# Dyadic semantics for many-valued logics

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**Abstract**

This paper obtains an effective method which assigns two-valued semantics to every finite-valued truth-functional logic (in the direction of the so-called "Suszko's Thesis"), provided that its truth-values can be individualized by means of its linguistic resources. Such two-valued semantics permit us to obtain new tableau proof systems for a wide class of finite-valued logics, including the main many-valued paraconsistent logics.

## 1 Introduction

A *tarskian* logic is a set of formulas endowed with a reflexive, monotonic and transitive consequence relation. For those logics two reductive results apply: Wójcicki's Reduction shows that every tarskian logic $\mathcal{L}$ is $n$-valued, for some $n$ bounded by the cardinality of $\mathcal{L}$, while Suszko's Reduction shows that every tarskian logic can also be characterized as 2-valued.

As a somewhat surprising consequence, which resulted in a philosophical standpoint known as *Suszko's Thesis* (see the companion paper [5]), finitely-valued logics turn out to be bivalued, apparently threatening the original motivations for introducing many-valuedness.

In this paper we obtain an effective method which assigns a two-valued semantics to every finite-valued truth-functional logic provided that its truth-values can be individualized by means of its language.

The paper is organized as follows: Section 2 introduces the techniques of separating truth-values, which is a cornerstone of our procedure. Section 3 obtains, in an effective way, bivalued semantics for many-valued logics, from the separation of truth-values guaranteed by the main Theorem 3.4. Several detailed examples are given in Section 4. The question of obtaining 'bivalent' tableaux for such logics is treated in Section 5. Finally, Section 6 briefly summarizes the obtained results.

## 2 Separating truth-values

Let $ats = \{p_1, p_2, \ldots\}$ be a denumerable set of *atomic sentences*, and let $\Sigma = \{\Sigma_n\}_{n\in\mathbb{N}}$ be a propositional signature, where each $\Sigma_n$ is a set of *connectives* of arity $n$. Let $cct = \bigcup_{n\in\mathbb{N}} \Sigma_n$ be the set of connectives. The set of formulas $\mathbb{L}$ is then defined as the algebra freely generated by $ats$ over $\Sigma$. From now on, $\mathcal{L}$ will stand for a propositional finite-valued logic, and $\mathbb{L}$ for its set of formulas. Additionally, $\mathbb{V}$ is a fixed $\Sigma$-algebra defining a truth-functional semantics for $\mathcal{L}$ over a finite non-empty set of truth-values $\mathcal{V} = \mathcal{D} \cup \mathcal{U}$. Assume that $\mathcal{D} = \{d_1, \ldots, d_i\}$ and $\mathcal{U} = \{u_1, \ldots, u_j\}$ are the sets of designated and undesignated truth-values, respectively, with $\mathcal{D} \cap \mathcal{U} = \emptyset$. Assume also that the valuations composing the semantics of *genuinely $n$-valued* logics (logics having $n$-valued characterizing matrices, but no $m$-valued such matrices, for $m < n$) are given by the homomorphisms $\S : \mathbb{L} \to \mathbb{V}$. A *uniform substitution* is an endomorphism $\varepsilon : \mathbb{L} \to \mathbb{L}$. Let us denote by $\varphi(p_1, \ldots, p_n)$ a formula $\varphi$ whose set of atomic sentences appear among $p_1, \ldots, p_n$. From now on, we write $\varphi(p_1/\alpha_1, \ldots, p_n/\alpha_n)$ instead of $\varepsilon(\varphi(p_1, \ldots, p_n))$ whenever $\varepsilon(p_k) = \alpha_k$. Given a genuinely $n$-valued logic $\mathcal{L} = \langle \mathcal{V}, cct, \mathcal{D}\rangle$, we shall denote by $\mathcal{L}^{\mathbf{c}}$ any functionally complete genuinely $n$-valued (conservative) extension of it, that is, a logic $\mathcal{L}^{\mathbf{c}}$ with the same number of (un)designated values as $\mathcal{L}$, but which can define all $n$-valued matrices —if they were not already defined from the start.

**Def. 2.1** A set of *interpretation* maps $[.] : \mathcal{V}^n \to \mathcal{V}$ over $\mathbb{L}$, for each $n \in \mathbb{N}^+$, is defined as follows, given $\vec{v} = (v_1, \ldots, v_n) \in \mathcal{V}^n$:

(i) $[p_k](\vec{v}) = v_k$, if $1 \leq k \leq n$;

(ii) $[\otimes(\varphi_1, \ldots, \varphi_m)](\vec{v}) = \otimes([\varphi_1](\vec{v}), \ldots, [\varphi_m](\vec{v}))$, if $\otimes$ is an $m$-ary connective and we identify $\otimes$ with the corresponding operator in the algebra $\mathbb{V}$.

**Remark 2.2** Given formulas $\varphi(p)$ and $\alpha$ of $\mathcal{L}$, and a homomorphism $\S : \mathbb{L} \to \mathbb{V}$, then:

$$[\varphi](\S(\alpha)) = \S(\varphi(p/\alpha)). \qquad (*)$$

**Def. 2.3** Let $v_1, v_2 \in \mathcal{V}$. We say that $v_1$ and $v_2$ are *separated*, and we write $v_1 \sharp v_2$, if $v_1 \in \mathcal{D}$ and $v_2 \in \mathcal{U}$ (or vice-versa). Given some genuinely $n$-valued logic $\mathcal{L}$, there is always some formula $\varphi(p)$ of $\mathcal{L}^{\mathbf{c}}$ which *separates* $v_1$ and $v_2$, that is, such that $[\varphi](v_1)\sharp[\varphi](v_2)$ (or else one of these two values would be redundant, and the logic would thus not be genuinely $n$-valued). Equivalently, one can say that $\varphi(p)$ separates $v_1$ and $v_2$ if the truth-values obtained in the truth-table for $\varphi$ when $p$ takes the values $v_1$ and $v_2$ are separated. We say that $v_1$ and $v_2$ are *effectively separated* by a logic $\mathcal{L}$ in case there is some separating formula $\varphi(p)$ to be found among the original set of formulas of $\mathcal{L}$. In that case we will say that the values $v_1$ and $v_2$ of $\mathcal{L}$ are *(effectively) separable*.

