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LOGICS OF FORMAL INCONSISTENCY

1 INTRODUCTION

1.1 Contradictoriness and inconsistency, consistency and non-contradictoriness

In traditional logic, contradictoriness (the presence of contradictions in a theory or in a body of knowledge) and triviality (the fact that such a theory entails all possible consequences) are assumed inseparable, granted that negation is available. This is an effect of an ordinary logical feature known as 'explosiveness': According to it, from a contradiction 'α and $\neg\alpha$' everything is derivable. Indeed, classical logic (and many other logics) equate 'consistency' with 'freedom from contradictions'. Such logics forcibly fail to distinguish, thus, between contradictoriness and other forms of inconsistency. Paraconsistent logics are precisely the logics for which this assumption is challenged, by the rejection of the classical 'consistency presupposition'. The *Logics of Formal Inconsistency*, **LFI**s, object of this chapter, are the paraconsistent logics that neatly balance the equation:

CONTRADICTIONS + CONSISTENCY = TRIVIALITY

The **LFI**s have a remarkable way of reintroducing consistency into the non-classical picture: They internalize the very notions of consistency and inconsistency at the object-language level. The result of that strategy is the design of very expressive logical systems, whose fundamental feature is the ability to recover all consistent reasoning right on demand, while still allowing for some inconsistency to linger, otherwise.

Paraconsistency is the study of contradictory yet non-trivial theories.[1] The significance of paraconsistency as a philosophical program which dares to go beyond consistency lies in the possibilities (formal, epistemological and mathematical) to take profit from the distinctions and contrasts between asserting opposites (either in a formal or in a natural language) and ensuring non-triviality (in a theory, formal or not). A previous entry [Priest, 2002] in this Handbook was dedicated to paraconsistent logics. Although partaking in the same basic views on paraconsistency, our approach is oriented towards investigating and exhibiting the features of an ample and very expressive class of paraconsistent logics — the above mentioned **LFI**s.

[1] Paraconsistency has the meaning of 'besides, beyond consistency', just as paradox means 'besides, beyond opinion' and 'paraphrase' means 'to phrase in other words'.

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Moreover, our chapter starts from clear-cut abstract definitions of the terms involved (triviality, consistency, paraconsistency, etc.) and analyzes both proof-theoretical and model-theoretical aspects of **LFI**s, insisting on their special interest and hinting about their near ubiquity in the paraconsistent realm.

Once inconsistency is locally allowed, the chief value of a useful logical system (understood as a derivability formalism reflecting some given theoretical or pragmatical constraints) turns out to be its capability of doing what it is supposed to do, namely, to set acceptable inferences apart from unacceptable ones. The least one would ask for is, thus, that the system *does* separate propositions (into two non-empty classes, the derivable ones and the non-derivable) or, in other words, that it be non-trivial. Therefore, the most fundamental guiding criterion for choosing theories and systems worthy of investigation, as suggested by [Jaśkowski, 1948], [Nelson, 1959] and [da Costa, 1959], and extended in [Marcos, 2005c], should indeed be their abstract character of non-triviality, rather than the mere absence of contradictions.

The big challenge for paraconsistentists is to avoid allowing contradictory theories to explode and derive anything else (as they do in classical logic) and *still* to reserve resources for designing a respectful logic. For that purpose they must weaken their logical machinery by abandoning explosion in order to be able to draw reasonable conclusions from those theories, and *yet* come up with a legitimate logical system. A current trend in logic has been that of internalizing metatheoretical notions and devices at the object-language level, in order to build ever more expressive logical systems, as in the case of labeled deductive systems, hybrid logics, or the logics of provability. The **LFI**s constitute exactly the class of paraconsistent logics which can internalize the metatheoretical notions of consistency and inconsistency. As a consequence, despite constituting fragments of consistent logics, the **LFI**s can canonically be used to faithfully encode all consistent inferences. We will in this chapter present and discuss these logics, illustrating their uses, properties and representations.

Most of the material for the chapter is based on the article [Carnielli and Marcos, 2002], which founds the formal distinctions between contradictoriness, inconsistency and triviality, which we here utilize. In some cases we correct here the definitions and proofs presented there. Another central reference is the book [Marcos, 2005], where most of the examples and proposals hereby defended may be found, in extended form. The **LFI**s, central topic of the present chapter, are carefully introduced in Subsection 3.1. All necessary concepts and definitions showing how we approach the property of explosion and how this reflects on the principles of logic will be found in Section 2. Subsection 1.2 serves as vestibular to the more technical sections that follow.

The main **LFI**s are presented in Sections 3 and 4. One of their primary subclasses, the **C**-systems, is introduced as containing those **LFI**s in which consistency may be expressed as a single formula of the object language. Moreover, the **dC**-systems are introduced as those **C**-systems in which this same formula may be explicitly expressed in terms of other more usual connectives (see Definition 32). In Section 3 we study in detail a fundamental example of **LFI**, the logic **mbC**, where consistency is rendered expressible by means of a specific new primitive connective. This logic is compared to the stronger logic C_1 (cf. [da Costa, 1963] and [da Costa, 1974]), a logic of the early paraconsistent vintage. We provide Hilbert-style axiomatizations, as well as bivaluation semantics and adequate tableau systems for **mbC** and C_1. Additionally, adequate possible-translations semantics are proposed for **mbC**.

LFIs are typically based on previously given consistent logics. The fundamental feature enjoyed by classically-based **LFI**s of being able to recover classical reasoning (despite constituting themselves deductive fragments of classical logic) is explained in Subsection 3.6.

In Section 4 we extend the logic **mbC** by adding further axioms which permit us to talk about inconsistency and consistency in more symmetric guises inside the logic. A brief study of the thereby obtained logics follows, extending the results obtained in Section 3.

Section 5 explores additional topics on **LFI**s. In Subsection 5.1 some fundamental **dC**-systems are studied. Particular cases of **dC**-systems are da Costa's logics C_n, $1 \leq n < \omega$, Jaśkowski's logic **D2**, and all usual normal modal logics (under convenient formulations). Conveniently extending the previously obtained **LFI**s it is possible to introduce a large family of such logics by controlling the propagation of consistency (cf. Subsection 5.2). This procedure adds flexibility to the game, allowing one to propose tailor-suited **LFI**s; we illustrate the case by defining literally thousands of logics, including an interesting class of maximal logics in Subsection 5.3. We end this subsection by a brief note on the possibilities of algebraizing **LFI**s, in general, concluding a series of similar notes and results to be found along the paper, dedicated especially to the difficulties surrounding the so-called replacement property, the metatheoretical result that guarantees equivalent formulas to be logically indistinguishable.

Section 6 examines some perspectives on the research about Logics of Formal Inconsistency. The chapter ends by a list of axioms and systems given in Section 7.

It goes without saying that the route we will follow in this chapter corresponds not only to our preferences on how to deal with paraconsistency, but it brings also a personal choice of topics we consider to be of special philosophical and mathematical relevance.

1.2 The import of the Logics of Formal Inconsistency

Should the presence of contradictions make it impossible to derive anything sensible from a theory or a logic where such contradictions appear, as the classical logician would maintain? Or are there maybe situations in which contradictions are at least temporarily admissible, if only their wild behavior can somehow be controlled? The theoretical and practical relevance of such questions shows paraconsistency to be a bold programme in the foundations of formal sciences. As time goes by, the problems and methods of formal logic, traditionally connected to mathematics and philosophy, can more and more be seen to affect and influence several other areas of knowledge, such as computer science, information systems, formal philosophy, theoretical linguistics, and so forth. In such areas, certainly more than in mathematics, contradictions are presumably unavoidable: If contradictory theories appear only by mistake, or are due to some kind of resource-boundedness on computers, or depend on an altered state of reality, contradictions can hardly be prevented from at least being taken into consideration, as they often show up as gatecrashers. The pragmatic point thus is not whether contradictory theories *exist*, but *how to deal with them*.

Regardless of the disputable status of contradictory theories, it is hard to deny that they are, in many cases, quite *informative*, it being desirable to establish *well-reasoned* judgements even when contradictions are present. Consider, for instance, the following situation (adapted from [Carnielli and Marcos, 2001a]) in which you ask a yes-no question to two people: 'Does Jeca Tatu live in São Paulo?' Exactly one of the three following distinct scenarios is possible: They might both say 'yes', they might both say 'no', or else one of them might say 'yes' while the other says 'no'. Now, it happens that in no situation you can be sure whether Jeca Tatu lives in São Paulo or not (unless you trust one of the interviewees more than the other), but only in the last scenario, where a contradiction appears, you are sure to have received wrong information from one of your sources.

A challenge to any study on paraconsistency is to oppugn the tacit assumption that contradictory theories necessarily contain false sentences. Thus, if we can build models of structures in which some (but not all) contradictory sentences are simultaneously true, we will have the possibility of maintaining contradictory sentences inside a given theory and still be able, in principle, to perform reasonable inferences from that theory. The problem will not be that of *validating falsities*, but rather of *extending our notion of truth* (an idea further explored, for instance, in [Bueno, 1999]).

In the first half of the last century, some authors, including Łukasiewicz and Vasiliev, proposed a new approach to the idea of non-contradiction, offering interpretations to formal systems in which contradictions could make sense. Between the 1940s and the 60s the first systems of paraconsistent logic appeared (cf. [Jaśkowski, 1948], [Nelson, 1959], and [da Costa, 1963]). For historical notes on paraconsistency we suggest [Arruda, 1980], [D'Ottaviano, 1990], [da Costa and Marconi, 1989], the references mentioned in part 1 of [Priest *et al.*, 1989] and in section 3 of [Priest, 2002], as well as the book [Bobenrieth-Miserda, 1996] and the prolegomena to [Marcos, 2005].

