f \cup g)_{\Gamma_a}(x_1, \cdots, x_n), \end{aligned}$$

so it is also true for $f \cup g$.

If the theorem is true for a function f, then we have

$$\begin{aligned} a \cap (-f)_\Gamma(x_1, \cdots, x_n) &= a \cap -f_\Gamma(x_1, \cdots, x_n) \\ &= a \cap -[a \cap f_\Gamma(x_1, \cdots, x_n)] \\ &= a \cap -f_{\Gamma_a}(x_1, \cdots, x_n) \\ &= -_a f_{\Gamma_a}(x_1, \cdots, x_n) \\ &= (-f)_{\Gamma_a}(x_1, \cdots, x_n), \end{aligned}$$

so it is also true for $-f$.

If the theorem, finally, is true for a function f, then we have, making use of Theorem 1.8,

$$\begin{aligned} a \cap (\mathbf{C}f)_\Gamma(x_1, \cdots, x_n) &= a \cap \mathbf{C}f_\Gamma(x_1, \cdots, x_n) \\ &= a \cap \mathbf{C}[a \cap f_\Gamma(x_1, \cdots, x_n)] \\ &= a \cap \mathbf{C}f_{\Gamma_a}(x_1, \cdots, x_n) \\ &= \mathbf{C}_a f_{\Gamma_a}(x_1, \cdots, x_n) \\ &= (\mathbf{C}f)_{\Gamma_a}(x_1, \cdots, x_n), \end{aligned}$$

so that it is also true for the function $\mathbf{C}f$.

Hence by induction the theorem is true for all closure-algebraic functions.

The following theorem is given mainly for later reference.

THEOREM 4.9. *Let a be any open element of a closure algebra Γ, and let $x_1, \cdots, x_n$ be any mutually exclusive elements such that $a \subseteq x_1 \cup \cdots \cup x_n$ and $a \subseteq \mathbf{C}x_1 \cup \cdots \cup \mathbf{C}x_n$. Then, for any closure-algebraic function f of n variables,*

$$a \cap f_\Gamma(x_1, \cdots, x_n) = a \cap (x_{i_1} \cup \cdots \cup x_{i_r}),$$

where $x_{i_1} \cup \cdots \cup x_{i_r}$ is a partial sum of the elements $x_1, \cdots, x_n$ (in particular, this partial sum can have no terms and hence equal Λ).

PROOF. In case $a = V$, the theorem is rather obvious. We have here n mutually exclusive elements $x_1, \cdots, x_n$ such that: (i) they are mutually exclusive, (ii) their sum is V, and (iii) the closure of each of them is V. In view of (iii), every closure-algebraic function of $x_1, \cdots, x_n$ reduces to a Boolean-algebraic function (i.e., to a function constructed without the closure operation); and in view of (i) and (ii) every Boolean-algebraic function of $x_1, \cdots, x_n$ equals a partial sum of them.

From the particular case, we obtain the general case by considering instead of the elements $x_1, \cdots, x_n$ in the original closure algebra Γ, the elements $a \cap x_1, \cdots, a \cap x_n$ of the relativized subalgebra Γ_a, and by then applying Theorems 4.4 and 4.8.

It would of course also be easy to prove the general form of our theorem directly by applying the method of induction used in the proofs of Theorems 4.4 and 4.8.

DEFINITION 4.10. *The function f which correlates with each sequence on n elements of every closure algebra Γ the zero-element Λ of Γ is called the zero-function (of n variables), in symbols $f = \Lambda$. We then also say that the function f vanishes identically. If the formula $f(x_1, \cdots, x_n) = \Lambda$ holds for all $x_1, \cdots, x_n$ of a given closure algebra Γ, then we say that f vanishes identically in Γ.*

COROLLARY 4.11. *The function $f = \Lambda$ is a closure-algebraic function.*

In the proof of the next theorem (Theorem 4.12) we shall use one more notion from the domain of general algebra, namely that of the direct union of two algebras.[12] It is superfluous to formulate the definition of this notion explicitly. It is obvious that the direct union of two or more closure algebras is again a closure algebra. Some general remarks regarding direct unions will be found in Part V of the Appendix. As an example of specific results regarding the tcerid union of closure algebras, we want to quote without proof the following:

If a and b are two open elements of a closure algebra Γ such that $a \cup b = \mathrm{V}$, then Γ is isomorphic with a subalgebra of the direct union of the relativized algebras Γ_a and Γ_b ; if in addition $a \cap b = \Lambda$, then Γ is isomorphic with the direct union of Γ_a and Γ_b .

THEOREM 4.12. *If f and g are closure-algebraic functions (of the same number of variables) and if $\mathbf{C}f \cap \mathbf{C}g$ vanishes identically then either f or g vanishes identically.*[13]

PROOF. Suppose neither f nor g vanishes identically. Then there are elements $x_1, \cdots, x_n$ of a closure algebra Γ_1, and elements $y_1, \cdots, y_n$ of a closure algebra Γ_2, such that

$$f_{\Gamma_1}(x_1, \cdots, x_n) \neq \Lambda,$$

$$g_{\Gamma_2}(y_1, \cdots, y_n) \neq \Lambda.$$

Let Γ be the direct union of the algebras Γ_1 and Γ_2, and let

$$z_1 = \langle x_1, y_1 \rangle, \cdots, z_n = \langle x_n, y_n \rangle.$$

Then from the general algebraic properties of the operation of forming the direct union, we see that

$$f_\Gamma(z_1, \cdots, z_n) \neq \Lambda,$$

$$g_\Gamma(z_1, \cdots, z_n) \neq \Lambda.$$

Let

$$f_\Gamma(z_1, \cdots, z_n) = u_1$$

$$g_\Gamma(z_1, \cdots, z_n) = u_2 .$$

We now consider the algebra Γ^* described in Lemma 3.9, and we set

$$w_1 = \langle z_1, \Lambda \rangle, \cdots, w_n = \langle z_n, \Lambda \rangle;$$

and furthermore we set

$$v_1 = \langle \mathbf{C}u, \mathrm{V} \rangle \quad \text{and} \quad v_2 = \langle \mathbf{C}u_2, \mathrm{V} \rangle.$$

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[12] For notions and results in general algebra, see Garrett Birkhoff, *On the combination of subalgebras*, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 441–464.

[13] In view of the relationship between the C. I. Lewis sentential calculus and topology, which was established in McKinsey's paper referred to above, this theorem will be seen to be equivalent to a theorem regarding the Lewis calculus which was stated without proof by Kurt Gödel in *Eine Interpretation des intuitionistischen Aussagenakküls*, Ergebnisse eines mathematischen Kolloquiums, Heft 4, pp. 39–40.

Since $u_1 \neq \Lambda \neq u_2$, we see from the definition of Γ^* that

$$\mathbf{C}f_{\Gamma^*}(w_1, \cdots, w_n) = v_1 \neq \Lambda$$

$$\mathbf{C}g_{\Gamma^*}(w_1, \cdots, w_n) = v_2 \neq \Lambda.$$

Moreover

$$\begin{aligned} v_1 \cap v_2 &= \langle \mathbf{C}u_1, \mathrm{V} \rangle \cap \langle \mathbf{C}u_2, \mathrm{V} \rangle \\ &= \langle \mathbf{C}u_1 \cap \mathbf{C}u_2, \mathrm{V} \rangle \\ &\neq \Lambda. \end{aligned}$$

Hence

$$\mathbf{C}f_{\Gamma^*}(w_1, \cdots, w_n) \cap \mathbf{C}g_{\Gamma^*}(w_1, \cdots, w_n) \neq \Lambda.$$

Thus there is a closure algebra Γ^* in which $\mathbf{C}f \cap \mathbf{C}g$ does not vanish identically. Our theorem follows by contraposition.

THEOREM 4.13. *If $f_1, \cdots, f_p$ are closure-algebraic functions (of the same number of variables) such that*

$$\mathbf{C}f_1 \cap \cdots \cap \mathbf{C}f_p = \Lambda$$

then there is an $i \leqq p$ such that

$$f_i = \Lambda.$$

PROOF. We carry out the proof by an induction on p.

If $p = 1$, the theorem is obvious. Hence suppose the theorem true for $p = k$, and let $f_1, \cdots, f_k, f_{k+1}$ be $k + 1$ closure-algebraic functions such that

$$\mathbf{C}f_1 \cap \cdots \cap \mathbf{C}f_k \cap \mathbf{C}f_{k+1} = \Lambda.$$

Then by Corollary 1.7 (vi)

$$\mathbf{C}[\mathbf{C}f_1 \cap \cdots \cap \mathbf{C}f_k] \cap \mathbf{C}f_{k+1} = \Lambda.$$

And hence by Theorem 4.12 either

$$\mathbf{C}f_1 \cup \cdots \cup \mathbf{C}f_k = \Lambda \quad \text{or} \quad f_{k+1} = \Lambda,$$

and hence, by the induction hypothesis, there is an $i \leqq k + 1$ such that

$$f_i = \Lambda.$$

Thus our theorem is also true for $p = k + 1$, and hence by induction is true for all p.