**Example 2.4** Clearly, if $v_1 \sharp v_2$ then $p$ separates $v_1$ and $v_2$. Therefore, every pair of separated truth-values is always (effectively) separable. As another example, note that $\varphi(p) = \neg p$ separates 0 and $\frac{1}{2}$ in Łukasiewicz's logic Ł$_3$ (see the formulation of its matrices at Example 2.9), given that $[\neg p](0) = \neg 0 = 1$, $[\neg p](\frac{1}{2}) = \neg\frac{1}{2} = \frac{1}{2}$, and $1 \sharp \frac{1}{2}$.

The separability of the truth-values of a logic $\mathcal{L}$ surely depends on the original expressibility of this logic, i.e., the range of matrices that it can define by way of interpretations of its formulas. Take for instance a logic whose semantics is given by $\langle\{0, \frac{1}{2}, 1\}, \{\otimes\}, \{1\}\rangle$, where $v_1 \otimes v_2 = v_1$ if $v_1 = v_2$, otherwise $v_1 \otimes v_2 = 1$. The values 0 and $\frac{1}{2}$ of this logic are obviously not separable.

**Assumption 2.5** *(Separability)*
From this point on we will assume that every pair $\langle v_1, v_2\rangle \in \mathcal{D}^2 \cup \mathcal{U}^2$ such that $v_1 \neq v_2$ is effectively separable.

It follows from the last assumption that it is possible to individualize every truth-value in terms of membership to $\mathcal{D}$ (to be represented here by the 'classical' truth-value $T$) or to $\mathcal{U}$ (to be represented by the 'classical' truth-value $F$).

**Remark 2.6** Consider the mapping $t : \mathcal{V} \to \{T, F\}$ such that $t(v) = T$ iff $v \in \mathcal{D}$, for some logic $\mathcal{L}$. Note that:

$$\varphi \text{ separates } v_1 \text{ and } v_2 \text{ iff } t([\varphi](v_1)) \neq t([\varphi](v_2)). \qquad (**)$$

Now, suppose that $\varphi_{mn}$ separates $d_m$ and $d_n$ (for $1 \leq m < n \leq i$), and $\psi_{mn}$ separates $u_m$ and $u_n$ (for $1 \leq m < n \leq j$). Given a variable $x$ and $d \in \mathcal{D}$, consider the equation:

$$t([\varphi_{mn}](x)) = q^d_{mn}$$

where $q^d_{mn} = t([\varphi_{mn}](d))$. Observe that $q^d_{mn} \in \{T, F\}$ and $q^{d_m}_{mn} \neq q^{d_n}_{mn}$, using $(**)$. Thus, if $\vec{\varphi}_d(x)$ is the sequence $(t([\varphi_{mn}](x)) = q^d_{mn})_{1 \leq m<n \leq i}$, the distinguished truth-value $d$ can be characterized through the sequence of equations $Q_d(x) : \ (t(x) = T, \ \vec{\varphi}_d(x))$. That is:

$$x = d \ \text{ iff } \ t(x) = T \ \wedge \bigwedge_{1 \leq m<n \leq i} t([\varphi_{mn}](x)) = q^d_{mn}$$

characterizes $d$ in terms of membership to $\mathcal{D}$ or to $\mathcal{U}$ (or, equivalently, in terms of $T/F$), as desired. Analogously, if $r^u_{mn}$ is $t([\psi_{mn}](u))$ for $1 \leq m < n \leq j$ and $u \in \mathcal{U}$, then the sequence of equations $R_u(x) : \ (t(x) = F, \ \vec{\psi}_u(x))$ characterizes $u$ in terms of $T/F$, where $\vec{\psi}_u(x) = (t([\psi_{mn}](x)) = r^u_{mn})_{1 \leq m<n \leq j}$. That is:

$$x = u \ \text{ iff } \ t(x) = F \ \wedge \bigwedge_{1 \leq m<n \leq j} t([\psi_{mn}](x)) = r^u_{mn}$$

characterizes $u$ in terms of $T/F$ using $t$.

**Remark 2.7** If $\mathcal{D} = \{d\}$ then we simply write $x = d$ iff $t(x) = T$. Analogously, if $\mathcal{U} = \{u\}$ then we simply write $x = u$ iff $t(x) = F$.

**Remark 2.8** The composition $t \circ \S$ gives us the famed Suszko's 2-valued reduction of any given logic $\mathcal{L}$, viz. a 2-valued (usually non-truth-functional) semantical presentation of $\mathcal{L}$. Given a logic which respects our Separability Assumption 2.5, we will see in the next section how this 2-valued semantics can be mechanically written down in terms of 'dyadic semantics'. A later section will show how such semantics can provide us with classic-like tableaux for those same logics.

**Example 2.9** Consider the $n$-valued logics of Łukasiewicz, $n > 2$, which can be formulated by way of:

$$\text{Ł}_n = \langle \{0, \tfrac{1}{n-1}, \ldots, \tfrac{n-2}{n-1}, 1\}, \{\neg, \Rightarrow, \vee, \wedge\}, \{1\} \rangle.$$

The above operations over the truth-values can be defined as follows:

$$\begin{array}{ll} \neg v_1 := 1 - v_1; & (v_1 \Rightarrow v_2) := \mathsf{Min}(1, 1 - v_1 + v_2); \\ (v_1 \vee v_2) := \mathsf{Max}(v_1, v_2); & (v_1 \wedge v_2) := \mathsf{Min}(v_1, v_2). \end{array}$$

Consider now the particular case of $\text{Ł}_5$. Then we can take, for instance:

$$\psi_{0\frac{1}{4}} = \psi_{0\frac{2}{4}} = \psi_{0\frac{3}{4}} = \neg p; \quad \psi_{\frac{1}{4}\frac{2}{4}} = \psi_{\frac{1}{4}\frac{3}{4}} = (\neg p \Rightarrow p); \quad \psi_{\frac{2}{4}\frac{3}{4}} = (p \Rightarrow \neg p).$$

To save on notation, take $\triangle(p) = \psi_{\frac{1}{4}\frac{2}{4}}$ and $\triangledown(p) = \psi_{\frac{2}{4}\frac{3}{4}}$, and consider next the table:

| $v$ | $\neg v$ | $\triangle(v)$ | $\triangledown(v)$ |
|---|---|---|---|
| $0$ | $1$ | $0$ | $1$ |
| $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{2}{4}$ | $1$ |
| $\frac{2}{4}$ | $\frac{2}{4}$ | $1$ | $1$ |
| $\frac{3}{4}$ | $\frac{1}{4}$ | $1$ | $\frac{2}{4}$ |