Probably around the 40s, time was ripe for thinking about the role of negation in different terms: The falsificationism of K. Popper (cf. [Popper, 1959]) supported the idea (and stressed its role in the philosophy of science) that falsifying a proposition, as an epistemological step towards refuting it, is not the same as assuming the sentence to be false. This apparently led Popper to think about a paraconsistent-like logic dual to intuitionism in his [Popper, 1948], later to be rejected as somehow too weak as to be useful (cf. [Popper, 1989]). But it should be remarked that Popper never dismissed this kind of approach as nonsensical. His disciple D. Miller in [Miller, 2000] in fact argues that the logic for dealing with unfalsifiedness should be paraconsistent.[2] Another recent proposal by Y. Shramko also defends the paraconsistent character of falsificationism (cf. [Shramko, 2005]).

When proposing his first paraconsistent logics (cf. [da Costa, 1963]) da Costa's intuition was that the 'consistency' (which he dubbed 'good behavior') of a given formula would not only be a sufficient requisite to guarantee its explosive character, but that it could also be represented as an ordinary formula of the underlying language. For his initial logic, C_1, he chose to represent the consistency of a formula α by the formula $\neg(\alpha \wedge \neg\alpha)$, and referred to this last formula as a realization of the 'Principle of Non-Contradiction'.

In the present approach, as in [Carnielli and Marcos, 2002], we introduce consistency as a *primitive notion* of our logics: The Logics of Formal Inconsistency, **LFI**s, are paraconsistent logics that internalize the notions of consistency and inconsistency at the object-language level. In this chapter we will also study some significative subclasses of **LFI**s, the **C**-systems and **dC**-systems based on classical logic (and da Costa's logics C_n will be shown to constitute but particular samples from the latter subclass).

It is worth noting that, in general, paraconsistent logics do not validate contradictions nor, equivalently, invalidate the 'Principle of Non-Contradiction', in our reading of it (cf. the principle (1) in Subsection 2.1). Most paraconsistent logics, in fact, are proper fragments of (some version of) classical logic, and thus they cannot be *contradictory*.

Clearly, the concept of paraconsistency is related to the properties of a negation inside a given logic. In that respect, arguments can be found in the literature to the effect that 'negations' of paraconsistent logics would not be proper negation operators (cf. [Slater, 1995] and [Béziau, 2002a]). Béziau's argument amounts to a request for the definition of some mini-

[2]Indeed, Miller even proposes that the logic C_1 of da Costa's hierarchy could be used as a logic of falsification.

mal 'positive properties' in order to characterize paraconsistent negation as constituting a real *negation* operator, instead of something else. Slater argues for the *inexistence* of paraconsistent logics, given that their negation operator is not a 'contradictory-forming functor', but just a 'subcontrary-forming one', revisiting and extending an earlier argument from [Priest and Routley, 1989]. A reply to the latter kind of criticism is that it is as convincing as arguing that a 'line' in hyperbolic geometry is not a real line, since, through a given point not on the line, the 'parallel-forming functor' does not define a unique line.[3] In any case, this is not the only possible counter-objection, and the development of paraconsistent logic is not deterred by this discussion. Investigations about the general properties of paraconsistent negations include [Avron, 2002], [Béziau, 1994] and [Lenzen, 1998], among others. Those studies are surveyed in [Marcos, 2005c], where also a minimal set of 'negative properties' for negation is advanced as a new starting point for a unifying study of negation.

2 WHY'S AND HOW'S: CONCEPTS AND DEFINITIONS

2.1 The principles of logic revisited

Our presentation in what follows is situated at the level of a general theory of consequence relations. Let $\wp(X)$ denote the powerset of a set X. As usual, given a set *For* of formulas, we say that $\Vdash\, \subseteq \wp(For) \times For$ defines a (single-conclusion) *S-consequence relation* over *For* (where S stands for *standard*) if the following clauses hold, for any choice of formulas α and β, and of subsets Γ and Δ of *For* (formulas and commas at the left-hand side of $\Vdash$ denote, as usual, sets and unions of sets of formulas):

(Con1) $\alpha \in \Gamma$ implies $\Gamma \Vdash \alpha$ (reflexivity)
(Con2) $(\Delta \Vdash \alpha$ and $\Delta \subseteq \Gamma)$ implies $\Gamma \Vdash \alpha$ (monotonicity)
(Con3) $(\Delta \Vdash \alpha$ and $\Gamma, \alpha \Vdash \beta)$ implies $\Delta, \Gamma \Vdash \beta$ (cut)

So, an *S-logic* **L** will here be defined simply as a structure of the form $\langle For, \Vdash \rangle$, containing a set of formulas *For* and an S-consequence relation $\Vdash$ defined over this set. An additional useful property of a logic is *compactness*, defined as:

(Con4) $\Gamma \Vdash \alpha$ implies $\Gamma^{\mathsf{fin}} \Vdash \alpha$, for some finite $\Gamma^{\mathsf{fin}} \subseteq \Gamma$ (compactness)

We will assume that the language of every logic **L** is defined over a propositional signature $\Sigma = \{\Sigma_n\}_{n\in\omega}$, where Σ_n is the set of connectives of arity n. We will also assume that $\mathcal{P} = \{p_n : n \in \omega\}$ is the set of propositional

[3] In hyperbolic geometry the following property, known as the Hyperbolic Postulate, holds good: For every line l and point p not on l, there exist at least two distinct lines parallel to l that pass through p.

variables (or atomic formulas) from which we freely generate the algebra For of formulas using Σ. Along most of the present paper, the least we will suppose on a *logic* is that its consequence relation satisfies the clauses defining an S-consequence.

Another usual property of a logic is *structurality*. Let ε be an endomorphism in For, that is, ε is the unique homomorphic extension of a mapping from $\mathcal{P}$ into For. A logic is structural if its consequence relation preserves endomorphisms:

(Con5) $\Gamma \Vdash \alpha$ implies $\varepsilon(\Gamma) \Vdash \varepsilon(\alpha)$ (structurality)

In syntactical terms, structurality corresponds to the rule of uniform substitution or, alternatively, to the use of schematic axioms and rules.

Any set $\Gamma \subseteq For$ is here called a *theory* of $\mathbf{L}$. A theory Γ is said to be *proper* if $\Gamma \neq For$, and a theory Γ is said to be *closed* if it contains all of its consequences, that is, for a closed theory Γ we have $\Gamma \Vdash \alpha$ iff $\alpha \in \Gamma$, for every formula α. If $\Gamma \Vdash \alpha$ for all Γ, we will say that α is a *thesis* (of $\mathbf{L}$).

Unless explicitly stated to the contrary, we will from now on be working with some fixed arbitrary logic $\mathbf{L} = \langle For, \Vdash \rangle$ where For is written in a signature containing a unary 'negation' connective $\neg$ and $\Vdash$ satisfies (Con1)–(Con3) and (Con5).

Let Γ be a theory of $\mathbf{L}$. We say that Γ is *contradictory with respect to* $\neg$, or simply *contradictory*, if it satisfies:

$$\exists\alpha(\Gamma \Vdash \alpha \text{ and } \Gamma \Vdash \neg\alpha)$$

(The formal framework to deal with this kind of metaproperties can be found in [Coniglio and Carnielli, 2002].) For any such formula α we may also say that Γ is *α-contradictory*.

A theory Γ is said to be *trivial* if it satisfies:

$$\forall\alpha(\Gamma \Vdash \alpha)$$

Of course the theory For is trivial, given (Con1). We can immediately conclude that contradictoriness is a necessary (but, in general, not a sufficient) condition for triviality in a given theory, since a trivial theory derives everything.

A theory Γ is said to be *explosive* if:

$$\forall\alpha\forall\beta(\Gamma, \alpha, \neg\alpha \Vdash \beta)$$

Thus, a theory is called explosive if it trivializes when exposed to a pair of contradictory formulas. Evidently, if a theory is trivial, then it is explosive by (Con2). Also, if a theory is contradictory and explosive, then it is trivial by (Con3).

The above definitions may be immediately upgraded from theories to logics. We will say that **L** is *contradictory* if all of its theories are contradictory, that is:

$$\forall\Gamma\exists\alpha(\Gamma \Vdash \alpha \text{ and } \Gamma \Vdash \neg\alpha)$$

In the same spirit, we will say that **L** is *trivial* if all of its theories are trivial, and **L** is *explosive* if all of its theories are explosive.

Because of the monotonicity property (Con2), it is clear that an S-logic **L** is contradictory / trivial / explosive if, and only if, its empty theory is contradictory / trivial / explosive.

We are now in position to give a formal definition for some *logical principles* as applied to a generic logic **L**:

Principle of Non-Contradiction (**L** is non-contradictory)

$$\exists\Gamma\forall\alpha(\Gamma \nVdash \alpha \text{ or } \Gamma \nVdash \neg\alpha) \tag{1}$$

Principle of Non-Triviality (**L** is non-trivial)

$$\exists\Gamma\exists\alpha(\Gamma \nVdash \alpha) \tag{2}$$

Principle of Explosion (**L** is explosive)

$$\forall\Gamma\forall\alpha\forall\beta(\Gamma, \alpha, \neg\alpha \Vdash \beta) \tag{3}$$

The last principle is also often referred to as *Pseudo-Scotus* or Principle of *Ex Contradictione Sequitur Quodlibet*.[4]

It is clear that the three principles are interrelated:

THEOREM 1.
(i) A trivial logic is both contradictory and explosive.
(ii) An explosive logic fails the Principle of Non-Triviality if, and only if, it fails the Principle of Non-Contradiction. ■

The logics disrespecting (1) are sometimes called *dialectical*. However, the immense majority of the paraconsistent logics in the literature (including the ones studied here) are *not* dialectical. Indeed, they usually have non-contradictory empty theories, and thus their axioms are non-contradictory, and their inference rules do not generate contradictions from these axioms. All paraconsistent logics which we will present here are in some sense more careful than classical logic, once they extract less consequences than classical logic extracts from the same given theory, or at most the

[4]In fact, single-conclusion logics as those we work with here cannot see the difference between *Pseudo-Scotus* and *Ex Contradictione*, but those principles can be sharply distinguished in a multiple-conclusion environment. Moreover, in such an environment, several forms of triviality, or *overcompleteness*, may be very naturally set apart (cf. [Marcos, 2005c] and [Marcos, 2007a]).

same set of consequences, but never more. The paraconsistent logics studied in the present chapter (as most paraconsistent logics in the literature) do not validate any bizarre form of reasoning, and do not beget contradictory consequences if such consequences were already not derived in classical logic.