LEMMA 4.14. *Let $\Gamma = (K, \cup, \cap, -, \mathbf{C})$ be a closure algebra, and let $a_1, \cdots, a_r$ be a set of elements of K. Then there exists a subset K_1 of K, and a closure operation $\mathbf{C}_1$, which satisfy the following conditions:*

(i) *$(K_1, \cup, \cap, -, \mathbf{C}_1)$ is a closure algebra,*

(ii) *$a_i \in K_1$ for $i = 1, \cdots, r$,*

(iii) *K_1 contains at most 2^{2^r} elements,*

(iv) *If $x \in K_1$ and $\mathbf{C}x \in K_1$, then $\mathbf{C}_1 x = \mathbf{C}x$.*

PROOF. Let K_1 be the set of elements of K which can be obtained from the elements $a_1, \cdots, a_n$ by applying the Boolean operations: $\cup$, $\cap$, and $-$. It is immediately seen that K_1 is a subset of K which contains at most 2^{2^r} elements, and that $a_i \in K_1$ for $i = 1, \cdots, r$; thus we see that (ii) and (iii) are satisfied. Moreover, since the class K_1 is finite, the Boolean algebra $(K_1, \cup, \cap, -)$ is completely additive. Let K_2 be the set consisting of those elements x of K_1 such that $\mathbf{C}x$ is in K_1. Then we see that K_1, $\mathbf{C}$ and K_2 satisfy the hypothesis of Lemma 2.3. Hence by Lemma 2.3 we see that there is a closure operation $\mathbf{C}_1$ which satisfies (i) and (iv).

THEOREM 4.15. *If a closure-algebraic function f vanishes identically in every finite closure algebra, then it vanishes identically (in every closure algebra). If in addition f is a function of n variables and order r, it suffices to assume that f vanishes identically in every closure algebra with at most $2^{2^{n+r}}$ elements.*

PROOF. Suppose f does not vanish identically. Then there is a closure algebra $\Gamma = (K, \cup, \cap, -, \mathbf{C})$ and a set of elements $a_1, \cdots, a_n$ of K such that

$$f_\Gamma(a_1, \cdots, a_n) \neq \Lambda.$$

Since the order of f is r, we see by Definition 4.3 that there is a chain $f^{(1)}, \cdots, f^{(r)}$ of length r, such that $f^{(r)} = f$. We set

$$b_1 = f_\Gamma^{(1)}(a_1, \cdots, a_n), \cdots, b_r = f_\Gamma^{(r)}(a_1, \cdots, a_n).$$

Then clearly

$$b_r = f_\Gamma^{(r)}(a_1, \cdots, a_n) = f_\Gamma(a_1, \cdots, a_n) \neq \Lambda.$$

By Lemma 4.15 there is a finite closure algebra $\Gamma_1 = (K_1, \cup, \cap, -, \mathbf{C})$, where: (i) $a_1, \cdots, a_n$ and $b_1, \cdots, b_r$ are all in K_1; (ii) K_1 contains at most $2^{2^{n+r}}$ elements; and (iii) if x and $\mathbf{C}x$ are in K_1 then $\mathbf{C}_1 x = \mathbf{C}x$. From these conditions we see that

$$b_1 = f_{\Gamma_1}^{(1)}(a_1, \cdots, a_n), \cdots, b_r = f_{\Gamma_1}^{(r)}(a_1, \cdots, a_n),$$

and hence

$$f_{\Gamma_1}^{(r)}(a_1, \cdots, a_n) \neq \Lambda.$$

Hence f does not vanish identically in the finite algebra Γ_1. Our theorem now follows by contraposition.

THEOREM 4.16. *If a closure-algebraic function f vanishes identically in every finite well-connected closure algebra, then it vanishes identically (in every closure algebra).*

PROOF. If f does not vanish identically, then by Theorem 4.15 there is a finite closure algebra Γ in which it does not vanish identically. It is then easily verified that f does not vanish identically either in the well-connected finite closure algebra Γ^* described in Lemma 3.9.

The above theorem can also be proved in another way; namely, by making use of the result about direct unions which was stated before Theorem 4.14, we can show that any algebra with a minimum number of elements in which a function does not vanish identically, is well-connected.

THEOREM 4.17. *The class of closure-algebraic functions of n variables is countably infinite.*

PROOF. It is easily seen, in the first place, that there are only a finite number of closure-algebraic functions of n variables and order k. For there are just n functions of first order (the identity functions); and if the number of functions of order at most k is finite, then the number of functions of order at most $k + 1$ is also finite. Since there are only a countable infinity of orders, we see therefore that there are at most $\aleph_0$ closure-algebraic functions of n variables (of all orders).

Every function of one variable can also be considered as a function of n variables. Hence if there are infinitely many closure-algebraic functions of one variable, then there are certainly also infinitely many closure-algebraic functions of n variables (for each n). Thus to complete the proof of our theorem it suffices to show there are infinitely many closure-algebraic functions of one variable.

We definite[14] an infinite sequence $f_1, f_2, \cdots, f_k, \cdots$ of closure-algebraic functions (of one variable) as follows:

$$f_1(x) = x \cap \mathbf{C}(\mathbf{C}x \cap -x), \qquad f_2(x) = f_1f_1(x), \qquad f_3(x) = f_1f_1f_1(x), \cdots$$

We shall complete our proof by showing that no two of these functions are identical. To do this we find it necessary to describe a certain special topological space.

Let S be the set of all ordinal numbers $\alpha < \omega^\omega$. If A is any subset of S, let $\mathbf{C}(A)$ be the set A together with all upper limits (less that ω^ω) of sequences of elements of A. It is seen that S is a topological space with respect to $\mathbf{C}$.

Each element of S is expressible in the form of a polynomial in ω,

$$\omega^m \cdot n_1 + \omega^{m-1} \cdot n_2 + \cdots + \omega \cdot n_m + n_{m+1}$$

where m is a positive integer, and $n_1, n_2, \cdots, n_{m+1}$ are positive integers or zero. Let B be the set of those elements of S which are expressible in the form

$$\omega^m \cdot n_1 + \cdots + \omega^{2k} \cdot n_{m-2k+1} \quad \text{where} \quad n_{m-2k+1} \neq 0;$$

(in this expression we can have $k = 0$).

It is then easily shown that the sets

$$f_1(B), f_2(B), \cdots, f_k(B), \cdots$$

are all different. From this it follows that the functions $f_1, f_2, \cdots, f_k, \cdots$ are all different, as was to be shown.

§5. Free Closure Algebras

Near the beginning of the last section, we explained what is meant by saying that a subalgebra L of a closure algebra K is generated by a set S of elements.

[14] This construction and proof are taken from Kuratowski's paper referred to above.

It is easily seen that the subalgebra L consists of all elements y of the form

$$y = f(x_1, \cdots, x_p)$$

where f is a closure-algebraic function and $x_1, \cdots, x_p$ are elements of S; if S is a finite set consisting of n elements $a_1, \cdots, a_n$ then we can confine ourselves to functions f of n variables and elements y of the form

$$y = f(a_1, \cdots, a_n).$$

It may happen, in particular, that $L = K$ so that our algebra K is itself generated by S.

A closure algebra K is called a *free algebra generated by* $n \neq 0$ *elements* (or a *free algebra with n generators*) if there is a set S of n elements with the following properties: (i) K is generated by S, (ii) if $a_1, \cdots, a_p$ is any finite sequence of distinct elements of S, and f and g are two closure algebraic functions of p variables for which

$$f(a_1, \cdots, a_p) = g(a_1, \cdots, a_p),$$

then f and g are identically equal. This definition applies whether n is finite or infinite.

In case n is finite, we obtain as a particular case of a result of general algebra[12]:

THEOREM 5.1. *The class of all closure-algebraic functions of n variables is a free closure algebra with n generators; and every other free closure algebra with n generators is isomorphic with this one.*

By a similar construction we can establish the existence of a free closure algebra with n generators when n is infinite. In fact, let ν be the smallest ordinal number such that the set of all ordinal numbers which are smaller than ν has the power n.[15] Consider functions which correlate with every closure algebra Γ and with every sequence $x_1, x_2, \cdots, x_\xi, \cdots, (\xi < \nu)$ of elements of Γ a new element $f(x_1, x_2, \cdots, x_\xi, \cdots)$ of Γ. We can define for these functions operations corresponding to all the fundamental operations of closure algebra (exactly as in Definition 4.1). Then we can single out, from among all functions of the type considered, the *closure-algebraic* functions of n variables; these are the functions which can be obtained from the *identity functions*

$$f(x_1, x_2, \cdots, x_\xi, \cdots) = x_\eta \quad \text{for some } \eta < \nu$$

by applying finitely many times the fundamental operations. Finally we can show that the class of these closure-algebraic functions of n variables is a free closure algebra generated by n elements.