Note that (the reduced version of) each $\vec{\psi}_k(x)$ is as follows:

$$\vec{\psi}_0(x) = \langle t(\neg x) = T,\ t(\triangle(x)) = F,\ t(\triangledown(x)) = T \rangle,$$

$$\vec{\psi}_{\frac{1}{4}}(x) = \langle t(\neg x) = F,\ t(\triangle(x)) = F,\ t(\triangledown(x)) = T \rangle,$$

$$\vec{\psi}_{\frac{2}{4}}(x) = \langle t(\neg x) = F,\ t(\triangle(x)) = T,\ t(\triangledown(x)) = T \rangle,$$

$$\vec{\psi}_{\frac{3}{4}}(x) = \langle t(\neg x) = F,\ t(\triangle(x)) = T,\ t(\triangledown(x)) = F \rangle.$$

We obtain thus the following characterizations of the truth-values:

$$x = 0 \text{ iff } t(x) = F \wedge t(\neg x) = T \wedge t(\triangle(x)) = F \wedge t(\triangledown(x)) = T,$$

$$x = \tfrac{1}{4} \text{ iff } t(x) = F \wedge t(\neg x) = F \wedge t(\triangle(x)) = F \wedge t(\triangledown(x)) = T,$$

$$x = \tfrac{2}{4} \text{ iff } t(x) = F \wedge t(\neg x) = F \wedge t(\triangle(x)) = T \wedge t(\triangledown(x)) = T,$$

$$x = \tfrac{3}{4} \text{ iff } t(x) = F \wedge t(\neg x) = F \wedge t(\triangle(x)) = T \wedge t(\triangledown(x)) = F.$$

Of course, the sole distinguished truth-value 1 is characterized simply by:

$$x = 1 \text{ iff } t(x) = T.$$

A similar procedure can be applied to all the remaining finite-valued logics of Łukasiewicz, making use for instance of the well-known Rosser-Turquette (definable) functions so as to produce the appropriate effective separations of truth-values.

# 3 From finite matrices to dyadic valuations

Using the assumptions and the ideas from the last section, we will now show how to mechanically obtain 2-valued semantical counterparts of a large bunch of finite-valued logics. The axioms of the bivaluations produced by our method will in fact follow a very specific format, characterizing what we shall call 'dyadic semantics'. To that effect, we shall be making use of an appropriate equational language, made explicit in the following.

**Def. 3.1** A *gentzenian semantics* for a logic $\mathcal{L}$ is an adequate (sound and complete) set of 2-valued valuations $b : \mathbb{L} \to \{T, F\}$ given by conditional clauses $(\Phi \to \Psi)$ where both $\Phi$ and $\Psi$ are (meta)formulas of the form $\top$ (top), $\bot$ (bottom) or:

$$b(\varphi_1^1) = w_1^1, \ldots, b(\varphi_1^{n_1}) = w_1^{n_1} \mid \ldots \mid b(\varphi_m^1) = w_m^1, \ldots, b(\varphi_m^{n_m}) = w_m^{n_m}. \quad (G)$$

Here, $w_i^j \in \{T, F\}$, each $\varphi_i^j$ is a formula of $\mathcal{L}$, commas "," represent conjunctions, and bars "|" represent disjunctions. The (meta)logic governing these clauses is FOL, First-Order Classical Logic. We can alternatively write a clause of the form $(G)$ as $\bigvee_{1 \leq k \leq m} \bigwedge_{1 \leq s \leq n_m} b(\varphi_k^s) = w_k^s$.

Now, a dyadic semantics will be just a specialization of gentzenian semantics, in a deliberate intent to capture the computable class of such semantics, as follows. (A rigorous definition of dyadic semantics is given in the companion paper [5]).

**Def. 3.2** A gentzenian semantics $\mathcal{B}$ for a logic $\mathcal{L}$ is said to constitute a *dyadic semantics* for $\mathcal{L}$ in case the consequence relation $\models_{\mathcal{B}}$ (given by the valuations in $\mathcal{B}$) is recursive.

For instance, if it is possible to obtain a tableau decision procedure from a gentzenian semantics $\mathcal{B}$ for a logic $\mathcal{L}$ then $\mathcal{B}$ is a dyadic semantics for $\mathcal{L}$.

Now, let $\otimes$ be some connective of $\mathcal{L}$; for the sake of simplicity, suppose that $\otimes$ is binary. If an entry of the truth-table for $\otimes$ states that $\otimes(v_1, v_2) = v$ then we can express this situation as follows:

$$\text{if } x = v_1 \text{ and } y = v_2, \text{ then } \otimes(x, y) = v.$$

Now, recall from Remark 2.6 the mapping $t : \mathcal{V} \to \{T, F\}$ such that $t(v) = T$ iff $v \in \mathcal{D}$. If the previous situation is expressed in terms of $T/F$ using this mapping, we will get, respectively, systems of equations $E_{v_1}(x)$, $E_{v_2}(y)$ and $E_v(\otimes(x, y))$, and consequently the following statement in terms of $T/F$:

$$\text{if } E_{v_1}(x) \text{ and } E_{v_2}(y) \text{ then } E_v(\otimes(x, y)).$$

In the formal (meta)language of a gentzenian semantics (Def. 3.1), this statement is of the form:

$$\begin{array}{rl} & t([\beta_1](x)) = w_1, \ldots, t([\beta_m](x)) = w_m, \\ & t([\gamma_1](y)) = w'_1, \ldots, t([\gamma_{m'}](y)) = w'_{m'} \\ \to & t([\delta_1](\otimes(x, y))) = w''_1, \ldots, t([\delta_{m''}](\otimes(x, y))) = w''_{m''}, \end{array} \qquad (***)$$

where $w_n, w'_{k'}, w''_{s''} \in \{T, F\}$ for $1 \leq n \leq m$, $1 \leq k' \leq m'$ and $1 \leq s'' \leq m''$.