2.2 Paraconsistency: Between inconsistency and triviality

As mentioned before, some decades ago, Stanisław Jaśkowski ([Jaśkowski, 1948]), David Nelson ([Nelson, 1959]), and Newton da Costa ([da Costa, 1963]), the founders of paraconsistent logic, proposed, independently, the study of logics which could accommodate contradictory yet non-trivial theories. For da Costa, a logic is *paraconsistent*[5] *with respect to* $\neg$ if it can serve as a basis for $\neg$-contradictory yet non-trivial theories, that is:

$$\exists\Gamma\exists\alpha\exists\beta(\Gamma \Vdash \alpha \text{ and } \Gamma \Vdash \neg\alpha \text{ and } \Gamma \nVdash \beta) \tag{4}$$

Notice that, in our present framework, the notion of a paraconsistent logic has, in principle, nothing to do with the rejection of the Principle of Non-Contradiction, as it is commonly held. On the other hand, it is intimately connected to the rejection of the Principle of Explosion. Indeed, Jaśkowski defined a $\neg$-paraconsistent logic as a logic in which (3) fails, that is:

$$\exists\Gamma\exists\alpha\exists\beta(\Gamma, \alpha, \neg\alpha \nVdash \beta) \tag{5}$$

Using (Con1) and (Con3) it is easy to prove that (4) and (5) are equivalent ways of defining a paraconsistent logic. Whenever it is clear from the context, we will omit the $\neg$ symbol and refer simply to *paraconsistent logics*.

It is very important to observe that a logic where all contradictions are equivalent cannot be paraconsistent. To understand that point it is convenient first to make precise the concept of equivalence between sets of formulas: Γ and Δ are said to be *equivalent* if

$$\forall\alpha \in \Delta(\Gamma \Vdash \alpha) \text{ and } \forall\alpha \in \Gamma(\Delta \Vdash \alpha)$$

In particular, we say that two formulas α and β are *equivalent* if the sets $\{\alpha\}$ and $\{\beta\}$ are equivalent, that is:

$$(\alpha \Vdash \beta) \text{ and } (\beta \Vdash \alpha)$$

We denote these facts by writing, respectively, $\Gamma \dashVdash \Delta$, and $\alpha \dashVdash \beta$. The equivalence between formulas is clearly an equivalence relation, because of (Con1) and (Con3). However, the equivalence between sets is not, in

[5]As a matter of fact, this appellation would be coined only in the 70s by the Peruvian philosopher Francisco Miró Quesada.

general, an equivalence relation, unless the following property holds in **L**:

(Con6) $[\forall\beta \in \Delta(\Gamma \Vdash \beta)$ and $\Delta \Vdash \alpha]$ implies $\Gamma \Vdash \alpha$ (cut for sets)

Logics based on consequence relations that respect clauses (Con1), (Con2) and (Con6) will here be called (single-conclusion) *T-logics* (where T stands for *Tarskian*).

REMARK 2. (i) In logics defined by way of a collection of finite-valued truth-tables or by way of Hilbert calculi with schematic axioms and finitary rules, (Con1)–(Con6) all hold good. This is the case of most logics mentioned in the present paper.

(ii) (Con1) and (Con6) guarantee that $\dashv\Vdash$ defines an equivalence relation over sets of formulas.

(iii) Condition (Con3) follows from {(Con1), (Con2), (Con6)}. Indeed, suppose that (a) $\Delta \Vdash \alpha$ and (b) $\Gamma, \alpha \Vdash \beta$. By (Con1) we can further assume that (c) $\Delta, \Gamma \Vdash \gamma$, for every $\gamma \in \Gamma$. But if we apply (Con2) to hypothesis (a) it follows that (d) $\Delta, \Gamma \Vdash \alpha$. Using (Con6) on (c), (d) and (b) it follows that $\Delta, \Gamma \Vdash \beta$.

(iv) Condition (Con2) follows from {(Con1), (Con6)}. Indeed, suppose that (a) $\Delta \Vdash \alpha$ and (b) $\Delta \subseteq \Gamma$. From (b) and (Con1), we conclude that (c) $\Gamma \Vdash \delta$, for all $\delta \in \Delta$. Then, using (Con6) on (c) and (a) it follows that $\Gamma \Vdash \alpha$.

(v) Condition (Con3) follows from {(Con1), (Con6)}. To check that, compose (iii) and (iv).

(vi) Condition (Con6) does not follow from {(Con1), (Con2), (Con3)}. Indeed, consider for instance the logic $\mathbf{L}_{\mathbb{R}} = \langle \mathbb{R}, \Vdash \rangle$ such that $\mathbb{R}$ is the set of real numbers, and $\Vdash$ is defined as follows:

$$\Gamma \Vdash x \quad \text{iff} \quad x \in \Gamma, \text{ or } x = \tfrac{1}{n} \text{ for some } n \in \mathbb{N},\ n \geq 1, \text{ or there is a sequence } (x_n)_{n\in\mathbb{N}} \text{ contained in } \Gamma \text{ such that } (x_n)_{n\in\mathbb{N}} \text{ converges to } x.$$

It is easy to see that $\mathbf{L}_{\mathbb{R}}$ satisfies (Con1), (Con2) and (Con3). But (Con6) is not valid in $\mathbf{L}_{\mathbb{R}}$. Indeed, take $\Gamma = \varnothing$, $\Delta = \{\frac{1}{n} : n \in \mathbb{N},\ n \geq 1\}$ and $\alpha = 0$. Then the antecedent of (Con6) is true: Every element of Δ is a thesis, and Δ contains the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$, which converges to 0. However, the consequent of (Con6) is false: 0 is not a thesis of $\mathbf{L}_{\mathbb{R}}$.

Observe, by the way, that in $\mathbf{L}_{\mathbb{R}}$ the relation $\dashv\Vdash$ between sets of formulas is not transitive: Take Δ as above, and consider $\Delta_0 = \{0\}$ and $\Delta_1 = \{1\}$. Then $\Delta_0 \dashv\Vdash \Delta$ and $\Delta \dashv\Vdash \Delta_1$, but it is not the case that $\Delta_0 \dashv\Vdash \Delta_1$, because $\Delta_1 \nVdash 0$.

Do remark that, as a particular consequence of the above items, T-logics may be seen as specializations of S-logics. ■

Most logics we will study in the present paper are natural examples of T-logics. For many proofs that will be presented below, however, the assumption of an S-logic will suffice.

THEOREM 3. Let **L** be a T-logic. Then, in case all contradictions are equivalent in **L**, it follows that **L** is not paraconsistent.

Proof. Take an arbitrary set Γ in **L**. Suppose that all contradictions are equivalent, that is, for arbitrary α and β, $\{\alpha, \neg\alpha\} \dashv\Vdash \{\beta, \neg\beta\}$. Then, using (Con2), $\Gamma \cup \{\alpha, \neg\alpha\}$ is β-contradictory for an arbitrary β, and in particular $\Gamma, \alpha, \neg\alpha \Vdash \beta$. ■

By contrapositive reasoning, the above theorem may be rephrased as stating the following: If a T-logic **L** is paraconsistent, then there exist pairs of non-equivalent contradictions in **L**.

DEFINITION 4. The logic **L** is called *consistent* if it is both explosive and non-trivial, that is, if **L** respects both (3) and (2). **L** is called *inconsistent*, otherwise. ■

Paraconsistent logics are inconsistent, in that they control explosiveness, but they can do so in a variety of ways. Trivial logics are also inconsistent, by the above definition. What distinguishes a paraconsistent logic from a trivial logic is that a trivial logic does not disallow any inference: It accepts everything. As a consequence of the above definition of consistency, a third equivalent approach to the notion of paraconsistency may be proposed, parallel to those from definitions (4) and (5):

A logic is paraconsistent if it is inconsistent yet non-trivial. (6)

The compatibility of paraconsistency with the existence of some suitable explosive or trivial proper theories makes some paraconsistent logics able to recover classical reasoning, as we will see in Section 3.6. We will from now on introduce some specializations on the above definitions and principles.

A logic **L** is said to be *finitely trivializable* when it has finite trivial theories. Evidently, if a logic is explosive, then it is finitely trivializable. Non-explosive logics might be finitely trivializable or not.

A formula ξ in **L** is a *bottom particle* if it can, by itself, trivialize the logic, that is:

$$\forall\Gamma\forall\beta(\Gamma, \xi \Vdash \beta)$$

A bottom particle, when it exists, will here be denoted by $\bot$. This notation is unambiguous in the following sense: Any two bottom particles are equivalent. If in a given logic a bottom particle is also a thesis, then the logic is trivial — in which case, of course, all formulas turn out to be bottom particles.

The existence of bottom particles inside a given logic **L** is regulated by the following principle:

Principle of *Ex Falso Sequitur Quodlibet*

$$\exists\xi\forall\Gamma\forall\beta(\Gamma, \xi \Vdash \beta)(\textbf{L} \text{ has a bottom particle}) \quad (7)$$

As it will be seen, the existence of logics that do not respect (3) while still respecting (7) (as all **LFI**s of the present chapter) shows that *ex contradictione* does not need to be identified with *ex falso*, contrary to what is commonly held in the literature.

The dual concept of a bottom particle is that of a *top particle*, that is, a formula ζ which follows from every theory:

$$\forall\Gamma(\Gamma \Vdash \zeta)$$

We will denote any fixed such particle, when it exists, by $\top$ (again, this notation is unambiguous). Evidently, given a logic, any of its theses will constitute such a top particle (and logics with no theses, like Kleene's 3-valued logic, have no such particles). It is easy to see that the addition of a top particle to a given theory is pretty innocuous, for in that case $\Gamma, \top \Vdash \alpha$ if and only if $\Gamma \Vdash \alpha$.