It might seem that we could modify the notion of free algebra generated by n elements, in case of an infinite n, by allowing in condition (ii) above, not merely

[15] In some systems of set theory, a cardinal number n is simply identified with the smallest ordinal number ν for which the set of all ordinal numbers $<\nu$ has the power n. See for example Paul Bernays, *A system of axiomatic set theory*, Journal of Symbolic Logic, vol. 7 (1942), p. 142.

finite, but also infinite sequences of the generating set S, and the corresponding functions of infinitely many variables. However, in view of the finite character of the fundamental operations of closure algebra, it can be easily seen that the definition thus modified would be equivalent to the original one.

Several properties of free closure algebras, in addition to those formulated in Theorem 5.1, can be obtained by a direct application of certain well-known theorems of general algebra. Thus, for instance, every closure algebra generated by at most n elements is a homomorphic image of a free closure algebra generated by n elements. It is also seen that a free algebra generated by n elements contains, for every number $p < n$, a free subalgebra generated by p elements. Some specific properties of free closure algebras are stated in the following theorems.

THEOREM 5.2. *A free closure algebra with any number of generators is infinite.*

PROOF. In case the number of generators is infinite, the theorem is of course obvious. If the number of generators is finite, on the other hand, the theorem is a consequence of Theorems 5.1 and 4.17.

We can also conclude from Theorems 5.1 and 4.17 that a free closure algebra with a finite number of generators is denumerably infinite.

THEOREM 5.3. *A free closure algebra generated by any number of elements is well-connected.*

PROOF. Let us first consider the case that the number of generators is finite. As in the preceeding theorem, we can confine ourselves to the algebra constituted by all closure-algebraic functions of n variables. But the fact that this algebra is well-connected follows directly from Theorem 4.12 and Definition 1.10.

If now, the number of generators is infinite, we first extend Theorem 4.12 to the functions of infinitely many variables introduced above. From this extended theorem our conclusion follows.

We see thus that the notion of a well-connected closure algebra, which may have seemed rather artificial, intervenes in a natural way in the study of free closure algebras.

A free closure algebra K generated by n elements has obviously the following property: if two closure algebraic functions f and g of $p < n$ variables are equal in K—i.e., if

$$f(x_1, \cdots, x_p) = g(x_1, \cdots, x_p)$$

for all $x_1, \cdots, x_p$ in K—then f and g are identically equal (i.e., the above formula holds for all elements of every closure algebra). We express this briefly by saying that every free closure algebra with n generators is *functionally free in the order n*. The converse is clearly not true. For instance if n is a finite number, consider a free algebra generated by a non-denumerable number p of elements. Such an algebra is functionally free in the order p, and hence also in every order less than p, in particular n. On the other hand, this algebra has the power p, and therefore cannot be a free algebra generated by a finite number of elements.

In the case just considered, a functionally free algebra in the order n turned out not to be a free algebra generated by n elements because its power was too large. Outside the domain of closure algebra, on the other hand, we can easily find examples of functionally free algebras in the order n which are not free algebras generated by n elements because their power is too small. Thus for example it is easily seen that a Boolean algebra with at least two elements is functionally free in every order (so that the notion of a functionally free Boolean algebra is trivial); while such an algebra is certainly not a free algebra with n generators if n exceeds the power of the algebra.

On the other hand, by means of a functionally free algebra in the order n we can easily construct a free algebra with n generators, without considering functions defined over all closure algebras. In fact let K be a functionally free algebra in the order n, and consider all closure-algebraic functions of n variables defined exclusively for the elements of K; thus two such functions are equal if they assume equal values for all sequences $x_1, \cdots, x_n$ of elements of K. It can easily be shown that the class of all these functions is a free algebra generated by n elements.

The notion of a functionally free closure algebra in the order n seems interesting not only in view of its connection with the notion of a free algebra with n generators. If K is such a functionally free algebra, and if we succeed in proving that a topological equation with at most n variables holds in K, then we can conclude that it holds in every other closure algebra (and hence in every topological space).

Here we shall be concerned exclusively with functionally free closure algebras in the order $\aleph_0$. In view of the finite character of the fundamental operations of closure algebra, it can be easily shown that a functionally free closure algebra in the order $\aleph_0$ is also functionally free in every other infinite order, and conversely. It is also seen that for a closure algebra K to be functionally free in the order $\aleph_0$ it is sufficient (and necessary) that it be functionally free in every finite order. In view of these considerations, we shall refer to functionally free algebras in the order $\aleph_0$ simply as functionally free; and we shall state the formal definition of this notion in the following way:

DEFINITION 5.4. *A closure algebra K is called functionally free if every two closure algebraic functions f and g of n variables, where n is an arbitraty finite integer, which are equal in K are also identically equal (i.e., in every closure algebra).*

Hereafter when speaking of closure-algebraic functions of n variables, we shall use this term as it was originally defined in §4; i.e., we shall have in mind functions of finitely many variables, even when this is not mentioned specifically.

In view of the fact that every equation in Boolean algebra (and hence also in closure algebra) of the form

$$a = b$$

can be transformed into an equivalent equation of the form

$$(a \cap -b) \cup (-a \cap b) = \Lambda,$$

we can simplify Definition 5.4 as follows:

COROLLARY 5.5. *For a closure algebra Γ to be functionally free, it is necessary and sufficient that every closure-algebraic function which vanishes identically in Γ also vanishes identically in every other closure algebra.*

From the remarks made above it follows that there exist functionally free algebras which do not have any finite number of generators. Now as the first result regarding functionally free closure algebras, we shall give Theorem 5.6, which considerably strengthens these remarks. This theorem also shows an essential difference between closure algebra and Boolean algebra.

THEOREM 5.6. *No functionally free closure algebra is generated by a finite number of elements.*

PROOF. Suppose, if possible, that a functionally free closure algebra K is generated by the m elements $w_1, \cdots, w_m$. Let $e_1, \cdots, e_r$ be those of the 2^m products $w_1 \cap \cdots \cap w_m, w_1 \cap \cdots \cap w_{m-1} \cap -w_m, \cdots, -w_1 \cap \cdots \cap -w_m$ which are not equal to Λ. Since each of the elements $w_1, \cdots, w_m$ can be expressed as a partial sum of the elements $e_1, \cdots, e_r$, we see that K is also generated by the elements $e_1, \cdots, e_r$. Moreover, we have $e_1 \cup \cdots \cup e_r = V$, and $e_i \cap e_j = \Lambda$ if $i \neq j$.

Let $n = 2^{r+1}$. Let the closure-algebraic function f of n variables be defined as follows:

$$(1) \qquad f(x_1, \cdots, x_n) = \mathbf{I}[\mathbf{C}x_1 \cap \cdots \cap \mathbf{C}x_n \cap (x_1 \cup \cdots \cup x_n) \cap -((x_1 \cap x_2) \cup \cdots \cup (x_1 \cap x_n) \cup (x_2 \cap x_3) \cup \cdots \cup (x_{n-1} \cap x_n))].$$

We notice first that f does not vanish identically. For let $A_1, \cdots, A_n$ be n mutually exclusive subsets of the Euclidean line such that $\mathbf{C}(A_1) = \cdots = \mathbf{C}(A_n) = V$ and $A_1 \cup \cdots \cup A_n = V$. Then it is easily seen that we have $f(A_1, \cdots, A_n) = V \neq \Lambda$.

Since by hypothesis K is a functionally free closure algebra, we therefore conclude that there are elements $x_1, \cdots, x_n$ and x of K such that

$$(2) \qquad f(x_1, \cdots, x_n) = x \neq \Lambda.$$

By an argument of an elementary nature we can now conclude from (1) and (2) that

$$(3) \qquad f(x \cap x_1, \cdots, x \cap x_n) = x,$$

and that

$$(4) \qquad x \cap x_i \cap x_j = \Lambda \qquad \text{for } i \neq j.$$

Let $x \cap x_1 = y_1, \cdots, x \cap x_n = y_n$. Then from (3) and (4) we have

$$(5) \qquad f(y_1, \cdots, y_n) = x \neq \Lambda,$$

$$(6) \qquad y_i \cap y_j = \Lambda \qquad \text{for } i \neq j.$$

Since K is generated by $e_1, \cdots, e_r$, we see that there are closure-algebraic functions $f_1, \cdots, f_n$ of r variables such that

$$(7) \qquad y_1 = f_1(e_1, \cdots, e_r), \cdots, y_n = f_n(e_1, \cdots, e_r).$$

If there is any i such that

$$x \cap -\mathbf{C}e_i \neq \Lambda,$$

then we can suppose without loss of generality that $e_1, \cdots, e_r$ occur in such an order that

$$(8) \quad x \cap -\mathbf{C}e_1 = \Lambda, \cdots, x \cap -\mathbf{C}e_s = \Lambda, \quad x \cap -\mathbf{C}e_{s+1} \neq \Lambda, \cdots, x \cap -\mathbf{C}e_r \neq \Lambda.$$