Now, suppose that $v$ is $\S(\alpha)$ for some formula $\alpha$. But then, using $(*)$ (check Remark 2.2, but also 2.8) we obtain:

$$t([\varphi](v)) = t([\varphi](\S(\alpha))) = t(\S(\varphi(p/\alpha))) = b(\varphi(p/\alpha))$$

for every formula $\varphi(p)$. Using this in $(***)$ we obtain an axiom for $\mathcal{B}$ of the form:

$$\begin{array}{rl} & b(\beta_1(p/\alpha)) = w_1, \ \ldots, \ b(\beta_m(p/\alpha)) = w_m, \\ & b(\gamma_1(p/\beta)) = w'_1, \ \ldots, \ b(\gamma_{m'}(p/\beta)) = w'_{m'} \\ \to & b(\delta_1(p/\otimes(\alpha, \beta))) = w''_1, \ \ldots, \ b(\delta_{m''}(p/\otimes(\alpha, \beta))) = w''_{m''}, \end{array}$$

for $w_n, w'_{k'}, w''_{s''} \in \{T, F\}$ etc. Of course we can repeat this process for each entry of each connective $\otimes$ of $\mathcal{L}$. For 0-ary connectives there is no imput at the left-hand side —so, you should write conditional clauses $(\Phi \to \Psi)$ where $\Phi$ is $\top$.

**Example 3.3** In Ł$_5$ we have, for instance, the following entry in the truth-table for $\wedge$: if $v_1 = \frac{2}{4}$ and $v_2 = 1$ then $v_1 \wedge v_2 = \frac{2}{4}$. Or, in other words: if $\S(\alpha) = \frac{2}{4}$ and $\S(\beta) = 1$ then $\S(\alpha \wedge \beta) = \frac{2}{4}$, for any formulas $\alpha$ and $\beta$, and any homomorphism $\S$. By Example 2.9 we obtain, using $t$ and $b = t \circ \S$:

$$\begin{array}{rl} & b(\alpha) = F, \ b(\neg\alpha) = F, \ b(\triangle(\alpha)) = T, \ b(\nabla(\alpha)) = T, \ b(\beta) = T \\ \to & b(\alpha \wedge \beta) = F, \ b(\neg(\alpha \wedge \beta)) = F, \ b(\triangle(\alpha \wedge \beta)) = T, b(\nabla(\alpha \wedge \beta)) = T. \end{array}$$

So, each entry of the truth-table for each connective $\otimes$ of $\mathcal{L}$ determines an axiom for a gentzenian valuation mapping $b : \mathbb{L} \to \{T, F\}$. We obtain thus, through the above method, a kind of unique (partial) 'binary print' of the original truth-functional logic.

**Theorem 3.4** Given a logic $\mathcal{L}$, let $\mathcal{B}$ be the set of gentzenian valuations $b : \mathbb{L} \to \{T, F\}$ satisfying the axioms obtained from the truth-tables of $\mathcal{L}$ using the above method, plus the following axioms:

**(C1)** $\top \ \to \ b(\alpha) = T \mid b(\alpha) = F$;

**(C2)** $b(\alpha) = T,\ b(\alpha) = F \ \to\ \bot$;

**(C3)** $b(\alpha) = T \ \to\ \bigvee_{d\in\mathcal{D}} \bigwedge_{1\le m<n\le i} b(\varphi_{mn}(p/\alpha)) = q^d_{mn}$;

**(C4)** $b(\alpha) = F \ \to\ \bigvee_{u\in\mathcal{U}} \bigwedge_{1\le m<n\le j} b(\psi_{mn}(p/\alpha)) = r^u_{mn}$

for every $\alpha \in \mathbb{L}$ (here, $q^d_{mn}$ and $r^u_{mn}$ are as in Remark 2.6). Then $b \in \mathcal{B}$ iff $b = t \circ \S$ for some homomorphism $\S : \mathbb{L} \to \mathbb{V}$.

**Proof:** Given $b \in \mathcal{B}$, define a homomorphism $\S : \mathbb{L} \to \mathbb{V}$ such that:

(i) $\S(\alpha) = d$ iff $b(\alpha) = T$ and $b(\varphi_{mn}(p/\alpha)) = q^d_{mn}$ for every $1 \le m < n \le i$;

(ii) $\S(\alpha) = u$ iff $b(\alpha) = F$ and $b(\psi_{mn}(p/\alpha)) = r^u_{mn}$ for every $1 \le m < n \le j$,

where $\alpha$ ranges over the atomic sentences $ats \in \mathbb{L}$. Note that $\S$ is well-defined as a total functional assignment because $b \in \mathcal{B}$ satisfies conditions (**C1**)–(**C2**) above. Since $b$ satisfies all the axioms obtained from all the entries of the truth-tables of $\mathcal{L}$, it is straightforward to prove, by induction on the complexity of the formula $\alpha \in \mathbb{L}$, that (i) and (ii) hold when $\alpha$ ranges over all the formulas in $\mathbb{L}$. (Indeed, note that, in the light of conditions (**C3**)–(**C4**), given $b \in \mathcal{B}$ and $b(\alpha) = T$ we can conclude that there exists a unique $d \in \mathcal{D}$ such that $\bigwedge_{1\le m<n\le i} b(\varphi_{mn}(p/\alpha)) = q^d_{mn}$; on the other hand, given $b(\alpha) = F$ we can conclude that there exists a unique $u \in \mathcal{U}$ such that $\bigwedge_{1\le m<n\le j} b(\psi_{mn}(p/\alpha)) = r^u_{mn}$.) From this we obtain that $\S(\varphi) \in \mathcal{D}$ iff $b(\varphi) = T$, therefore $b = t \circ \S$ as desired.

The converse —if $b = t \circ \S$ for some homomorphism $\S$, then $b \in \mathcal{B}$— is immediate. QED

So, while the initial many-valued semantics seemed to defeat the so-called 'principle of bivalence', the new bivalued adequate semantics based on but two 'logical values' finally restored bivalence by way of its specific choice of designated / undesignated truth-values and by way of clauses (**C1**)–(**C2**).

**Corollary 3.5** (i) For every bivaluation $b : \mathbb{L} \to \{T, F\}$ in $\mathcal{B}$ there exists a homomorphism $\S_b : \mathbb{L} \to \mathbb{V}$ such that:

$$\S_b(\alpha) \in \mathcal{D} \ \text{ iff } \ b(\alpha) = T, \text{ for any } \alpha \in \mathbb{L}; \tag{1}$$

(ii) for every $\S : \mathbb{L} \to \mathbb{V}$ there exists a $b_\S \in \mathcal{B}$ such that:

$$b_\S(\alpha) = T \ \text{ iff } \ \S(\alpha) \in \mathcal{D}, \text{ for any } \alpha \in \mathbb{L}. \tag{2}$$

We now have two notions of semantic entailment for $\mathcal{L}$: the first one, $\models$, uses the truth-tables given by $\mathbb{V}$ and the corresponding homomorphic valuations $\S$, whereas the second one, $\models_\mathcal{B}$, uses the related gentzenian semantics $\mathcal{B}$.