Henceforth, a formula φ of **L** constructed using all and only the variables $p_0, \ldots, p_n$ will be denoted by $\varphi(p_0, \ldots, p_n)$. This formula will be said to *depend only* on the variables that occur in it. The notation may be generalized to sets, and the result is denoted by $\Gamma(p_0, \ldots, p_n)$. If $\gamma_0, \ldots, \gamma_n$ are formulas then $\varphi(\gamma_0, \ldots, \gamma_n)$ will denote the (simultaneous) substitution of p_i by γ_i in $\varphi(p_0, \ldots, p_n)$ (for $i = 0, \ldots, n$). Given a set of formulas $\Gamma(p_0, \ldots, p_n)$, we will write $\Gamma(\gamma_0, \ldots, \gamma_n)$ with an analogous meaning.

DEFINITION 5. We say that a logic **L** has a *supplementing negation* if there is a formula $\varphi(p_0)$ such that:

(a) $\varphi(\alpha)$ is not a bottom particle, for some α;

(b) $\forall\Gamma\forall\alpha\forall\beta(\Gamma, \alpha, \varphi(\alpha) \Vdash \beta)$ ■

Observe that the same logic might have several non-equivalent supplementing negations (check Remark 43).

Consider a logic having a supplementing negation, and denote it by $\sim$. Parallel to the definition of contradictoriness with respect to $\neg$, we might now define a theory Γ to be *contradictory with respect to* $\sim$ if it is such that:

$$\exists\alpha(\Gamma \Vdash \alpha \text{ and } \Gamma \Vdash {\sim}\alpha)$$

Accordingly, a logic **L** could be said to be *contradictory with respect to* $\sim$ if all of its theories were contradictory with respect to $\sim$. Obviously, by design, no logic can be $\sim$-paraconsistent, or even $\sim$-contradictory, if $\sim$ is a supplementing negation, and a logic that has a supplementing negation must satisfy the Principle of Non-Contradiction with respect to this negation. The main logics studied in this paper are all endowed with supplementing negations. The availability of some specific supplementing negations makes some paraconsistent logics able to easily emulate classical negation (see Subsection 3.6).

Here we may of course introduce yet another variation on (3):

Supplementing Principle of Explosion

$$\mathbf{L} \text{ has a supplementing negation} \tag{8}$$

Supplementing negations are very common. We will show here some sufficient conditions for their definition. The presence of a convenient implication in our logics is often convenient so as to help explicitly internalizing the definition of new connectives.

DEFINITION 6. We say that a logic **L** has a *deductive implication* if there is a formula $\psi(p_0, p_1)$ such that:

(a) $\psi(\alpha, \beta)$ is not a bottom particle, for some choice of α and β;

(b) $\forall\alpha\forall\beta\forall\Gamma(\Gamma \Vdash \psi(\alpha,\beta)$ implies $\Gamma, \alpha \Vdash \beta)$;

(c) $\psi(\alpha, \beta)$ is not a top particle, for some choice of α and β;

(d) $\forall\alpha\forall\beta\forall\Gamma(\Gamma, \alpha \Vdash \beta$ implies $\Gamma \Vdash \psi(\alpha,\beta))$. ■

Inside the most usual logics, condition (b) is usually guaranteed by the validity of the rule of *modus ponens*, while condition (d) is guaranteed by the so-called 'deduction theorem' (when this theorem holds). Obviously, any logic having a deductive implication will be non-trivial, by condition (a).

THEOREM 7. Let **L** be a non-trivial logic endowed with a bottom particle $\bot$ and a deductive implication $\to$.
(i) Let $\neg$ be some negation symbol, and suppose that it satisfies:
(a) $\Gamma, \neg\alpha \Vdash \alpha \to \bot$;
(b) $\Gamma, \neg\alpha \to \bot \Vdash \alpha$.
Then, this $\neg$ is a supplementing negation.
(ii) Suppose, otherwise, that the following is the case:
(c) $\alpha \to \bot \nVdash \bot$, for some formula α.
Then, a supplementing negation may be defined by setting $\neg\alpha \stackrel{\text{def}}{=} \alpha \to \bot$.

Proof. Item (i). By hypothesis (a) and the properties of the bottom and the implication, we have $\Gamma, \alpha, \neg\alpha \Vdash \beta$. Now, suppose $\neg\alpha$ defines a bottom particle, for any choice of α. Then, by the deduction theorem, $\Gamma \Vdash \neg\alpha \to \bot$, for an arbitrary Γ. Thus, by (b) and (Con3), $\Gamma \Vdash \alpha$. But this cannot be the case, as **L** is non-trivial.
Item (ii) is a straightforward consequence of the above definitions, and we leave it as an exercise for the reader. ■

One might also consider the dual of a supplementing negation:

DEFINITION 8. We say that a logic **L** has a *complementing negation* if there is a formula $\psi(p_0)$ such that:

(a) $\psi(\alpha)$ is not a top particle, for some α;

(b) $\forall\Gamma\forall\alpha(\Gamma, \alpha \Vdash \psi(\alpha)$ implies $\Gamma \Vdash \psi(\alpha))$.

We say that **L** has a *classical negation* if it has some (primitive or defined) negation connective that is both supplementing and complementing. As a particular consequence of this definition, it can be easily checked that for any classical negation $\urcorner$ the equivalence $(\urcorner\urcorner\alpha \dashv\Vdash \alpha)$ will be derivable. ∎

Yet some other versions of explosiveness can here be considered:

DEFINITION 9. Let **L** be a logic, and let $\sigma(p_0, \ldots, p_n)$ be a formula of **L**. (i) We say that **L** is *partially explosive with respect to* σ, or σ*-partially explosive*, if:

(a) $\sigma(\beta_0, \ldots, \beta_n)$ is not a top particle, for some choice of $\beta_0, \ldots, \beta_n$;

(b) $\forall\Gamma\forall\beta_0 \ldots \forall\beta_n\forall\alpha(\Gamma, \alpha, \neg\alpha \Vdash \sigma(\beta_0, \ldots, \beta_n))$.

(ii) **L** is *boldly paraconsistent* if there is no σ such that **L** is σ-partially explosive.

(iii) **L** is said to be *controllably explosive in contact with* σ, if:

(a) $\sigma(\alpha_0, \ldots, \alpha_n)$ and $\neg\sigma(\alpha_0, \ldots, \alpha_n)$ are not bottom particles, for some choice of $\alpha_0, \ldots, \alpha_n$;

(b) $\forall\Gamma\forall\alpha_0 \ldots \forall\alpha_n\forall\beta(\Gamma, \sigma(\alpha_0, \ldots, \alpha_n), \neg\sigma(\alpha_0, \ldots, \alpha_n) \Vdash \beta)$. ∎

EXAMPLE 10. A well-known example of a logic that is not explosive but is partially explosive, is provided by Kolmogorov & Johánsson's Minimal Intuitionistic Logic, *MIL*, obtained by the addition to the positive fragment of intuitionistic logic (see Remark 29 below) of some weak forms of *reductio ad absurdum* (cf. [Johánsson, 1936] and [Kolmogorov, 1967]). In this logic, the intuitionistically valid inference $(\Gamma, \alpha, \neg\alpha \Vdash \beta)$ fails, but $(\Gamma, \alpha, \neg\alpha \Vdash \neg\beta)$ holds good. This means that *MIL* is paraconsistent, but not boldly paraconsistent, as all negated propositions can be inferred from any given contradiction. A class of (obviously non-boldly) paraconsistent logics extending *MIL* is studied in [Odintsov, 2005]. ∎

The requirement that a paraconsistent logic should be boldly paraconsistent was championed by [Urbas, 1990]. The class of boldly paraconsistent logics is surely very natural and pervasive. From now on, we will be making an effort, as a matter of fact, to square our paraconsistent logics into this class (check Theorems 20, 38 and 130).

Most paraconsistent logics studied in this chapter are also controllably explosive (check, in particular, Theorem 79, but a particularly strong counterexample may be found in Example 17).

We should observe that conjunction may play a central role in relating contradictoriness and triviality.

DEFINITION 11. A logic **L** is said to be *left-adjunctive* if there is a formula $\psi(p_0, p_1)$ such that:

(a) $\psi(\alpha, \beta)$ is not a bottom particle, for some α and β;

(b) $\forall\alpha\forall\beta\forall\Gamma\forall\gamma(\Gamma, \alpha, \beta \Vdash \gamma$ implies $\Gamma, \psi(\alpha, \beta) \Vdash \gamma)$. ■

The formula $\psi(\alpha, \beta)$, when it exists, will often be denoted by $(\alpha \wedge \beta)$, and the sign $\wedge$ will be called a *left-adjunctive conjunction* (but it will not necessarily have, of course, all properties of a classical conjunction). Similarly, we can define the following:

DEFINITION 12. A logic **L** is said to be *left-disadjunctive* if there is a formula $\varphi(p_0, p_1)$ such that:

(a) $\varphi(\alpha, \beta)$ is not a top particle, for some α and β;

(b) $\forall\alpha\forall\beta\forall\Gamma\forall\gamma(\Gamma, \varphi(\alpha, \beta) \Vdash \gamma$ implies $\Gamma, \alpha, \beta \Vdash \gamma)$. ■

In general, whenever there is no risk of misunderstanding or of misidentification of different entities, we might also denote the formula $\varphi(\alpha, \beta)$, when it exists, by $(\alpha \wedge \beta)$, and we will accordingly call $\wedge$ a *left-disadjunctive conjunction*. Of course, a logic can have just one of these conjunctions, or it can have both a left-adjunctive conjunction and a left-disadjunctive conjunction without the two of them coinciding. In natural deduction, clause (b) of Definition 11 corresponds to conjunction elimination, and clause (b) of Definition 12 corresponds to conjunction introduction.