Let

$$a = x \cap -\mathbf{C}e_r, \quad \text{and} \quad y_1 \cap a = z_1, \cdots, y_n \cap a = z_n.$$

Then by the same sort of argument used to derive (5) and (6) we can easily show that

$$(9) \qquad f(z_1, \cdots, z_n) = a \neq \Lambda,$$

$$(10) \qquad z_i \cap z_j = \Lambda \quad \text{if} \quad i \neq j.$$

Moreover, since $a \subseteq x$ we see from (8) that

$$(11) \qquad a \cap -\mathbf{C}e_1 = \Lambda, \cdots, a \cap -\mathbf{C}e_s = \Lambda.$$

From (7) we see that

$$(12) \qquad \begin{aligned} z_1 &= a \cap y_1 = a \cap f_1(e_1, \cdots, e_r) \\ &\cdot \\ &\cdot \\ &\cdot \\ z_n &= a \cap y_n = a \cap f_n(e_1, \cdots, e_r). \end{aligned}$$

Moreover, $a \subseteq -\mathbf{C}e_r \subseteq -e$, and hence $a \cap e_r = \Lambda$. Hence by Theorem 4.4 we have

$$(13) \qquad \begin{aligned} z_1 &= a \cap f_1(e_1, \cdots, e_{r-1}, \Lambda) = a \cap g_1(e_1, \cdots, e_{r-1}) \\ &\cdot \\ &\cdot \\ &\cdot \\ z_n &= a \cap f_n(e_1, \cdots, e_{r-1}, \Lambda) = a \cap g_n(e_1, \cdots, e_{r-1}), \end{aligned}$$

where $g_1, \cdots, g_n$ are closure-algebraic functions of $r - 1$ variables.

It will be seen that by repeating the above argument sufficiently many times, we can show the following: there exist elements $u_1, \cdots, u_n$ and u of K, and closure-algebraic functions $h_1, \cdots, h_n$ of s variables such that:

$$(14) \qquad f(u_1, \cdots, u_n) = u \neq \Lambda,$$

$$(15) \qquad u_i \cap u_j = \Lambda \quad \text{for} \quad i \neq j,$$

$$(16) \qquad u \cap -\mathbf{C}e_1 = \Lambda, \cdots, u \cap -\mathbf{C}e_s = \Lambda,$$

$$(17) \qquad u \subseteq -\mathbf{C}e_{s+1}, \cdots, u \subseteq -\mathbf{C}e_r,$$

$$(18) \qquad \begin{aligned} u_1 &= u \cap h_1(e_1, \cdots, e_s) \\ &\cdot \\ &\cdot \\ &\cdot \\ u_n &= u \cap h_n(e_1, \cdots, e_s). \end{aligned}$$

From (17) we conclude that $u \subseteq -e_{s+1}, \cdots, u \subseteq -e_r$, and hence that

$$u \subseteq -(e_{s+1} \cup \cdots \cup e_r) = e_1 \cup \cdots \cup e_s .$$

From (16) we see that

$$u \subseteq \mathbf{C}e_1, \cdots, u \subseteq \mathbf{C}e_s .$$

From the definition of the function f in (1) it is clear that u is open. Hence the hypothesis of Theorem 4.9 is satisfied, so we see that for each i we have

$$u_i = u \cap h_i(e_r, \cdots, e_s) = u \cap (e_{i_1} \cup \cdots \cup e_{i_k})$$

where $e_{i_1} \cup \cdots \cup e_{i_k}$ is a sum of some of the elements $e_1, \cdots, e_s$. Since there are only 2^s such sums, and since $n = 2^{r+1} \geqq 2^{s+1} > 2^s$, the elements $u_1, \cdots, u_n$ cannot all be different. Suppose $u_i = u_j$, where $i \neq j$; then by (15) we have $u_i = u_i \cap u_i = u_i \cap u_j = \Lambda$. But it is then immediately seen that

$$f(u_1, \cdots, u_n) = f(u_1, \cdots, u_{i-1}, \Lambda, u_{i+1}, \cdots, u_n) = \Lambda.$$

Since this result contradicts (14), we see that our original assumption that there exists a free closure algebra generated by a finite number of elements must be rejected.

THEOREM 5.7. *Every generalized universal algebra for the class of all finite closure algebras, is functionally free.*

PROOF. Let Γ be a generalized universal algebra for the class of all finite closure algebras, and let f be any closure-algebraic function of n variables which does not vanish identically. We are to show that there are elements $a_1, \cdots, a_n$ of Γ such that

$$f_\Gamma(a_1, \cdots, a_n) \neq \Lambda.$$

Since f does not vanish identically, we see by Theorem 4.15 that there is a finite closure algebra Δ which contains elements $b_1, \cdots, b_n$ such that

$$f_\Delta(b_1, \cdots, b_n) \neq \Lambda.$$

Since Γ is a generalized universal algebra for all finite algebras, we see by Definition 3.3 that there is an open element a of Γ such that Δ is isomorphic with a subalgebra of Γ_a. Let the elements $b_1, \cdots, b_n$ of Δ correspond respectively to elements $a_1, \cdots, a_n$ of Γ_a under this isomorphism. Then clearly we have

$$f_{\Gamma_a}(a_1, \cdots, a_n) \neq \Lambda.$$

By Theorem 4.8 we then have

$$a \cap f_\Gamma(a_1, \cdots, a_n) = f_{\Gamma_a}(a_1, \cdots, a_n) \neq \Lambda.$$

From this we conclude

$$f_\Gamma(a_1, \cdots, a_n) \neq \Lambda,$$

as was to be shown.

In a similar way, by making use of Theorem 4.16, we can prove the following:

THEOREM 5.8. *Every universal algebra (in the proper sense) for the class of all finite connected closure algebras (or even only for the class of all finite well-connected closure algebras) is functionally free.*

Either by applying Theorem 5.7 together with Theorem 3.12, or Theorem 5.8 together with Theorem 3.7, we obtain directly:

THEOREM 5.9. *Every dissectable closure algebra is functionally free*[16].

This theorem together with Theorem 3.5 leads immediately to the main result of this paper.

THEOREM 5.10. *The closure algebra over Euclidean space of any number of dimensions is functionally free; or, more generally, the closure algebra over any normal, dense-in-itself topological space with a countable basis, is functionally free.*

The implications of this theorem are obvious. From the point of view of the axiomatic foundations of topology, Theorem 5.10 shows that the system of postulates for closure algebra (or for topological space in terms of closure) has a certain completeness property: Every topological equation which is identically true in Euclidean space, can be derived from these postulates. In fact Theorem 5.10 implies that, if any topological equation is proved for a given Euclidean space, then it holds also in every other Euclidean space, and indeed in every topological space. Hence by contraposition, we see also that if a topological equation fails in some topological space, or even in some closure algebra (defined perhaps in the most artificial way), then we can be sure of finding a counter-example for it in any given Euclidean space, for instance on a straight line; and if we analyze the proof of the theorems on which Theorem 5.10 depends, we see that we are able not merely to prove the existence of the sets which do not satisfy the equation in question, but that we can even construct them in an effective way. Towards the end of this section we shall see that this result can be somewhat further strengthened.

It may be noticed that, in addition to the closure algebras explicitly mentioned in the last two theorems, we can construct many other closure algebras which are functionally free. This is seen, for instance, from the following simple theorem:

THEOREM 5.11. *If a is an open element of a closure algebra Γ, and if Γ_a is functionally free, then so is Γ.*

Continuing our study of functionally free algebras, we shall give here two

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[16] This theorem is closely related to the main result of Tarski's paper referred to above; in our present terminology this result can be formulated as follows: the class of all open elements of a dissectable closure algebra (and in particular, the family of all open sets of a Euclidean space) is a functionally free Brouwerian logic. We plan to present in a separate paper some applications of the results of the present paper to Brouwerian logic.

results about closure-algebraic functions in functionally free algebras, which follow almost immediately from a theorem of the preceeding section. They seem interesting if only for this reason, that in view of Theorem 5.10, they apply in particular to Euclidean space.

THEOREM 5.12. *If K is a functionally free closure algebra, and $f_1, \cdots, f_n$ any closure-algebraic functions of the same number p of variables such that*

$$\mathsf{C}f_1 \cap \cdots \cap \mathsf{C}f_n$$

vanishes identically in K, then at least one of these functions must vanish identically in K (and hence in every closure algebra).

COROLLARY 5.13. *If K is a functionally free closure algebra, and $f_1, \cdots, f_n$ are any closure-algebraic functions of p variables such that, for every set $x_1, \cdots, x_p$ of p elements of K either $f_1(x_1, \cdots, x_p) = \Lambda$ or $f_2(x_1, \cdots, x_p) = \Lambda$ or $\cdots$ or $f_n(x_1 \cdots, x_p) = \Lambda$, then at least one of the functions $f_1, \cdots, f_n$ vanishes identically in K (and hence in every closure algebra).*

It may be noticed that Theorem 5.12 and Corollary 5.13 are also true of free closure algebras generated by q elements, provided the number p of variables does not exceed q.