**Theorem 3.6** The set $\mathcal{B}$ of gentzenian valuations for $\mathcal{L}$ is adequate, that is, for any $\Gamma \cup \{\varphi\} \subseteq \mathbb{L}$:

$$\Gamma \models \varphi \ \text{ iff } \ \Gamma \models_\mathcal{B} \varphi.$$

**Proof:** Suppose that $\Gamma \models \varphi$, and let $b \in \mathcal{B}$ be such that $b(\Gamma) \subseteq \{T\}$, if possible. By Corollary 3.5(i) there exists a homomorphism $\S_b$ such that $\S_b(\Gamma) \subseteq \mathcal{D}$. By hypothesis we get $\S_b(\varphi) \in \mathcal{D}$, whence $b(\varphi) = T$ by (1). This shows that $\Gamma \models_{\mathcal{B}} \varphi$. The converse is proven in an analogous way, using Corollary 3.5(ii). QED

# 4 Some Examples

In this section we will give examples of gentzenian semantics for several genuinely finite-valued paraconsistent logics, obtained through an application of the 2-valued reduction algorithm proposed in the last section. Instead of writing extensive lists of bivaluation axioms, one for each entry of each truth-table, plus some complementing axioms, we shall be using First-Order Classical Logic, FOL, in what follows, in order to manipulate and simplify the clauses written in our equational (meta)language. Moreover, we will often seek to reformulate things so as to make them more convenient for a tableaux-oriented approach, as in the next section.

**Example 4.1** The paraconsistent logic $\mathbf{P}^1_3 = \langle \{0, \frac{1}{2}, 1\}, \{\neg, \Rightarrow\}, \{\frac{1}{2}, 1\}\rangle$, was introduced by Sette in [15] (where it was called $P^1$), having as truth-tables:

| | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $\neg$ | $1$ | $1$ | $0$ |

| $\Rightarrow$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ |
| $\frac{1}{2}$ | $0$ | $1$ | $1$ |
| $1$ | $0$ | $1$ | $1$ |

Note that $\neg p$ separates $\frac{1}{2}$ and 1. Indeed:

$$[\neg p](1) = 0, \ [\neg p](\tfrac{1}{2}) = 1$$

and $0\sharp 1$. Thus:

$x = 0$ iff $t(x) = F$;
$x = \frac{1}{2}$ iff $t(x) = T, \ t(\neg x) = T$;
$x = 1$ iff $t(x) = T, \ t(\neg x) = F$.

Applying our reduction algorithm to the truth-tables of $\neg$ and $\Rightarrow$ we obtain the following axioms for $b$:

(i) $b(\alpha) = F \ \rightarrow \ b(\neg\alpha) = T, \ b(\neg\neg\alpha) = F$;
(ii) $b(\alpha) = T, \ b(\neg\alpha) = T \ \rightarrow \ b(\neg\alpha) = T, \ b(\neg\neg\alpha) = F$;
(iii) $b(\alpha) = T, \ b(\neg\alpha) = F \ \rightarrow \ b(\neg\alpha) = F$;
(iv) $b(\alpha) = F \mid b(\beta) = T \ \rightarrow \ b(\alpha \Rightarrow \beta) = T, \ b(\neg(\alpha \Rightarrow \beta)) = F$;
(v) $b(\alpha) = T, \ b(\beta) = F \ \rightarrow \ b(\alpha \Rightarrow \beta) = F$.

In this case, axiom **(C3)** gives $b(\alpha) = T \ \rightarrow \ b(\neg\alpha) = T \mid b(\neg\alpha) = F$, which can be derived from **(C1)**. Axiom **(C4)** gives $b(\alpha) = F \ \rightarrow \ b(\neg\alpha) = T$, which is derivable from the above clause (i).

Using FOL we can rewrite clauses (i)–(v) equivalently as:

(4.1.1) $b(\neg\alpha) = F \;\to\; b(\alpha) = T$;
(4.1.2) $b(\neg\neg\alpha) = T \;\to\; b(\neg\alpha) = F$;
(4.1.3) $b(\alpha \Rightarrow \beta) = T \;\to\; b(\alpha) = F \mid b(\beta) = T$;
(4.1.4) $b(\alpha \Rightarrow \beta) = F \;\to\; b(\alpha) = T,\ b(\beta) = F$;
(4.1.5) $b(\neg(\alpha \Rightarrow \beta)) = T \;\to\; b(\alpha) = T,\ b(\beta) = F$.

Note that (4.1.3)–(4.1.5) axiomatize a sort of 'classic-like' implication.

Axioms (4.1.1)–(4.1.5) plus **(C1)**–**(C2)** characterize a dyadic semantics for $\mathbf{P}^1_3$.

**Example 4.2** The paraconsistent logic $\mathbf{P}^1_4 = \langle\{0, \frac{1}{3}, \frac{2}{3}, 1\}, \{\neg, \Rightarrow\}, \{\frac{1}{3}, \frac{2}{3}, 1\}\rangle$, was introduced in [7] and [11], and studied under the name $P^2$ in [10]. The truth-tables of its connectives are as follows:

| | $0$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $1$ |
|---|---|---|---|---|
| $\neg$ | $1$ | $\frac{2}{3}$ | $1$ | $0$ |

| $\Rightarrow$ | $0$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $1$ |
|---|---|---|---|---|
| $0$ | 1 | 1 | 1 | 1 |
| $\frac{1}{3}$ | 0 | 1 | 1 | 1 |
| $\frac{2}{3}$ | 0 | 1 | 1 | 1 |
| $1$ | 0 | 1 | 1 | 1 |

It is easy to see that $\neg p$ separates 1 and $\frac{1}{3}$, as well as 1 and $\frac{2}{3}$. On the other hand, $\neg\neg p$ separates $\frac{1}{3}$ and $\frac{2}{3}$. From this we get:

$x = 0$ iff $t(x) = F$;
$x = \frac{1}{3}$ iff $t(x) = T,\ t(\neg x) = T,\ t(\neg\neg x) = T$;
$x = \frac{2}{3}$ iff $t(x) = T,\ t(\neg x) = T,\ t(\neg\neg x) = F$;
$x = 1$ iff $t(x) = T,\ t(\neg x) = F,\ t(\neg\neg x) = T$.