It is straightforward to prove the following:

THEOREM 13. Let **L** be a left-adjunctive logic. (i) If **L** is finitely trivializable (in particular, if it has a supplementing negation), then it has a bottom particle. (ii) If **L** respects *ex contradictione*, then it also respects *ex falso*. ■

EXAMPLE 14. The 'pre-discussive' logic J proposed in [Jaśkowski, 1948], in the usual signature of classical logic, is such that:

$$\Gamma \Vdash_J \alpha \text{ iff } \Diamond\Gamma \Vdash_{S5} \Diamond\alpha,$$

where $\Diamond\Gamma = \{\Diamond\gamma : \gamma \in \Gamma\}$, $\Diamond$ denotes the possibility operator, and $\Vdash_{S5}$ denotes the consequence relation defined by the well-known modal logic $S5$. It is easy to see that $(\alpha, \neg\alpha \Vdash_J \beta)$ does not hold in general, though $(\alpha \wedge \neg\alpha) \Vdash_J \beta$ does hold good, for any formulas α and β. This phenomenon can only happen because J is left-adjunctive but not left-disadjunctive. Hence, Theorem 13 still holds for J, but this logic provides a simple example of a logic that respects the Principle of *Ex Falso Sequitur Quodlibet* (7) but not the Principle of *Ex Contradictione Sequitur Quodlibet* (3). ■

The literature on paraconsistency (cf. section 4.2 of [Priest, 2002]) traditionally calls *non-adjunctive* the logics failing left-disadjunctiveness. In the present paper, conjunctions that are both left-adjunctive and left-dis-adjunctive will be called *standard*.

3 LFIS AND THEIR RELATIONSHIP TO CLASSICAL LOGIC

3.1 Introducing *LFIs* and *C*-systems

From now on, we will concentrate on logics which are paraconsistent but nevertheless have some special explosive theories, as those discussed in the last section. With the help of such theories some concepts can be studied under a new light — this is the case of the notion of *consistency* (and its opposite, the notion of *inconsistency*), as we shall see. This section will introduce the Logics of Formal Inconsistency as the paraconsistent logics that respect a certain Gentle Principle of Explosion, to be clarified below. By way of motivation, we start with a few helpful definitions and concrete examples.

Given two logics $\mathbf{L1} = \langle For_1, \Vdash_1 \rangle$ and $\mathbf{L2} = \langle For_2, \Vdash_2 \rangle$, we will say that $\mathbf{L2}$ is a (proper) *linguistic extension* of $\mathbf{L1}$ if For_1 is a (proper) subset of For_2, and we will say that $\mathbf{L2}$ is a (proper) *deductive extension* of $\mathbf{L1}$ if $\Vdash_1$ is a (proper) subset of $\Vdash_2$. Finally, if $\mathbf{L2}$ is both a linguistic extension and a deductive extension of $\mathbf{L1}$, and if the restriction of $\mathbf{L2}$'s consequence relation $\Vdash_2$ to the set For_1 will make it identical to $\Vdash_1$ (that is, if $For_1 \subseteq For_2$, and for any $\Gamma \cup \{\alpha\} \subseteq For_1$ we have that $\Gamma \Vdash_2 \alpha$ iff $\Gamma \Vdash_1 \alpha$) then we will say that $\mathbf{L2}$ is a *conservative extension* of $\mathbf{L1}$ (and similarly for *proper* conservative extensions). In any of the above cases we can more generally say that $\mathbf{L2}$ is an *extension* of $\mathbf{L1}$, or that $\mathbf{L1}$ is a *fragment* of $\mathbf{L2}$. These concepts will be used here to compare a number of logics that will be presented. Most paraconsistent logics in the literature, and all of those studied here, are proper deductive fragments of classical logic written in a convenient signature.

REMARK 15. From here on, Σ will denote the signature containing the binary connectives $\wedge$, $\vee$, $\rightarrow$, and the unary connective $\neg$, such that $\mathcal{P} = \{p_n : n \in \omega\}$ is the set of atomic formulas. By *For* we will denote the set of formulas freely generated by $\mathcal{P}$ over Σ.

In the same spirit, Σ° will denote the signature obtained by the addition to Σ of a new unary connective $\circ$ to the signature Σ, and For° will denote the algebra of formulas for the signature Σ°. ■

DEFINITION 16. A *many-valued semantics* for a set of formulas For will here be any collection Sem of mappings $v_k\colon For \longrightarrow \mathcal{V}_k$, called *valuations*, where the set of *truth-values* in $\mathcal{V}_k$ is separated into *designated* values $\mathcal{D}_k$ (denoting the set of 'true values') and *undesignated* values $\mathcal{U}_k$ (denoting the set of 'false values'), that is, $\mathcal{V}_k$ is such that $\mathcal{V}_k = \mathcal{D}_k \cup \mathcal{U}_k$ and $\mathcal{D}_k \cap \mathcal{U}_k = \varnothing$, for each $v \in \mathsf{Sem}$. A (truth-preserving single-conclusion) many-valued *entailment relation* $\models_{\mathsf{Sem}} \subseteq \wp(For) \times For$ can then be defined by setting, for every choice of $\Gamma \cup \{\alpha\} \subseteq For$:

$$\Gamma \models_{\mathsf{Sem}} \alpha \text{ iff, for every } v \in \mathsf{Sem},\ v(\alpha) \in \mathcal{D} \text{ whenever } v(\Gamma) \subseteq \mathcal{D}.$$

A nice general abstract result can be proven to the effect that a consequence relation characterizes a T-logic (recall Subsection 2.2) if, and only if, it is determined by a many-valued entailment relation (check [Marcos, 2004; Caleiro *et al.*, 2005a], and the references therein). A distinguished class of many-valued semantics that will be much explored in the present paper, starting from Subsection 3.3, is the class of semantics in which $\mathcal{D}$ and $\mathcal{U}$ are fixed singletons (representing 'truth' and 'falsity') throughout every $v \in \mathsf{Sem}$. Those semantics are now known as *bivaluation semantics*.

A very usual particular class of many-valued semantics is the class of *truth-functional semantics*, which include those many-valued semantics such that $\mathcal{V}$, $\mathcal{D}$ and $\mathcal{U}$ are fixed sets of truth-values throughout every $v \in \mathsf{Sem}$, and such that the truth-values are organized into an algebra similar to the algebra of formulas, that is, for every κ-ary connective in the signature Σ that defines For there is a corresponding κ-ary operator over $\mathcal{V}$, where κ is the cardinality of $\mathcal{V}$. In case $\kappa < \omega$ we say that we are talking about a *finite-valued truth-functional logic*.

We will often present truth-functional T-logics below simply in terms of sets of truth-tables and corresponding designated values defining the behavior of the connectives from the signature, and take it for granted that the reader assumes that and understands how those tables characterize an entailment relation $\models$, defined as above. Not all logics, and not all paraconsistent logics, have truth-functional semantics, though. Partially explosive paraconsistent logics such as *MIL* (check Example 10) provide indeed prime examples of logics that are not characterizable by truth-functional semantics, neither finite-valued nor infinite-valued (for a discussion on that phenomenon, check [Marcos, 2007b], and the references therein).

Some useful generalizations of truth-functional semantics include *non-deterministic semantics* and *possible-translations semantics* based on truth-functional many-valued logics (presented below, starting from Subsection 3.4). ■

EXAMPLE 17. Consider the logic presented by way of the following truth-tables:

$\wedge$	1	$^1/_2$	0
1	1	$^1/_2$	0
$^1/_2$	$^1/_2$	$^1/_2$	0
0	0	0	0

$\vee$	1	$^1/_2$	0
1	1	1	1
$^1/_2$	1	$^1/_2$	$^1/_2$
0	1	$^1/_2$	0

$\to$	1	$^1/_2$	0
1	1	$^1/_2$	0
$^1/_2$	1	$^1/_2$	0
0	1	1	1

	$\neg$
1	0
$^1/_2$	$^1/_2$
0	1

where both 1 and $\frac{1}{2}$ are designated values. *Pac* is the name under which this logic appeared in [Avron, 1991] (Section 3.2.2), though it had previously appeared, for instance, in [Avron, 1986], under the denomination $RM_3^{\widetilde{\supset}}$, and, even before that, in [Batens, 1980], where it was called PI^s. The logic *Pac* conservatively extends the logic *LP* by the addition of a classical implication. *LP* is an early example of a 3-valued paraconsistent logic with classic-like operators for a standard conjunction and a standard disjunction, and it was introduced in [Asenjo, 1966] and investigated in [Priest, 1979].

In *Pac*, for no formula α it is the case that $\alpha, \neg\alpha \vdash_{Pac} \beta$ for all β. So, *Pac* is not a controllably explosive logic. A classical negation for *Pac* would be illustrated by the truth-table:

	$\sim$
1	0
$^1/_2$	0
0	1

However, it should be clear that such a negation is *not* definable in *Pac*. Indeed, any truth-function of this logic having only $\frac{1}{2}$'s as input will also have $\frac{1}{2}$ as output. As a consequence, *Pac* has no bottom particle (and this logic also cannot express the consistency of its formulas, as we shall see below). Being a left-adjunctive logic as well, *Pac* is, consequently, not finitely trivializable. ■

EXAMPLE 18. In adding to *Pac* either a supplementing negation as above or a bottom particle, one obtains a well-known conservative extension of it, obviously still paraconsistent, but this time a logic that has some interesting explosive theories: It satisfies, in particular, principles (7) and (8) from the previous subsection. This logic was introduced in [Schütte, 1960] for proof-theoretical reasons and independently investigated under the appellation $\mathbf{J}_3$ in [D'Ottaviano and da Costa, 1970] as a 'possible solution to the problem of Jaśkowski'. It also reappeared quite often in the literature after that, for instance as the logic **CLuNs** in [Batens and De Clercq, 2000]. In [D'Ottaviano and da Costa, 1970]'s first presentation of $\mathbf{J}_3$, a 'possibility connective' ∇ was introduced instead of the supplementing negation $\sim$. In [Epstein, 2000] this logic was reintroduced having also a sort of 'consistency connective' $\circ$ (originally denoted by ©) as primitive. The truth-tables of ∇ and $\circ$ are as follows:

	∇	$\circ$
1	1	1
$^1/_2$	1	0
0	0	1

The expressive and inferential power of this logic was more deeply explored in [Avron, 1999] and in [Carnielli *et al.*, 2000]. The latter paper also explores the possibility of applying this logic to the study of inconsistent databases (for a more technical perspective check [de Amo *et al.*, 2002]), abandoning $\sim$ and ∇ but still retaining $\circ$ as primitive. This logic (renamed **LFI1** in the signature Σ°) has been argued to be appropriate for formalizing the notion of consistency in a very convenient way, as discussed below. It is worth noticing that $\sim\alpha$ and $\nabla\alpha$ may be defined in **LFI1** as $(\neg\alpha \wedge \circ\alpha)$ and $(\alpha \vee \neg\circ\alpha)$, respectively. Alternatively, $\circ\alpha \stackrel{\text{def}}{=} (\neg\nabla\alpha \vee \neg\nabla\neg\alpha)$. A complete axiomatization for **LFI1** is presented in Theorem 127. ■

EXAMPLE 19. Paraconsistency and many-valuedness have often been companions. In [Sette, 1973] the following 3-valued logic, alias $\mathbf{P}^1$, was studied:

$\wedge$	1	$^1/_2$	0
1	1	1	0
$^1/_2$	1	1	0
0	0	0	0

$\vee$	1	$^1/_2$	0
1	1	1	1
$^1/_2$	1	1	1
0	1	1	0

$\to$	1	$^1/_2$	0
1	1	1	0
$^1/_2$	1	1	0
0	1	1	1

	$\neg$
1	0
$^1/_2$	1
0	1

where 1 and $\frac{1}{2}$ are the designated values. The truth-table of the consistency connective $\circ$ as in Example 18 can now be defined via $\circ\alpha \stackrel{\text{def}}{=} \neg\neg\alpha \vee \neg(\alpha \wedge \alpha)$. The logic $\mathbf{P}^1$ has the remarkable property of being controllably explosive in contact with arbitrary non-atomic formulas, that is, the paraconsistent behavior obtains only at the atomic level: $\alpha, \neg\alpha \vDash \beta$, for arbitrary non-atomic α. Moreover, another property of this logic is that $\vDash \circ\alpha$ holds for non-atomic α. Those two properties are in fact not related by a mere accident, but as an instance of Theorem 79. A complete axiomatization for the logic $\mathbf{P}^1$ is presented in Theorem 127. ■

We had committed ourselves to present paraconsistent logics that would be boldly paraconsistent (recall Definition 9(ii)). The logics from Examples 18 and 19 can indeed be seen to enjoy this property:

THEOREM 20. **LFI1** and $\mathbf{P}^1$ are boldly paraconsistent. And so are their fragments.

Proof. Assume $\Gamma \nvDash \sigma(p_0, \ldots, p_n)$ for some appropriate choice of formulas. In particular, by (Con2), it follows that $\nvDash \sigma(p_0, \ldots, p_n)$. Now, consider a variable p not in $p_0, \ldots, p_n$. Let p be assigned the value $\frac{1}{2}$, and extend this assignment to the variables $p_0, \ldots, p_n$ so as to give the value 0 to $\sigma(p_0, \ldots, p_n)$. It is obvious that, in this situation, $p, \neg p \nvDash \sigma(p_0, \ldots, p_n)$. ■

Paraconsistent logics are tools for reasoning under conditions which do not presuppose consistency. If we understand consistency as what might be lacking to a contradiction for it to become explosive, logics like **LFI1** and $\mathbf{P}^1$ are clearly able to express the consistency (of a formula) at the object-language level. This feature will permit consistent reasoning to be recovered from inside an inconsistent environment.

In formal terms, consider a (possibly empty) set $\bigcirc(p)$ of formulas which depends only on the propositional variable p, satisfying the following: There are formulas α and β such that

$$\text{(a)} \quad \bigcirc(\alpha), \alpha \nVdash \beta;$$
$$\text{(b)} \quad \bigcirc(\alpha), \neg\alpha \nVdash \beta.$$

We will call a theory Γ *gently explosive* (*with respect to* $\bigcirc(p)$) if:

$$\forall\alpha\forall\beta(\Gamma, \bigcirc(\alpha), \alpha, \neg\alpha \Vdash \beta).$$

A theory Γ will be said to be *finitely gently explosive* when it is gently explosive with respect to a finite set $\bigcirc(p)$.

A logic **L** will be said to be *(finitely) gently explosive* when there is a (finite) set $\bigcirc(p)$ such that all of the theories of **L** are (finitely) gently explosive (with respect to $\bigcirc(p)$). Notice that a finitely gently explosive theory is finitely trivialized in a very distinctive way.

We may now consider the following 'gentle' variations on the Principle of Explosion:

Gentle Principle of Explosion

$$\textbf{L} \text{ is gently explosive with respect to some set } \bigcirc(p) \tag{9}$$

Finite Gentle Principle of Explosion

$$\textbf{L} \text{ is gently explosive with respect to some finite set } \bigcirc(p) \tag{10}$$

For any formula α, the set $\bigcirc(\alpha)$ is intended to express, in a specific sense, the consistency of α relative to the logic **L**. When this set is a singleton, we will denote by $\circ\alpha$ the sole element of $\bigcirc(\alpha)$, and in this case $\circ$ defines a *consistency connective* or *consistency operator*. It is worth noting, however, that $\circ$ is not necessarily a primitive connective of the signature of **L**. In fact, several logics that will be studied below (namely, the so-called 'direct **dC**-systems', see Definition 32) present $\circ$ as a connective that is defined in terms of other connectives of a less complex underlying signature.

The above definitions are very natural, and paraconsistent logics with a consistency connective are in fact quite common. One way of seeing that is through the use of a classic-like (in fact, intuitionistic-like) disjunction:

DEFINITION 21. We say that a logic **L** *has a standard disjunction* if there is a formula $\psi(p_0, p_1)$ such that:

(a) $\psi(\alpha, \beta)$ is not a bottom particle, for some α and β;

(b) $\forall\alpha\forall\beta\forall\Gamma\forall\Delta\forall\gamma(\Gamma, \alpha \Vdash \gamma$ and $\Delta, \beta \Vdash \gamma$ implies $\Gamma, \Delta, \psi(\alpha, \beta) \Vdash \gamma)$;

(c) $\psi(\alpha, \beta)$ is not a top particle, for some α and β;

(d) $\forall\alpha\forall\beta\forall\Gamma\forall\gamma(\Gamma, \psi(\alpha, \beta) \Vdash \gamma$ implies $\Gamma, \alpha \Vdash \gamma$ and $\Gamma, \beta \Vdash \gamma)$.

In natural deduction, clause (b) corresponds to disjunction elimination, and clause (d) to disjunction introduction. The reader can now easily check that:

THEOREM 22. (i) Any non-trivial explosive theory / logic is finitely gently explosive, supposing that there is some formula α such that $\neg\alpha$ is not a bottom particle. (ii) Any left-adjunctive finitely gently explosive logic respects *ex falso*. (iii) Let **L** be a logic containing a bottom particle $\bot$, a standard disjunction $\vee$, an implication $\rightarrow$ respecting *modus ponens* and a negation $\neg$ such that there exists some formula α satisfying:
(a) $\alpha, (\neg\alpha \rightarrow \bot) \nVdash \bot$;
(b) $\neg\alpha, (\alpha \rightarrow \bot) \nVdash \bot$.
Then **L** defines a consistency operator $\circ\alpha \stackrel{\text{def}}{=} (\alpha \rightarrow \bot) \vee (\neg\alpha \rightarrow \bot)$. ■

We now define the Logics of Formal Inconsistency as the paraconsistent logics that can 'talk about consistency' in a meaningful way.

DEFINITION 23. A *Logic of Formal Inconsistency* (**LFI**) is any gently explosive paraconsistent logic, that is, any logic in which explosion, (3), does not hold, while gentle explosion, (9), holds good. ■

In other words, a logic **L** is an **LFI** (with respect to a negation $\neg$) if:

(a) $\exists\Gamma\exists\alpha\exists\beta(\Gamma, \alpha, \neg\alpha \nVdash \beta)$, and

(b) there exists a set of formulas $\bigcirc(p)$ depending exactly on the propositional variable p such that $\forall\Gamma\forall\alpha\forall\beta(\Gamma, \bigcirc(\alpha), \alpha, \neg\alpha \Vdash \beta)$.

Besides the 3-valued paraconsistent logics presented in the above examples, we will study in this chapter several other paraconsistent logics based on different kinds of semantics. Many will have been originally proposed without a primitive consistency connective, but, being sufficiently expressive, they will often be shown to admit of such a connective. Examples of that phenomenon were already presented above, for the cases of **LFI1** and $\mathbf{P}^1$. Another interesting and maybe even surprising example of that phenomenon is provided by Jaśkowski's Discussive Logic **D2** (cf. [Jaśkowski, 1948] and [Jaśkowski, 1949]), the first paraconsistent logic ever to be introduced as such in the literature:

EXAMPLE 24. Let $\Sigma^\Diamond$ be the extension of the signature Σ obtained by the addition of a new unary connective $\Diamond$, and let $For^\Diamond$ be the corresponding algebra of formulas. Let $\Vdash_{S5}$ be the consequence relation of modal logic *S5* over the language $For^\Diamond$. Consider a mapping $*\colon For \longrightarrow For^\Diamond$ such that:

1. $p^* = p$ for every $p \in \mathcal{P}$;
2. $(\neg\alpha)^* = \neg\alpha^*$;
3. $(\alpha \vee \beta)^* = \alpha^* \vee \beta^*$;
4. $(\alpha \wedge \beta)^* = \alpha^* \wedge \Diamond\beta^*$;

5. $(\alpha \to \beta)^* = \Diamond\alpha^* \to \beta^*$.