In the last theorems of this section we shall return to the question of the relation between functionally free algebras and free algebras with n generators. More specifically, we shall be concerned with the question under what conditions a functionally free algebra contains a free subalgebra generated by a given finite number of elements. We begin with a result of a negative character.

THEOREM 5.14. *There exists a functionally free closure algebra, which contains no free subalgebra generated by any number of elements.*

PROOF. Let us consider any closure algebra K which is universal for all finite algebras (the existence of such algebras can be derived, for instance, from Theorem 3.8, or can also be proved in a direct way). We arrange all finite subalgebras of this algebra in a transfinite sequence

$$K_1, K_2, \cdots, K_\xi, \cdots$$

and construct the direct union L of all these subalgebras. Thus the elements of L are infinite sequences

$$x_1, x_2, \cdots, x_\xi, \cdots$$

where $x_1 \in K_1, x_2 \in K_2, \cdots, x_\xi \in K_\xi, \cdots$. Let M be the set of all those sequences belonging to L in which either all elements except a finite number equal Λ or else all elements except a finite number equal V. It is easily seen that M is a subalgebra of L, and therefore a closure algebra. Furthermore it can be shown without difficulty that to every finite algebra N, there exists an open element a of M such that N is isomorphic with a subalgebra of M_a, and indeed even to M_a itself. In fact, since K is universal for all finite algebras, N is isomorphic with some algebra K_ξ; and we can take as a the sequence

$$x_1, x_2, \cdots, x_\xi, \cdots$$

where $x_\xi = V$ and the remaining terms are Λ. Hence by Definition 3.3, M is a generalized universal algebra for all finite algebras; and hence by Theorem 5.7 it is functionally free.

Now consider any particular element b of M. b is a sequence

$$x_1, x_2, \cdots, x_\xi, \cdots$$

in which:

(i) All elements except for a finite number

$$x_{\xi_1}, x_{\xi_2}, \cdots, x_{\xi_n}$$

either equal Λ or equal V; while

(ii) The exceptional elements

$$x_{\xi_1}, x_{\xi_2}, \cdots, x_{\xi_n}$$

belong to the finite algebras

$$K_{\xi_1}, K_{\xi_2}, \cdots, K_{\xi_n}.$$

Let P be the set of all sequences belonging to M which satisfy the conditions (i) and (ii)—the numbers $\xi_1, \cdots, \xi_n$ being the same for all these sequences. P can easily be proved to be a finite subalgebra of M; and, since the element b belongs to P, the subalgebra generated by b is contained in P, and is therefore also finite. Hence by Theorem 5.2 this subalgebra cannot be a free algebra generated by one element. Since b is quite an arbitrary element of M, it follows that M contains no free algebra with one generator.

The remarks at the beginning of this section regarding the general properties of free algebras imply that every free algebra with an arbitrary number of generators contains a free subalgebra with one generator. Therefore M cannot contain a free subalgebra with any number of generators. Thus M has all the desired properties.

In the preceeding proof we were not interested in the power of the closure algebra which is involved in this theorem. However from the proof it is easily seen that it is possible to construct an algebra satisfying the conditions of the conditions of the theorem, which is in addition denumerable. In fact it suffices for this purpose to omit from the sequence

$$K_1, K_2, \cdots, K_\xi, \cdots$$

of all finite subalgebras of K every subalgebra which is isomorphic with one which preceeds it in the sequence, and then to consider the direct union of the remaining subalgebras.

Now we shall show that under certain additional conditions a functionally free closure algebra must contain free subalgebras with any finite number of generators. It will be seen that these additional conditions are satisfied by all the most important examples of functionally free closure algebras.

LEMMA 5.15. *Every functionally free closure algebra K contains a countable infinity of mutually exclusive (non-empty) open elements.*

PROOF. n being an arbitrary natural number, consider the n closure-algebraic functions of n variables defined by means of the formulas:

$$f_1(x_1, \cdots, x_n) = \mathbf{I}(-x_1 \cap x_2 \cap \cdots \cap x_n)$$
$$f_2(x_1, \cdots, x_n) = \mathbf{I}(x_1 \cap -x_2 \cap \cdots \cap x_n)$$
$$\vdots$$
$$f_n(x_1, \cdots, x_n) = \mathbf{I}(x_1 \cap x_2 \cap \cdots \cap -x_n).$$

It is obvious that none of these functions vanish identically; for putting, for instance,

$$x_1 = \Lambda, \qquad x_2 = \cdots = x_n = \mathrm{V}$$

we obtain

$$f_1(x_1, \cdots, x_n) = \mathrm{V} \neq \Lambda.$$

Hence by Corollary 5.13, there exist elements $a_1, \cdots, a_n$ of K for which none of the functions $f_1, \cdots, f_n$ vanish. By putting

$$b_1 = f_1(a_1, \cdots, a_n), \cdots, b_n = f_n(a_1, \cdots, a_n)$$

and applying Corollary 1.4 (iii) and Definition 1.6 we see that our algebra contains n mutually exclusive open elements, for each n.

Hence we can conclude that K contains also a countable infinity of open elements, by applying a reasoning which is by no means specific for closure algebras.[17] In fact, let L be the class of all those elements of K which are non-empty, open, and do not contain any two non-empty mutually exclusive open elements. If L is not empty, we pick out any element a_1 of L, then any element a_2 of L which is disjoint with a_1 (if such an element exists), then any element a_3 of L which is disjoint with both a_1 and a_2, and so on: clearly we apply here the axiom of choice. It may happen that in this we succeed in obtaining infinitely many elements $a_1, a_2, \cdots, a_n, \cdots$ of L, in which case the conclusion of our theorem is obviously satisfied. Otherwise we arrive at a finite sequence $a_1, \cdots, a_k$ of elements of L such that no element of L different from all of them is disjoint with all of them.

In the latter case we make use of the fact which was established before, that for every n, and in particular for $n = k + 1$, there are n mutually exclusive non-empty open elements in K. Let $b_1, \cdots, b_{k+1}$ be such elements. It is easily seen that each of the elements a_i $(i = 1, \cdots, k)$ can have a non-empty product with at most one element b_j $(j = 1, \cdots, k + 1)$, for otherwise a_i could not belong to L. Hence it clearly follows that at least one element b_j is mutually exclusive with every element a_i. Consequently b_j cannot contain any element a

[17] See the paper by P. Erdös and A. Tarski, *On families of mutually exclusive sets*, these Annals, vol. 44 (1943), p. 315–329.

of L, for such an element would be disjoint with all elements $a_1, \cdots, a_k$, which as we know is impossible.

We have thus shown that there exists a non-empty open element $b = b_j$ of K which contains no element of L (this is also trivially true in case the class L is empty). Now b *a fortiori* is not itself such an element, and therefore it contains two non-empty mutually exclusive open elements c_1 and d_1. Again d_1 is not an element of L, and hence it contains two non-empty mutually exclusive open elements c_2 and d_2. We can continue this procedure indefinitely (using the axiom of choice) so as to obtain finally an infinite sequence $c_1, c_2, \cdots, c_n, \cdots$ of mutually exclusive open elements of K. Thus the proof is complete.

THEOREM 5.16. *Let Γ be a countably additive closure algebra such that, for every non-empty open element a of Γ, Γ_a is functionally free. Then Γ contains a free algebra with a countable infinity of generators (and hence also a free closure algebra with any arbitrary finite number of generators).*

PROOF. It clearly suffices to show that Γ contains a countably infinite sequence S of elements

$$b_1, b_2, \cdots, b_k, \cdots$$

such that:

(i) If f is any closure-algebraic function of any finite number p of variables which does not vanish identically, and if

$$b_{i_1}, \cdots, b_{i_p}$$

are any p distinct elements of S, then

$$f_\Gamma(b_{i_1}, \cdots, b_{i_p}) \neq \Lambda.$$

For if we let Δ be the subalgebra of Γ generated by S, then Δ will be a free algebra with a countable infinity of generators.

Moreover, the condition (i) is equivalent to the following condition:

(i′) If g is any closure-algebraic function of any finite number q of variables which does not vanish identically, then

$$g_\Gamma(b_1, b_2, \cdots, b_q) \neq \Lambda.$$

(i) clearly implies (i′), but it is easily seen that the implication in the opposite direction also holds. For in fact assume (i′) to hold. Given a function f of p variables, and a sequence

$$b_{i_1}, \cdots, b_{i_p}$$

of elements of S, we let q be the maximum of the numbers

$$i_1, \cdots, i_p,$$

and we define the closure-algebraic function g of q variables by putting

$$g(x_1, \cdots, x_q) = f(x_{i_1}, \cdots, x_{i_p}).$$

By applying (i′) to g, we obtain (i) for f.