From the truth-table for $\neg$ we obtain, after applying FOL:

(4.2.1) $b(\neg\alpha) = F \;\to\; b(\alpha) = T$;
(4.2.2) $b(\neg\neg\alpha) = T \;\to\; b(\alpha) = T$;
(4.2.3) $b(\neg\neg\neg\alpha) = T \;\to\; b(\neg\neg\alpha) = F$.

Now, axiom **(C3)** is derivable from **(C1)**, and axiom **(C4)** is derivable from the clauses above.

The implication $\Rightarrow$ is again 'classic-like', in the same sense as in the last example. Therefore, axioms (4.2.1)–(4.2.3), (4.1.3)–(4.1.5) and **(C1)**–**(C2)** characterize together a dyadic semantics for $\mathbf{P}^1_4$. Similar procedures can be applied to each paraconsistent logic of the hierarchy $\mathbf{P}^1_{n+2} (= P^n)$, for $n \in \mathbb{N}^+$, from [10].

**Example 4.3** Having already used negation in the two above examples in order to separate truth-values, let us now make it differently. Consider the paraconsistent propositional logic **LFI1** $= \langle\{0, \frac{1}{2}, 1\}, \{\neg, \bullet, \Rightarrow, \wedge, \vee\}, \{\frac{1}{2}, 1\}\rangle$, studied in detail in [9], whose matrices are:

|  | 0 | $\frac{1}{2}$ | 1 |
|---|---|---|---|
| $\neg$ | 1 | $\frac{1}{2}$ | 0 |
| $\bullet$ | 0 | 1 | 0 |

| $\Rightarrow$ | 0 | $\frac{1}{2}$ | 1 |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 |
| 1 | 0 | $\frac{1}{2}$ | 1 |

plus conjunction $\wedge$ and disjunction $\vee$ defined as in Łukasiewicz's logics (see Example 2.9). Clearly, $\bullet p$ separates 1 and $\frac{1}{2}$, then:

$x = 0$ iff $t(x) = F$;
$x = \frac{1}{2}$ iff $t(x) = T$, $t(\bullet x) = T$;
$x = 1$ iff $t(x) = T$, $t(\bullet x) = F$.

From the truth-table for $\neg$, and using FOL, we obtain:

(4.3.1) $b(\neg\alpha) = T \;\to\; b(\alpha) = F \mid b(\bullet\alpha) = T$;
(4.3.2) $b(\neg\alpha) = F \;\to\; b(\alpha) = T,\; b(\bullet\alpha) = F$.

Axiom **(C3)** is again derivable from **(C1)**; axiom **(C4)** is derivable from (4.3.2). Now, these are the axioms for $\bullet$:

(4.3.3) $b(\bullet\alpha) = T \;\to\; b(\alpha) = T$;
(4.3.4) $b(\bullet\bullet\alpha) = T \;\to\; b(\bullet\alpha) = F$;
(4.3.5) $b(\bullet\neg\alpha) = T \;\to\; b(\bullet\alpha) = T$;
(4.3.6) $b(\bullet\neg\alpha) = F \;\to\; b(\neg\alpha) = F \mid b(\alpha) = F$.

From the truth-tables for the binary connectives, and using FOL, we obtain:

(4.3.7) $b(\alpha \wedge \beta) = T \;\to\; b(\alpha) = T,\; b(\beta) = T$;
(4.3.8) $b(\alpha \wedge \beta) = F \;\to\; b(\alpha) = F \mid b(\beta) = F$;
(4.3.9) $b(\alpha \vee \beta) = T \;\to\; b(\alpha) = T \mid b(\beta) = T$;
(4.3.10) $b(\alpha \vee \beta) = F \;\to\; b(\alpha) = F,\; b(\beta) = F$;
(4.3.11) $b(\alpha \Rightarrow \beta) = T \;\to\; b(\alpha) = F \mid b(\beta) = T$;
(4.3.12) $b(\alpha \Rightarrow \beta) = F \;\to\; b(\alpha) = T, b(\beta) = F$;

To those we may add, furthermore:

(4.3.13) $b(\bullet(\alpha \wedge \beta)) = T$
$\to\; b(\alpha) = T,\; b(\bullet\beta) = T \mid b(\beta) = T,\; b(\bullet\alpha) = T$;
(4.3.14) $b(\bullet(\alpha \wedge \beta)) = F$
$\to\; b(\alpha) = F \mid b(\beta) = F \mid b(\alpha) = T,\; b(\bullet\alpha) = F,\; b(\beta) = T,\; b(\bullet\beta) = F$;
(4.3.15) $b(\bullet(\alpha \vee \beta)) = T$
$\to\; b(\alpha) = F,\; b(\bullet\beta) = T \mid b(\beta) = F,\; b(\bullet\alpha) = T \mid b(\bullet\alpha) = T,\; b(\bullet\beta) = T$;
(4.3.16) $b(\bullet(\alpha \vee \beta)) = F$
$\to\; b(\alpha) = F,\; b(\beta) = F \mid b(\alpha) = T,\; b(\bullet\alpha) = F \mid b(\beta) = T,\; b(\bullet\beta) = F$;
(4.3.17) $b(\bullet(\alpha \Rightarrow \beta)) = T \;\to\; b(\alpha) = T,\; b(\bullet\beta) = T$;
(4.3.18) $b(\bullet(\alpha \Rightarrow \beta)) = F \;\to\; b(\alpha) = F \mid b(\bullet\beta) = F$.

So, if the above axioms are taken together with **(C1)**–**(C2)**, then we obtain a natural dyadic semantics for **LFI1**. Also in [9] two slightly different (non-gentzenian) bivaluation semantics for **LFI1** were explored.

**Example 4.4** Belnap's paraconsistent and paracomplete 4-valued logic (cf. [2]), $B_4 = \langle\{0, \frac{1}{3}, \frac{2}{3}, 1\}, \{\neg, \wedge, \vee\}, \{\frac{2}{3}, 1\}\rangle$, can be presented by way of the following matrices:

| | $0$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $1$ |
|---|---|---|---|---|
| $\neg$ | $0$ | $\frac{2}{3}$ | $\frac{1}{3}$ | $1$ |

| $\wedge$ | $0$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $1$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $\frac{1}{3}$ | $0$ | $\frac{1}{3}$ | $0$ | $\frac{1}{3}$ |
| $\frac{2}{3}$ | $0$ | $0$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| $1$ | $0$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $1$ |

| $\vee$ | $0$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $1$ |
|---|---|---|---|---|
| $0$ | $0$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $1$ |
| $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $1$ | $1$ |
| $\frac{2}{3}$ | $\frac{2}{3}$ | $1$ | $\frac{2}{3}$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ |

Clearly, $\neg p$ separates 1 and $\frac{2}{3}$ and also separates $\frac{1}{3}$ and 1, so:

$x = 0$ iff $t(x) = F$, $t(\neg x) = F$;
$x = \frac{1}{3}$ iff $t(x) = F$, $t(\neg x) = T$;
$x = \frac{2}{3}$ iff $t(x) = T$, $t(\neg x) = F$;
$x = 1$ iff $t(x) = T$, $t(\neg x) = T$.