Given $\Gamma \subseteq For$, let Γ^* denote the subset $\{\alpha^* : \alpha \in \Gamma\}$ of $For^\Diamond$. For any $\Gamma \subseteq For^\Diamond$ let $\Diamond\Gamma = \{\Diamond\alpha : \alpha \in \Gamma\}$. Jaśkowski's Discussive logic **D2** is defined over the signature Σ as follows: $\Gamma \Vdash_{\mathbf{D2}} \alpha$ iff $\Diamond(\Gamma^*) \Vdash_{S5} \Diamond(\alpha^*)$, for any $\Gamma \cup \{\alpha\} \subseteq For$. Equivalently, **D2** may be introduced with the help of the pre-discussive logic J (recall Example 14), by setting $\Gamma \Vdash_{\mathbf{D2}} \alpha$ iff $\Gamma^* \Vdash_J \alpha^*$. With such definitions, **D2** can easily be seen to be non-explosive with respect to the negation $\neg$, that is, **D2** is paraconsistent (with respect to $\neg$). Consider now the following abbreviations defined on the set For (here, $\alpha \in For$):

$\top \stackrel{\text{def}}{=} (\alpha \vee \neg\alpha)$;
$\bot \stackrel{\text{def}}{=} \neg\top$;
$\blacksquare\alpha \stackrel{\text{def}}{=} (\neg\alpha \to \bot)$;
$\blacklozenge\alpha \stackrel{\text{def}}{=} \neg\blacksquare\neg\alpha$;
$\circ\alpha \stackrel{\text{def}}{=} (\blacklozenge\alpha \to \blacksquare\alpha)$.

It is an easy task to check now (say, using a Kripke semantics or tableaux for the logic $S5$) that in **D2** the formulas $\top$ and $\bot$ denote top and bottom particles, respectively, and $\circ$ behaves as a consistency operator (giving rise to gentle explosion). ■

THEOREM 25.
(i) Classical logic is not an **LFI**.

(ii) Pac (see Example 17) is also not an **LFI**.

(iii) **LFI1** (see Example 18) is an **LFI**.

(iv) $\mathbf{P}^1$ (see Example 19) is an **LFI**.

(v) Jaśkowski's Discussive Logic **D2** (see Example 24) is an **LFI**.

Proof. For item (i), note that explosion, (3), holds classically.

To check item (ii), let p be an atomic formula and let $\bigcirc(p)$ be the set of all formulas of Pac that depend only on p. The valuation from the truth-table that assigns $\frac{1}{2}$ to p and 0 to q is a model for $\bigcirc(p), p, \neg p$ but it invalidates gentle explosion (on q).

For item (iii), take consistency to be expressed in $\mathbf{J}_3$ by the connective $\circ$, as intended, that is, take $\bigcirc(\alpha) = \{\circ\alpha\}$. Obviously, $\bigcirc(\alpha), \alpha, \neg\alpha \vDash \beta$ holds. Take now a valuation from the truth-table that assigns 1 to p and notice that $\bigcirc(p), p \nvDash \beta$. Finally, take a valuation that assigns 0 to p and notice that $\bigcirc(p), \neg p \nvDash \beta$.

To check item (iv), again take consistency to be expressed in $\mathbf{P}^1$ by $\circ$ and note that $p, \neg p \nvDash q$, for atomic and distinct p and q.

Item (v) may be verified directly from the definitions in Example 24. ■

In accordance with definition (6) from Subsection 2.2, paraconsistent logics are the non-trivial logics whose negation fails the 'consistency presupposition'. Some inferences that depend on this presupposition, thus, will necessarily be lost. However, one might well expect that, if a sufficient number of 'consistency assumptions' are made, then those same inferences should be recovered. In fact, the **LFI**s are intended to be exactly the logics that can internalize this idea. To be more precise, and following [Marcos, 2005e]:

REMARK 26. Consider a logic $\mathbf{L1} = \langle For_1, \Vdash_1 \rangle$ in which explosion holds good for a negation $\neg$, that is, a logic that satisfies, in particular, the rule $(\alpha, \neg\alpha \Vdash_1 \beta)$. Let $\mathbf{L2} = \langle For_2, \Vdash_2 \rangle$ now be some other logic written in the same signature as **L**1 such that: (i) **L**2 is a proper deductive fragment of **L**1 that validates inferences of **L**1 only if they are compatible with the *failure* of explosion; (ii) **L**2 is *expressive* enough so as to be an **LFI**, therefore, in particular, there will be in **L**2 a set of formulas $\bigcirc(p)$ such that $(\bigcirc(\alpha), \alpha, \neg\alpha \Vdash_2 \beta)$ holds good; (iii) **L**1 can in fact be *recovered* from **L**2 by the addition of $\bigcirc(\alpha)$ as a new set of valid schemas / axioms. These constraints alone suggest that the reasoning of **L**1 might somehow be recovered from inside **L**2, if only a sufficient number of 'consistency assumptions' are added in each case. Thus, typically the following *Derivability Adjustment Theorem* (**DAT**) may be proven (as in [Marcos, 2005e]):

$$\forall\Gamma\forall\gamma\exists\Delta(\Gamma \Vdash_1 \gamma \;\text{ iff }\; \bigcirc(\Delta), \Gamma \Vdash_2 \gamma).$$

The **DAT** shows how the weaker logic **L**2 can be used to 'talk about' the stronger logic **L**1. The essential intuition behind such theorem was emphasized in [Batens, 1989], but an early version of that very idea can already be found in [da Costa, 1963] and [da Costa, 1974] (check our Theorem 112). On those grounds, **LFI**s are thus proposed and understood as the non-trivial inconsistent logics that can recover consistent inferences through convenient derivability adjustments. We will come back to this idea in Subsection 3.6 and Theorems 96, 112 and 113. ■

To get a bit more concrete, and at the same time specialize from the broad Definition 23 of **LFI**s, we introduce now the concept of a **C**-system.

DEFINITION 27. Let **L**1 and **L**2 be two logics defined over signatures Σ_1 and Σ_2, respectively, such that Σ_2 extends Σ_1, and Σ_2 contains a unary negation connective $\neg$ that does not belong to Σ_1. We say that **L**2 is a **C**-*system based on* **L**1 *with respect to* $\neg$ (in short, a **C**-*system*) if:

(a) **L**2 is a conservative extension of **L**1,
(b) **L**2 is an **LFI** (with respect to $\neg$), such that the set $\bigcirc(p)$ is a singleton $\{\circ p\}$, that is, consistency may be defined as a formula $\varphi(p)$ in **L**2,[6]

[6]In particular, $\varphi(p)$ could be of the form $\circledast(p)$ for $\circledast$ a unary connective of Σ_2.

(c) the non-explosive negation $\neg$ cannot be defined in **L**1,
(d) **L**1 is non-trivial. ∎

All **C**-systems we will be studying below are examples of non-contradictory $\neg$-paraconsistent logical systems. Furthermore, they are equipped with supplementing negations and bottom particles, and they are based on classical propositional logic (in a convenient signature which includes an explicit connective for classical negation). Accordingly, they will all respect Principles (1), (2), (7), (8) and (9), but they will obviously disrespect (3).

As it will be seen in the following, the hierarchy of logics C_n, $1 \leq n < \omega$ (cf. [da Costa, 1963] or [da Costa, 1974]) provide clear illustrations of **C**-systems based on classical logic, provided that each C_n is presented in an extended signature including a connective for classical negation. The cautious reader should bear in mind that C_ω (cf. Definition 40 below), the logic proposed as a kind of 'limit' for the hierarchy is *not* a **C**-system, not even an **LFI**. The real deductive limit for the hierarchy, the logic C_{Lim}, is an interesting example of a gently explosive **LFI** that is not finitely so, and it was studied in [Carnielli and Marcos, 1999]. The next definition will recall the hierarchy C_n, $1 \leq n < \omega$, in an axiomatic formulation of our own:

DEFINITION 28. Recall, once more, the signature Σ from Remark 15. For every formula α, let $\circ\alpha$ be an abbreviation for the formula $\neg(\alpha \wedge \neg\alpha)$. The logic $C_1 = \langle For, \vdash_{C_1} \rangle$ may be axiomatized by the following schemas of a Hilbert calculus:

Axiom schemas:

(Ax1) $\alpha \to (\beta \to \alpha)$

(Ax2) $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$

(Ax3) $\alpha \to (\beta \to (\alpha \wedge \beta))$

(Ax4) $(\alpha \wedge \beta) \to \alpha$

(Ax5) $(\alpha \wedge \beta) \to \beta$

(Ax6) $\alpha \to (\alpha \vee \beta)$

(Ax7) $\beta \to (\alpha \vee \beta)$

(Ax8) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

(Ax9) $\alpha \vee (\alpha \to \beta)$

(Ax10) $\alpha \vee \neg\alpha$

(Ax11) $\neg\neg\alpha \to \alpha$

(bc1) $\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$

(ca1) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \wedge \beta)$

(ca2) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \vee \beta)$

(ca3) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \to \beta)$

Inference rule:

(MP) $\dfrac{\alpha,\ \alpha \to \beta}{\beta}$

In general, given a set of axioms and rules of a logic **L**, we write $\Gamma \vdash_{\mathbf{L}} \alpha$ to say that there is proof in **L** of α from the premises in Γ. The subscript will be omitted when obvious from the context. If Γ is empty we say that α is a *theorem* of **L**.

The logic C_1 is a **LFI** such that $\bigcirc(p) = \{\circ p\} = \{\neg(p \wedge \neg p)\}$. We shall see that axioms (bc1), and (ca1)–(ca3) can be stated in a new fashion by taking $\circ$ as a primitive connective instead of as an abbreviation. From these new axioms different logics will emerge. Moreover, since it is possible to define a classical negation $\sim$ in C_1 (namely, $\sim\alpha = \neg\alpha \wedge \circ\alpha$), this logic may be rewritten in an extended signature which contains $\sim$ as a primitive connective (and adding the obvious axioms identifying $\sim\alpha$ with $\neg\alpha \wedge \circ\alpha$), and so it is easy to see that C_1 (presented in the extended signature) is a **C**-system based on classical logic (see Remark 29 below).