Hence in order to prove our theorem it suffices to show that Γ contains a sequence S which satisfies condition (i′).

By Theorem 4.17 we see that there are only a countable infinity of closure-algebraic functions of a finite number of variables. Let the set of all those closure-algebraic functions which do not vanish identically be ordered in a sequence

$$f^{(1)}, f^{(2)}, \cdots, f^{(k)}, \cdots.$$

For each k we shall suppose that $f^{(k)}$ is a function of p_k variables.

It is evident from the hypothesis of our theorem that Γ is a functionally free closure algebra. Hence by Lemma 5.15 Γ contains a countable infinity

$$a_1, a_2, \cdots, a_k, \cdots$$

of mutually exclusive non-empty open elements.

By the hypothesis we then see that each of the relativized subalgebras

$$\Gamma_{a_1}, \Gamma_{a_2}, \cdots, \Gamma_{a_k}, \cdots$$

is functionally free. Hence, for each k, there are p_k elements

$$x_{k,1}, x_{k,2}, \cdots, x_{k,p_k}$$

of Γ_{a_k} such that

(1) $$f^{(k)}_{\Gamma_{a_k}}(x_{k,1}, \cdots, x_{k,p_k}) \neq \Lambda.$$

From (1) and Theorem 4.8 we conclude that

(2) $$a_k \cap f^{(k)}_{\Gamma}(x_{k,1}, \cdots, x_{k,p_k}) \neq \Lambda.$$

Remembering that by hypothesis Γ is countably additive, we now set, for each i,

$$b_i = x_{1,i} \cup x_{2,i} \cup \cdots \cup x_{k,i} \cup \cdots.$$

We shall show that the sequence S of elements

$$b_1, b_2, \cdots, b_k, \cdots$$

satisfies condition (i′).

Let f be any closure-algebraic function which does not vanish identically. Then for some k we have

$$f = f^{(k)}.$$

Since a_k is open, we see by Theorem 4.4 that

$$a_k \cap f^{(k)}_{\Gamma}(b_1, \cdots, b_{p_k}) = a_k \cap f^{(k)}_{\Gamma}(b_1 \cap a_k, \cdots, b_{p_k} \cap a_k).$$

From the way in which b was defined, however, we have, for each i,

$$b_i \cap a_k = x_{k,i}.$$

Thus we conclude that

$$a_k \cap f_\Gamma^{(k)}(b_1, \cdots, b_{p_k}) = a_k \cap f_\Gamma^{(k)}(x_{k,1}, \cdots, x_{k,p_k}).$$

Since the elements

$$x_{k,1}, \cdots, x_{k,p_k}$$

were so chosen as to satisfy (2), we therefore conclude that

$$f_\Gamma^{(k)}(b_1, \cdots, b_{p_k}) \neq \Lambda.$$

Thus our sequence

$$b_1, b_2, \cdots, b_k, \cdots$$

satisfies condition (i′), as was to be shown.

The question remains open whether this theorem can be strengthened to apply either to all functionally free closure algebras which are countably additive, or to all those algebras in which every subalgebra relativized to a non-empty open element is functionally free. It may be noticed that the functionally free algebra M which was constructed in the proof of Theorem 5.14 satisfies neither of these conditions: it is not countably additive, and for no open element $a \neq V$ is the algebra M_a functionally free.

By means of Theorems 5.16, 5.9, and Corollary 3.5, we obtain directly:

THEOREM 5.17. *Every countably additive dissectable closure algebra contains a free algebra with a countable infinity of generators; in particular, the closure algebra over Euclidean space (of any number of dimensions) contains a free algebra with a countable infinity of generators, and hence also a free algebra generated by any finite number of elements.*

This theorem exhibits a stronger property of Euclidean space than that given by Theorem 5.10. Consider, for instance, topological equations with one variable; from Theorem 5.10 it follows that when such an equation is not an identity then we can construct in Euclidean space a counter-example for it. Theorem 5.17 implies much more; it shows that in Euclidean space there exists a set which is, so to speak, a universal counter-example for all these equations, and we can even construct such a set effectively.

APPENDIX

I. Derivative Algebra

Like the topological operation of closure, other topological operations can also be treated in an algebraic way.[18] This may be especially interesting in regard to those operations which are not definable in terms of closure—i.e., which are not closure-algebraic operations. An especially important notion is that of the

[18] The results in this section are largely to be found, in a somewhat different form, in Kuratowski's paper referred to above. The theorem stated as an analogue of 4.15, however, is of course new.

derivative of a point set A, which will be denoted by $\mathbf{D}(A)$. For the purpose of an algebraic investigation of this notion, we define a new class of algebras, which we shall refer to as derivative algebras.

We say that a set K is a derivative algebra with respect to the operations $\cup, \cap, -$, and $\mathbf{D}$, when:

(i) K is a Boolean algebra with respect to $\cup, \cap, -$

(ii) If x is in K, then $\mathbf{D}x$ is in K.

(iii) If x is in K, then $\mathbf{DD}x \subseteq \mathbf{D}x$.

(iv) If x and y are in K, then $\mathbf{D}(x \cup y) = \mathbf{D}x \cup \mathbf{D}y$.

(v) $\mathbf{D}\Lambda = \Lambda$.

As regards the question of the relation between derivative algebra and closure algebra, the following can be remarked. As is known, in a topological space the closure of a set can be defined in terms of derivative by means of the formula:

$$\mathbf{C}(A) = A \cup \mathbf{D}(A).$$

If we introduce a corresponding definition into derivative algebra,

$$\mathbf{C}x = x \cup \mathbf{D}x,$$

we can easily show that the derivative algebra becomes a closure algebra with respect to the Boolean operations $\cup, \cap, -$, and the operation $\mathbf{C}$ just defined. On the other hand, it can easily be shown that there is no unary closure-algebraic operation O such that

$$\mathbf{D}(A) = O(A)$$

holds for every set A of Euclidean space.

The methods of proof used to establish some of our results regarding closure algebras can be applied almost without change to prove analogous theorems about derivative algebras. For example, if we define *derivative-algebraic* functions in a way analogous to that used to define closure-algebraic functions, then we can easily prove the following analogue of Theorem 4.15:

If a derivative-algebraic function f vanishes in every finite derivative algebra, then it vanishes identically. If in addition f is a function of n variables and order r, then it suffices to assume that f vanishes in every derivative algebra with at most $2^{2^{n+r}}$ elements.

It is also immediately seen that there are infinitely many derivative-algebraic functions, since, as is well known, the functions

$$\mathbf{D}x, \mathbf{DD}x, \mathbf{DDD}x, \cdots$$

are all different.

As regards the central problem of determining conditions for a functionally free derivative algebra, we shall here consider only the question whether the derivative algebra over Euclidean space is functionally free. The answer is trivially negative, since the equation

$$\mathbf{D}(x \cup -x) = x \cup -x,$$

or simply

$$\mathbf{D}V = V$$

is identically satisfied in the derivative algebra over Euclidean space, and in general over every dense-in-itself space, but is not identically satisfied in every derivative algebra.

In view of this fact, we include the equation

(vi) $\mathbf{D}V = V$

into the postulate system of derivative algebra. We call a derivative algebra satisfying this additional postulate a *dense-in-itself* derivative algebra. We modify in an obvious way the notion of being functionally free, so as to make it apply to dense-in-itself derivative algebras, and now ask whether the derivative algebra over Euclidean space is functionally free in the new sense. The answer however is known to be negative even in this case, at least as regards Euclidean space of 2 or more dimensions. For the equation

$$\mathbf{D}[(x \cap \mathbf{D} - x) \cup (-x \cap \mathbf{D}x)] = \mathbf{D}x \cap \mathbf{D} - x$$

is identically satisfied in the derivative algebra over Euclidean space of 2 or more dimensions, but is not satisfied in the derivative algebra over the Euclidean straight line. The problem whether the latter is a functionally free dense-in-itself derivative algebra remains open. The same applies to the derivative algebra over Cantor's discontinuum, for instance, or the space of the rational numbers. We do not know, either, whether the derivative algebra over the Euclidean plane would become a free derivative algebra if in addition to the density postulate we included into the postulate system also the last equation stated above. Finally the problem remains open whether it is possible to distinguish the derivative algebras over the various Euclidean spaces (of dimension greater than 1) by means of equations.

II. A Modification of the Definitions of Closure-Algebraic Functions and Free Algebras

Some of the constructions of §§4–5 may raise considerable doubts from the point of view of the foundations of set-theory, and may seem to have an antinomial character. In fact there are axiomatic systems of set-theory in which the assumption that there exists the set of all closure algebras, or a function defined over all closure algebras, would lead to a contradiction (the notion of a function being here taken in its set-theoretical meaning; i.e., a function of n variables being regarded as a set of ordered $n+1$-tuples). There are other systems in which a distinction is made between proper and improper sets; improper sets, which are also called classes, are defined to be those sets which are not elements of other sets. In such systems it would be possible to prove the existence of the class of all closure algebras, and of functions over all closure algebras; however we should be unable to form classes of these functions, and we should meet with difficulty in trying to distinguish from among all functions, those functions we have called closure-algebraic (especially as regards functions of infinitely many variables). Thus even in this case most of the developments of §§4–5 would seem to be based on rather shaky foundations.