Now, from the truth-table for $\neg$, and using FOL, we obtain:

(4.4.1) $b(\neg\neg\alpha) = T \;\rightarrow\; b(\alpha) = T$;
(4.4.2) $b(\neg\neg\alpha) = F \;\rightarrow\; b(\alpha) = F$.

Both axioms **(C3)** and **(C4)** are now derivable from **(C1)**. From the truth-tables of conjunction and disjunction, using FOL, we obtain the positive clauses (4.3.7)–(4.3.10) again, but also:

(4.4.3) $b(\neg(\alpha \wedge \beta)) = T \;\rightarrow\; b(\alpha) = F,\ b(\neg\alpha) = T,\ b(\beta) = F,\ b(\neg\beta) = T \mid b(\alpha) = F,\ b(\neg\alpha) = T,\ b(\beta) = T,\ b(\neg\beta) = T \mid b(\alpha) = T,\ b(\neg\alpha) = T,\ b(\beta) = F,\ b(\neg\beta) = T \mid b(\alpha) = T,\ b(\neg\alpha) = T\ b(\beta) = T,\ b(\neg\beta) = T$;

(4.4.4) $b(\neg(\alpha \wedge \beta)) = F \;\rightarrow\; b(\alpha) = F,\ b(\neg\alpha) = F \mid b(\alpha) = T,\ b(\neg\alpha) = F \mid b(\beta) = F,\ b(\neg\beta) = F \mid b(\beta) = T,\ b(\neg\beta) = F$;

(4.4.5) $b(\neg(\alpha \vee \beta)) = T \;\rightarrow\; b(\alpha) = F,\ b(\neg\alpha) = T \mid b(\alpha) = T,\ b(\neg\alpha) = T \mid b(\beta) = F,\ b(\neg\beta) = T \mid b(\beta) = T,\ b(\neg\beta) = T$;

(4.4.6) $b(\neg(\alpha \vee \beta)) = F \;\rightarrow\; b(\alpha) = F,\ b(\neg\alpha) = F,\ b(\beta) = F,\ b(\neg\beta) = F \mid b(\alpha) = F,\ b(\neg\alpha) = F,\ b(\beta) = T,\ b(\neg\beta) = F \mid b(\alpha) = T,\ b(\neg\alpha) = F,\ b(\beta) = F,\ b(\neg\beta) = F \mid b(\alpha) = T,\ b(\neg\alpha) = F\ b(\beta) = T,\ b(\neg\beta) = F$.

A dyadic semantics for $B_4$ is given by the above axioms, plus **(C1)**–**(C2)**.

# 5 Application: tableaux for logics with dyadic semantics

In the examples from the last section we found axioms for the set of bivaluation mappings $b$ —defining a gentzenian semantics for a genuinely finite-valued logic $\mathcal{L}$— expressed as conditional clauses of the form:

$b(\alpha) = w$
$\quad \to b(\alpha_1^1) = w_1^1, \ldots, b(\alpha_1^{n_1}) = w_1^{n_1} \mid \ldots \mid b(\alpha_m^1) = w_m^1, \ldots, b(\alpha_m^{n_m}) = w_m^{n_m},$

where $w, w_k^s \in \{T, F\}$ and $\alpha_k^s$ has smaller complexity, under some appropriate measure (cf. [5]), than $\alpha$. Each clause as above generates a tableau rule for $\mathcal{L}$ as follows: Translate $b(\beta) = T$ as the signed formula $T(\beta)$, and $b(\beta) = F$ as the signed formula $F(\beta)$. Then, a conditional clause such as the one above induces the following tableau-rule:

$$\begin{array}{ccccc} & & w(\alpha) & & \\ & \swarrow & \cdots & \searrow & \\ w_1^1(\alpha_1^1) & & & & w_m^1(\alpha_m^1) \\ \vdots & & & & \vdots \\ w_1^{n_1}(\alpha_1^{n_1}) & & & & w_m^{n_m}(\alpha_m^{n_m}) \end{array}$$

where $w, w_k^s \in \{T, F\}$. In that case, it is straightforward to prove that the set of tableau rules for $\mathcal{L}$ obtained from the clauses for $\mathcal{B}$ characterizes a sound and complete tableau system for $\mathcal{L}$. The structural similarity between the tableau rules so obtained and the classical ones is not fortuitous. Once we have obtained an appropriate dyadic semantics for a many-valued logic, then we can forget the many truth- values which might have been in use as labels (as in [6]), and work only with the 'logical values' $T$ and $F$, just like in the classical case. While the former many-signed tableaux have the so-called *subformula property*, according to which each formula $\alpha_k^s$ obtained from the application to $\alpha$ of a tableau rule as the one above is a subformula of the initial formula $\alpha$ —the latter related two-signed tableaux obtained through our method will lose this property, paralleling the lost of truth-functionality of the many-valued homomorphisms by the two-valued valuations. We will still have, though, a *shortening property* which is as good for efficiency as the subformula property: each formula $\alpha_k^s$ will be *less complex* (under some appropriate measure, cf. [5]) than the initial formula $\alpha$ being analyzed by the tableau rules.