In view of these difficulties we want to indicate here a way of modifying these constructions, which will make them unobjectionable from the point of view of practically all systems of set-theory, even of those logically weaker systems which do not allow us to prove the existence of improper sets. The changes we are going to outline imply certain complications in formulating most of our definitions and theorems, but on the other hand they allow us to state some of our results in a somewhat stronger and more general way. In what follows we shall always use the word "class" synomously with the word "set", taking this to mean what we have before called a "proper set".

The notions which we introduced at the beginning of §4 will now be relativized to an arbitrary non-empty class $\mathfrak{K}$ of closure algebras (which of course does not in any way presuppose the existence of the class of *all* closure algebras). Thus we consider functions f which correlate with every algebra Γ of $\mathfrak{K}$ and every n-tuple $x_r, \cdots, x_n$ of elements of Γ, a new element $f(x_r, \cdots, x_n)$ of Γ; we shall refer to such functions briefly as $\mathfrak{K}$-functions. We define for these functions, the fundamental operations of closure algebra (as in Definition 4.1), and we distinguish among these functions the *closure-algebraic* $\mathfrak{K}$-functions. For any given $\mathfrak{K}$, the class of all $\mathfrak{K}$-functions of n variables (whose existence is insured by any ordinary axiom system of set-theory) clearly forms a closure algebra; the subclass of closure-algebraic $\mathfrak{K}$-functions constitutes a subalgebra of this algebra, and will be referred to as the *function algebra of* n^{th} *order over* $\mathfrak{K}$. Definition 4.10 is to be modified in an obvious way, so as to make precise the notion of an *identically vanishing* $\mathfrak{K}$-function.

It is interesting to observe that the notion of a function algebra is only a special case of the notion of the direct union of algebras. In order to arrive at the general notion of the direct unions of closure algebras, we consider an arbitrary set J, and we assume that a closure algebra Γ_i is correlated with every element i of J. The direct union of all Γ_i (for i in J) is constituted by all functions f defined over J and correlating with each i in J an element $f(i)$ belonging to Γ_i; we must of course define in the familiar way all the fundamental operations of closure algebra. Now suppose first that the class $\mathfrak{K}$ consists of just one algebra Γ. If we take J to be the set of all elements of the given closure algebra Γ and if each Γ_i, for i in J, coincides with Γ, then our direct union goes over into the algebra of all $\mathfrak{K}$-functions of one variable. If J is taken to be the set of all n-termed sequences of elements of Γ, and if each Γ_i is again Γ, then the direct union goes over into the algebra of all $\mathfrak{K}$-functions of n variables. Considering now the general case, suppose that $\mathfrak{K}$ consists of various algebras

$$\Gamma^{(1)}, \Gamma^{(2)}, \cdots .$$

Let J_1 consist of all n-termed sequences of elements of $\Gamma^{(1)}$, J_2 of all n-termed sequences of elements of $\Gamma^{(2)}$, and so on; and let

$$J = J_1 \cup J_2 \cup \cdots .$$

If i is any element of J then there is some $\Gamma^{(j)}$ such that i is an n-termed sequence of elements of $\Gamma^{(j)}$; we take Γ_i to be $\Gamma^{(j)}$. It is easily seen that the direct union then again goes over into the algebra of all $\mathfrak{K}$-functions of n variables. The algebra constituted by the closure-algebraic $\mathfrak{K}$-functions is of course a subalgebra of this direct union.

Theorems and Corollaries 4.4–4.7, 4.9, and 4.11 undergo an obvious relativization: an arbitrary class $\mathfrak{K}$ of closure algebras is introduced, which contains the closure algebra Γ involved, and the function f is assumed to be a closure-algebraic $\mathfrak{K}$-function. In 4.8 we have to assume that $\mathfrak{K}$, in addition to Γ, includes also Γ_a. The situation is somewhat more involved as regards the remaining theorems of §4. In each of these theorems we introduce again a class $\mathfrak{K}$ of closure algebras, and we assume that all the algebras involved belong to $\mathfrak{K}$ and that all the functions involved are closure-algebraic $\mathfrak{K}$-functions. However we have to make some additional assumptions regarding $\mathfrak{K}$. It turns out that one common assumption suffices for all these theorems; in fact, that for every finite closure algebra Γ there corresponds an isomorphic algebra in $\mathfrak{K}$ (but $\mathfrak{K}$ may contain also infinite algebras). For this assumption clearly suffices for the proof of 4.16, as is easily seen from the proof of this theorem; and by an essential use of 4.16 relativized in this way we can easily see that this assumption suffices also for the other theorems. It may be remarked that a direct analysis of 4.12 and 4.13 shows that these theorems hold under a different assumption regarding $\mathfrak{K}$; namely, under the hypothesis that $\mathfrak{K}$ is closed under the operation of forming direct unions, and the operation Γ^* of Lemma 3.9.

In §5 the first change is in the definition of a free algebra with n generators. We shall now say that a closure algebra is a free closure algebra with n generators if: (i) it is an algebra generated by a set S consisting of n elements; and (ii) for any sequence $a_1, \cdots, a_p$ of distinct elements of S, for any class $\mathfrak{K}$ of closure algebras to which our algebra belongs, and for any closure-algebraic $\mathfrak{K}$-functions f and g, the formula

$$f(a_r, \cdots, a_p) = g(a_r, \cdots, a_p)$$

implies that f and g are identically equal. The first part of Theorem 5.1 assumes the form:

If $\mathfrak{K}$ is a class of closure algebras, which contains an algebra isomorphic with each finite algebra, then, for every positive integer n, the function algebra over $\mathfrak{K}$ of order n is a free algebra with n generators.

It is easy to prove the existence of a class which satisfies the hypothesis of the theorem just stated. For instance we can take as $\mathfrak{K}$ the class of all closure algebras

$$\Gamma = (K, \cup, \cap, -, \mathbf{C})$$

where K is a finite set of natural numbers (and the operations vary). Hence this theorem implies the existence of free closure algebras with any finite number of generators. The extension of this construction and result to the case where n

is infinite can be carried through in a way quite analogous to that sketched in §5. It may be noticed that in the existence theorem for free algebras with infinitely many generators the assumption regarding the class $\mathfrak{K}$ remains unchanged.

The definition of a functionally free algebra (and more generally, of a functionally free algebra in order n) must be modified in a similar way. In fact, Definition 5.4 assumes now the form:

A closure algebra Γ is called functionally free if for any class $\mathfrak{K}$ of algebras which contains Γ, and for every positive integer n, any two closure-algebraic $\mathfrak{K}$-functions of n variables which are equal in Γ are also identically equal (i.e., in every other algebra of $\mathfrak{K}$).

The relations between functionally free algebras of order n and free algebras with n generators remain unchanged. In particular, by using the newly introduced notion of a function algebra we can state the theorem:

If Γ is a functionally free algebra, then for every positive integer n, the function algebra of order n over Γ (i.e., over the class consisting of Γ alone) is a free algebra with n generators.

Theorem 5.12 and Corollary 5.13 must clearly be formulated for closure-algebraic $\mathfrak{K}$-functions where $\mathfrak{K}$ is an arbitrary class of closure algebras which contains the functionally free closure algebra in question. The remaining theorems of §5, such as Theorem 5.14 and Theorem 5.16, remain quite unchanged, since they do not involve the notion of closure-algebraic functions.

It should be emphasized that the construction outlined above depends on no special properties of closure algebra, and hence can be carried over to the domain of general algebra; of course we have in mind here only definitions and immediate consequences of these definitions, and not specific theorems on closure algebra. It must be noticed however, that the existence theorem for free algebras with n generators which was formulated above would require in general algebra a stronger assumption regarding the class $\mathfrak{K}$; in fact, the assumption that for every denumerable (and not only finite) algebra of the kind considered, $\mathfrak{K}$ contains an isomorphic algebra. If we were concerned with algebras with infinite operations, or with a non-denumerable number of operations, still stronger assumptions would be needed.

In various discussions of this subject in the literature, one can find quite a different definition of a free algebra with a given number of generators, and also a different proof of the existence of such algebras. Both the definition and the proof use certain terms of a meta-mathematical character. Thus for instance a free algebra with n generators is sometimes defined as one in which every equation which holds between generators is a consequence of the postulates defining this algebra; and in order to prove the existence of such an algebra one constructs an algebra of "functions", which are, however, not interpreted in a set-theoretical way, but as certain expressions for which "equality" is defined in an appropriate manner. It could be shown that this procedure, if quite rigorously described, is equivalent to that outlined here. However our considerations show that the introduction of meta-mathematical notions into the discussion of free algebras is quite superfluous. In the case of algebras with infinite operations, moreover, the meta-mathematical procedure could hardly be carried through in a rigorous way.

III. Absolutely Free Closure Algebras

We want to consider here a property of closure algebras which is stronger than that of being functionally free (see Definition 5.4). As was mentioned at the end of Part II, a meta-mathematical terminology is often used in discussing free algebras with a given number of generators; the same terminology can be applied to functionally free algebras. For the sake of simplicity, we shall use this terminology here in introducing the new notion we want to use, though it would also be possible to express what we want to say in purely mathematical terms.

A functionally free closure algebra can be characterized meta-mathematically as one in which only those closure-algebraic equations are identically satisfied which are identically satisfied in every closure algebra. (We use the term *closure-algebraic equations* here, in the same sense in which we have previously used the term *topological equations*). Let us now consider, instead of the class of closure-algebraic equations, a wider class of sentences which we shall call *universal sentences* (or more correctly, *sentential functions*). These are sentences which, roughly speaking, express the fact that arbitrary elements of the algebra have a certain property, and do not involve the existence of special elements. All these sentences are built up from closure-algebraic equations and inequalities (i.e., negations of equations) by means of the words "if $\cdots$ then", "or", "and", and the like. As an example, we can give the sentence:

$$\text{If } x \neq \Lambda \quad \text{and} \quad -x \neq \Lambda, \qquad \text{then } \mathbf{C}x \cap \mathbf{C}-x \neq \Lambda.$$

As we know, a closure algebra is called connected if every element x of the algebra satisfies this sentence.

Now we shall call a closure algebra *absolutely free* if every universal sentence which is satisfied by all elements of this algebra is also satisfied by all elements of every closure algebra. Obviously every absolutely free closure algebra is also functionally free. The converse does not hold; for instance the closure algebra over Euclidean space is, as we know, functionally free; but it is not absolutely free since it is connected, while not all closure algebras are connected. The problem arises whether there are any absolutely free closure algebras at all, and whether we can find any interesting examples and formulate any general sufficient conditions.

The answer to this question is positive. This can be shown in the following way. Theorem 4.15 (in its first part) implies that a closure-algebraic equation which is identically satisfied in every finite closure algebra is also identically satisfied in every closure algebra. By analyzing the proof of this result we see that it can easily be extended to arbitrary general sentences. Hence by the definition of universal algebras we conclude that every closure algebra which is universal for all finite closure algebras, is absolutely free. Therefore, by using Theorem 3.8, we obtain the following result:

Every totally disconnected dissectable closure algebra is absolutely free.

This applies in particular to the closure algebra over Cantor's discontinuum, or any dense-in-itself and denumerable topological space.

IV. Decision-Procedure in Closure Algebra

Theorem 4.15 implies a *decision-procedure* for all closure-algebraic equations; i.e., it provides a method which enables us in any particular case to decide in a finite number of steps whether a given closure-algebraic equation is identically satisfied in every closure algebra, or (what amounts to the same thing in view of Theorem 2.6) in every topological space. In fact, given an equation

$$A = B$$

we first replace it by the equivalent equation

$$(A \cap - B) \cup (-A \cap B) = \Lambda.$$

Furthermore we determine the number n of different variables occurring in the left member of the latter equation, and the order r of the function represented by this left member. Instead of the order of the function, we can take r to be any upper bound of the order; it is easily seen that such an upper bound can be found by counting the number of variable and constant symbols in the left member, each symbol being counted as many times as it actually occurs. Next we construct all closure algebras $(K, \cup, \cap, -, \mathbf{C})$ where K is a subset of the set of the first 2^{2^n+r} positive integers. The number of these algebras is clearly finite, and we can check whether our given equation is identically satisfied in all of them or not. In the second case, our equation is obviously not identically satisfied in *every* closure algebra. In the first case, on the other hand, it follows from Theorem 4.15 that it is satisfied in every closure algebra; for it is seen at once that every closure algebra with at most 2^{2^n+r} elements is isomorphic with one of those constructed. In this way the decision procedure is established.

In Part III of this appendix we have pointed out that the first part of Theorem 4.15 can be extended from equations to what we called universal sentences. It is seen that the second part can also be so extended. For n we take now the number of different variables occurring in the sentence. In order to compute r, we first replace as above, all equations and inequalities occurring in this sentence by equivalent equations and inequalities having Λ for their right members, and then we take for r the number of all constant and variable symbols in the sentence transformed in this way. It is evident that Theorem 4.15 thus extended implies a similar extension of the decision-procedure, which can now be applied, not only to equations, but to arbitrary universal sentences.

As was stated in Part I, Theorem 4.15 can be extended to derivative algebras. Hence we obtain a decision-procedure for algebraic equations in derivative algebras, which in turn can be extended to arbitrary universal sentences in these algebras[19].

V. Direct Unions in General Algebras

In §4 we made use of the notion of direct union of closure algebras, and applied the fact that the direct union of two closure algebras is itself a closure algebra. We take this opportunity to make some remarks and state some elementary results regarding this notion as applied to arbitrary algebras.

An algebra can be defined in general as a system constituted by a certain set K and certain operations O_1, O_2, $\cdots$ (in most cases a finite number of finite operations), with the assumption that the operations can be performed on arbitrary elements of K, and yield always again an element of K. Two algebras $(K, O_1, O_2, \cdots)$ and $(K', O_1', O_2', \cdots)$ are called *similar* if the number of operations is the same in both algebras and if the corresponding operations O_1 and O_1', O_2 and O_2', etc., are operations with the same number of terms.

We are usually interested not in individual algebras, but in certain classes of similar algebras, such as the class of groups, that of Boolean algebras, and the like. From the point of view of general algebraic properties, the simplest sort of algebra-classes are those which can be called *equationally definable*[20]. To every such class $\mathfrak{K}$ there corresponds a set of algebraic equations such that $\mathfrak{K}$ consists of just those algebras which satisfy all the given equations. A class of algebras which is characterized in a similar way by a set of universal sentences (in the sense of Part III), which may or may not be equations, will here, for want of a better term, be called *universally definable.* Every equationally definable class of algebras is of course universally definable, but not conversely. For instance the class of groups (defined as systems with a binary operation $a \cdot b$ and a unary operation a^{-1}) and that of closure algebras are equationally definable. On the other hand from the ordinary definitions of the class of semi-groups (as systems with one binary operation which is associative and satisfies the cancellation laws) it is seen only that this class is universally definable; and the same can be shown for the class of integrity domains on the basis of a suitable definition of this class. There are of course classes of algebras which are not equationally definable, or even universally definable.

As is well-known, the direct union of two or more algebras of an equationally

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[19] A decision-procedure in this domain was found by S. Jaśkowski in 1939, but the result was not published. So far as Tarski remembers, the procedure applied at any rate to universal sentences in derivative algebras, and probably also to closure algebras. Jaśkowski also showed, on the other hand, that no decision-procedure can be found for the sentences of closure algebra and derivative algebra which contain also so-called bound variables and quantifiers.

In view of the present war conditions it has been impossible to verify this information in detail.

[20] For a more detailed study of this notion, see the paper by Garrett Birkhoff, *On the the structure of abstract algebras*, Proceedings of the Cambridge Philosophical Society, vol. 31 (1935), pp. 433–454.

definable class is also an algebra of the same class. This is no longer true of arbitrary universally definable classes. A more detailed investigation of the notion shows that the following elementary theorems can be established:

An algebraic equation or conjunction of such equations holds in the direct union of certain algebras (i.e., is satisfied by any elements of this algebra) if and only if it holds in each of the given algebras.

An inequality holds in the direct union of certain algebras if and only if it holds in at least one of the given algebras.

If a conditional equation (i.e., an implication whose hypothesis is a conjuntion of equations, and whose conclusion is a single equation) holds in each of certain algebras, then it holds also in their direct union, but not conversely.

If, on the other hand, a disjunction of equations holds in the direct union, then it holds also in each of the given algebras, but not conversely.

Hence it is seen that not only equationally definable classes of algebras are closed under the operation of forming the direct union, but the same is true also for all those algebra-classes which are defined by an arbitrary set of universal sentences having the form of equations, inequalities, and conditional equations. For instance the class of semi-groups is of this type, so that a direct union of semi-groups is a semi-group. Nevertheless it can be shown in another way that the class of semi-groups is not equationally definable (since a homomorphic image of a semi-group is not necessarily a semi-group). On the other hand it can be easily shown that the direct union of two integrity domains is in general not an integrity domain (in fact, the direct union is never an integrity domain if each of the given integrity domains has at least two elements); hence we conclude that the class of integrity domains, although universally definable, cannot be characterized by a set of postulates each of which has one of the three forms mentioned above. *A fortiori*, the class of integrity domains is not equationally definable.

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