**Example 5.1** The following set of rules characterizes a tableau system for the paraconsistent logic $\mathbf{P}_3^1$, according to clauses (4.1.1)–(4.1.5) of Example 4.1:

$$(5.1.1)\ \frac{F(\neg\alpha)}{T(\alpha)} \qquad (5.1.2)\ \frac{T(\neg\neg\alpha)}{F(\neg\alpha)}$$

$$(5.1.3)\ \frac{T(\alpha \Rightarrow \beta)}{F(\alpha) \mid T(\beta)} \qquad (5.1.4)\ \frac{F(\alpha \Rightarrow \beta)}{T(\alpha),\ F(\beta)} \qquad (5.1.5)\ \frac{T(\neg(\alpha \Rightarrow \beta))}{T(\alpha),\ F(\beta)}$$

**Example 5.2** Following Example 4.2, an adequate set of tableau rules for the paraconsistent logic $\mathbf{P}_4^1$ is given by (5.1.3)–(5.1.5) plus:

$$(5.2.1)\ \frac{F(\neg\alpha)}{T(\alpha)} \qquad (5.2.2)\ \frac{T(\neg\neg\alpha)}{T(\alpha)} \qquad (5.2.3)\ \frac{T(\neg\neg\neg\alpha)}{F(\neg\neg\alpha)}$$

**Example 5.3** Now we exhibit a tableau system for the paraconsistent logic **LFI1** (see Example 4.3, based on its dyadic semantics):

$$(5.3.1)\ \frac{T(\neg\alpha)}{F(\alpha)\mid T(\bullet\alpha)} \qquad (5.3.2)\ \frac{F(\neg\alpha)}{T(\alpha),F(\bullet\alpha)} \qquad (5.3.3)\ \frac{T(\bullet\alpha)}{T(\alpha)}$$

$$(5.3.4)\ \frac{T(\bullet\bullet\alpha)}{F(\bullet\alpha)} \qquad (5.3.5)\ \frac{T(\bullet\neg\alpha)}{T(\bullet\alpha)} \qquad (5.3.6)\ \frac{F(\bullet\neg\alpha)}{F(\neg\alpha)\mid F(\alpha)}$$

$$(5.3.7)\ \frac{T(\alpha\wedge\beta)}{T(\alpha),\ T(\beta)} \qquad (5.3.8)\ \frac{F(\alpha\wedge\beta)}{F(\alpha)\mid F(\beta)}$$

$$(5.3.9)\ \frac{T(\alpha\vee\beta)}{T(\alpha)\mid T(\beta)} \qquad (5.3.10)\ \frac{F(\alpha\vee\beta)}{F(\alpha),\ F(\beta)}$$

$$(5.3.11)\ \frac{T(\alpha\Rightarrow\beta)}{F(\alpha)\mid T(\beta)} \qquad (5.3.12)\ \frac{F(\alpha\Rightarrow\beta)}{T(\alpha),F(\beta)}$$

$$(5.3.13)\ \frac{T(\bullet(\alpha\wedge\beta))}{\begin{array}{c|c} T(\alpha), & T(\beta), \\ T(\bullet\beta) & T(\bullet\alpha) \end{array}} \qquad (5.3.14)\ \frac{F(\bullet(\alpha\wedge\beta))}{\begin{array}{c|c|c} F(\alpha) & F(\beta) & T(\alpha),T(\beta), \\ & & F(\bullet\alpha),F(\bullet\beta) \end{array}}$$

$$(5.3.15)\ \frac{T(\bullet(\alpha\vee\beta))}{\begin{array}{c|c|c} F(\alpha), & F(\beta), & T(\bullet\alpha), \\ T(\bullet\beta) & T(\bullet\alpha) & T(\bullet\beta) \end{array}} \qquad (5.3.16)\ \frac{F(\bullet(\alpha\vee\beta))}{\begin{array}{c|c|c} F(\alpha), & T(\alpha), & T(\beta), \\ F(\beta) & F(\bullet\alpha) & F(\bullet\beta) \end{array}}$$

$$(5.3.17)\ \frac{T(\bullet(\alpha\Rightarrow\beta))}{T(\alpha),T(\bullet\beta)} \qquad (5.3.18)\ \frac{F(\bullet(\alpha\Rightarrow\beta))}{F(\alpha)\mid F(\bullet\beta)}$$

Compare this tableau system for **LFI1** with the tableau systems for this same logic presented in [8]. The points of departure from the latter were non-gentzenian semantics, and then (decidable) tableaux without the shortening property (in fact, tableaux allowing for loops) were obtained.

**Example 5.4** A tableau system for Belnap's 4-valued logic (see Example 4.4), $B_4$, can be obtained by adding to (5.3.7)–(5.3.10) the following rules:

$$(5.4.1)\ \frac{T(\neg\neg\alpha)}{T(\alpha)} \qquad (5.4.2)\ \frac{F(\neg\neg\alpha)}{F(\alpha)}$$

$$(5.4.3)\ \frac{T(\neg(\alpha\wedge\beta))}{\begin{array}{c|c|c|c} F(\alpha),T(\neg\alpha), & F(\alpha),T(\neg\alpha), & T(\alpha),T(\neg\alpha), & T(\alpha),T(\neg\alpha), \\ F(\beta),T(\neg\beta) & T(\beta),T(\neg\beta) & F(\beta),T(\neg\beta) & T(\beta),T(\neg\beta) \end{array}}$$

$$(5.4.4)\ \frac{F(\neg(\alpha\wedge\beta))}{F(\alpha),F(\neg\alpha)\mid T(\alpha),F(\neg\alpha)\mid F(\beta),F(\neg\beta)\mid T(\beta),F(\neg\beta)}$$

$$(5.4.5)\ \frac{T(\neg(\alpha\vee\beta))}{F(\alpha),T(\neg\alpha)\mid T(\alpha),T(\neg\alpha)\mid F(\beta),T(\neg\beta)\mid T(\beta),T(\neg\beta)}$$

$$(5.4.6)\ \frac{F(\neg(\alpha\vee\beta))}{\begin{array}{c|c|c|c} F(\alpha),F(\neg\alpha), & F(\alpha),F(\neg\alpha), & T(\alpha),F(\neg\alpha), & T(\alpha),F(\neg\alpha), \\ F(\beta),F(\neg\beta) & T(\beta),F(\neg\beta) & F(\beta),F(\neg\beta) & T(\beta),F(\neg\beta) \end{array}}$$

# 6 Conclusions

In this paper we have exhibited a method to transform any finite-valued truth-functional semantics in which truth-values can be individualized in the sense of Assumption 2.5 into 2-valued semantics. The specific form of the gentzenian axioms we obtain permits us then to define automatically a (decidable) tableau system for each logic subjected to the 2-valued reduction. The same methods can be applied to many other well-known logics such as Łukasiewicz's $Ł_n$, Kleene's $K_3$, Gödel's $G_3$ etc. The reduction method builds bulk in the results from [13, 14] and [1]. Similar procedures and also sequent systems for the 2-valued semantics hereby produced can be found in [3, 4] and [12].

It is an open problem to extend our 2-valued reduction procedure so as to cover other classes of logics such as modal or infinite-valued logics.

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