Let α^1 abbreviate the formula $\neg(\alpha\wedge\neg\alpha)$, and α^{n+1} abbreviate the formula $(\neg(\alpha^n \wedge \neg\alpha^n))^1$. Then, each logic C_n of the hierarchy $\{C_n\}_{1\leq n<\omega}$ may be obtained by assuming $\bigcirc(p) = \{p^1, \ldots, p^n\}$. This is equivalent, of course, to setting $\circ\alpha \stackrel{\text{def}}{=} \alpha^1 \wedge \ldots \wedge \alpha^n$ in axioms (bc1) and (ca1)–(ca3). It is immediate to see that every logic C_n is an **LFI**. Moreover, by considering the definable classical negation $\sim$ as a primitive connective, each C_n (presented in the extended signature) is a **C**-system based on classical logic. It is well known that each C_n properly extends each C_{n+1}. ■

REMARK 29. Let the signature Σ^+ denote the signature Σ without the symbol $\neg$, and For^+ be the corresponding $\neg$-free fragment of For. *Positive classical logic*, from now on denoted as $\mathbf{CPL}^+$, may be axiomatized in the signature Σ^+ by axioms (Ax1)–(Ax9), plus (MP). *Classical propositional logic*, from now on denoted by **CPL**, is an extension of $\mathbf{CPL}^+$ in the signature Σ, where $\neg$ is governed by two dual axioms, (Ax10) and the following 'explosion law':

(exp) $\alpha \to (\neg\alpha \to \beta)$

That axiomatization should come as no surprise, if you only recall the notion of a classical negation from Definition 8. Clearly, for any logic **L** extending $\mathbf{CPL}^+$ a (primitive or defined) unary connective $\urcorner$ of **L** is a classical negation iff the schemas $(\alpha \vee \urcorner\alpha)$ and $(\alpha \to (\urcorner\alpha \to \beta))$ are provable.

CPL is also the minimal consistent extension of C_1. Indeed, an alternative way of axiomatizing **CPL** is by adding $\circ\alpha$ to C_1 as a new axiom schema, and (exp) then follows from (bc1) and this new axiom, by (MP). On the other hand, *positive intuitionistic logic* may be axiomatized from $\mathbf{CPL}^+$ by dropping (Ax9).

As mentioned above, C_1 may be considered as a deductive fragment of **CPL** (in the signature Σ), whereas **CPL** may be considered as a deductive fragment of C_1 in the signature $\Sigma^\sim$ obtained from Σ by adding a symbol $\sim$ for classical negation, and where $\neg$ denotes the paraconsistent negation of C_1.

As it is well known (cf. [Mendelson, 1997]), any logic having (Ax1) and (Ax2) as axioms, and *modus ponens* (MP) as its only primitive inference rule has a deductive implication.[7]

In any logic endowed with a deductive implication, the Principle of Explosion, (3), and the explosion law, (exp), are interderivable. So, for any such logic, if paraconsistency is to be obtained, (exp) must fail.

As usual, bi-implication $\leftrightarrow$ will be defined here by setting $(\alpha \leftrightarrow \beta) \stackrel{\text{def}}{=} ((\alpha \to \beta) \wedge (\beta \to \alpha))$. Note that, in the presence of a deductive implication $\to$, $\vdash (\alpha \leftrightarrow \beta)$ if, and only if, $\alpha \vdash \beta$ and $\beta \vdash \alpha$, that is, iff α and β are equivalent. Nevertheless, the equivalence of two formulas, in the logics we will study here, does not necessarily guarantee that these formulas may be freely inter-substituted for each other, as we shall see below. ■

Recall that the definition of a **C**-system (Definition 27) mentioned **LFI**s in which the set $\bigcirc(p)$ could be taken as a singleton. The easiest way of realizing this intuition is by extending the original language of our logics so as to count from the start with a primitive connective $\circ$ for consistency.

REMARK 30. Recall the signature Σ° from Remark 15. Consider the following (innocuous, but linguistically relevant) extension of **CPL** that presupposes all formulas to be consistent, obtained by the addition of the following new axiom:

(ext) $\circ\alpha$

In practice, this will constitute of course just another version of **CPL** in a different signature, where any formula of the form $\circ\alpha$ is assumed to be a top particle. This logic, which we will here call *extended classical logic* and denote by **eCPL**, will come in handy below when we start building **C**-systems based on classical logic. ■

Sometimes our Logics of Formal Inconsistency can dismiss the new consistency connective (by replacing it by a formula built from the other connectives already present in the signature). Before defining this class of logics,

[7]This is not always true, though, for logics extending (Ax1), (Ax2) and (MP) by the addition of new primitive inference rules.

it is convenient to make a little detour and present a fundamental notion that will have a role to play in several parts of this chapter, namely, the concept of translation between logics.

DEFINITION 31. Let **L**2 and **L**1 be logics with sets of formulas For_2 and For_1, respectively. A mapping t: $For_2 \longrightarrow For_1$ is said to be a *translation from* **L**2 *to* **L**1 if, for every set $\Gamma \cup \{\alpha\}$ of **L**2-formulas,

$$\Gamma \vdash_{\mathbf{L2}} \alpha \ \text{ implies } \ t(\Gamma) \vdash_{\mathbf{L1}} t(\alpha).$$

Here, $t(\Gamma)$ stands for $\{t(\gamma) : \gamma \in \Gamma\}$.

If 'implies' is replaced by 'iff' in the definition above, then t is called a *conservative translation*. See [da Silva *et al.*, 1999], [Coniglio and Carnielli, 2002] and [Coniglio, 2005] for a general account of translations and conservative translations. ■

Now, having the notion of translations at hand, the special kind of **C**-systems mentioned above is defined as follows:

DEFINITION 32. Let **L**2 be a **C**-system with respect to $\neg$, based on a logic **L**1, and let $\varphi(p)$ represent the formula schema with respect to which **L**2 is gently explosive, that is, such that $\varphi(\alpha)$ represents in **L**2 the consistency of the formula α with respect to the non-explosive negation $\neg$. Where Σ_2 represents the signature of the logic **L**2, let $\mathsf{cnt}[\varphi(\mathsf{p})]$ represent the set of connectives involved in the formulation of $\varphi(p)$. Let Σ' be any signature obtained by dropping from Σ_2 all the connectives that appear in $\mathsf{cnt}[\varphi(\mathsf{p})]$, that is, Σ' is a restriction of the signature of **L**2 in which consistency can no more be expressed in the same way as in the original logic **L**2. Now, in case it is still possible to express the consistency of the formulas of **L**2 with the help of the remaining connectives in $\Sigma' \subsetneq \Sigma_2$, say, by way of a set of formulas $\varphi'(p)$ over Σ', then we say that **L**2 is a **dC**-*system* based on **L**1 (or simply a **dC**-*system*). So, **dC**-systems are **C**-systems with respect to some negation and some consistency schema $\varphi(p)$ where it is also possible to express consistency alternatively by way of a formula $\varphi'(p)$ such that $\varphi(p)$ and $\varphi'(p)$ have no common structure, that is, such that $\mathsf{cnt}[\varphi'(\mathsf{p})] \cap \mathsf{cnt}[\varphi'(\mathsf{p})] = \varnothing$. This is typically the case when $\varphi(p)$ has the form $\circ(p)$, where $\circ$ is a primitive unary connective of Σ_2, but where, at the same time, $\circ$ can be explicitly defined by way of the connectives in $\Sigma_2 \setminus \{\circ\}$ (see examples below). In that case we say that **L**2 is a *direct* **dC**-*system* based on **L**1 (or simply a *direct* **dC**-*system*). As we will see below, there are **dC**-systems that are not direct (they will from here on be called *indirect*). In those indirect **dC**-systems, consistency cannot be expressed by a unary connective $\circ$, primitive or defined, but only by way of a complex formula φ, depending on a single variable.

DEFINITION 33. Let Σ be the signature of an indirect **dC**-system **L**, and consider the direct **dC**-system **L**$'$ defined over the signature Σ', such that:

(a) $\mathbf{L}'$ is a conservative extension of $\mathbf{L}$ obtained by the addition of a new unary connective $\circ$, that is, such that $\Sigma' = \Sigma \cup \{\circ\}$ (so, in particular, the consistency of a formula α can be expressed in $\mathbf{L}'$ exactly as in $\mathbf{L}$, namely, by way of the formula $\varphi(\alpha)$);

(b) $\mathbf{L}'$ is an **LFI** with respect to $\circ \in \Sigma'$, and a **C**-system with respect to some $\neg \in \Sigma$ (so, in particular, the consistency of α can also be expressed in $\mathbf{L}'$ by way of the formula $\circ\alpha$);

(c) In $\mathbf{L}'$ the connective $\circ$ plays the same role as the formula φ plays in $\mathbf{L}$, more specifically, there is a translation $\star : For_{\mathbf{L}'} \longrightarrow For_{\mathbf{L}}$ respecting the following clauses:

(c.1) $t(p) = p$, for p a propositional variable

(c.2) $t(\circledast(\alpha_1, \ldots, \alpha_n)) = \circledast(t(\alpha_1), \ldots, t(\alpha_n))$, for every n-ary connective $\circledast$ in Σ' distinct from $\circ$, and for any choice of formulas $\alpha_1, \ldots, \alpha_n$ from $For_{\mathbf{L}'}$

(c.3) $t(\circ\alpha) = \varphi(t(\alpha))$, for any formula α in $For_{\mathbf{L}'}$

In a case like this we may say that the direct **dC**-system $\mathbf{L}'$ *corresponds* to the indirect **dC**-system $\mathbf{L}$. Indirect **dC**-systems appear typically when we are talking about **C**-systems for which the replacement property fails to such an extent that it might turn out to be impossible to give an explicit definition of the consistency connective in terms of other, more usual connectives. (Examples follow below.) ■

The next example and the subsequent theorem will show that **dC**-systems are even more ubiquitous than one might initially imagine.

EXAMPLE 34. Let $\Sigma^{\Diamond\Box}$ be the signature obtained by the addition of the new unary connectives $\Diamond$ and $\Box$ to the signature Σ, where the connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$ of Σ are interpreted as in classical logic and the new connectives are interpreted as usual in normal modal logics. So, $\Diamond\alpha$ (respectively, $\Box\alpha$) will be true in a given world iff α is true in some (respectively, any) world accessible to the former. The most obvious degenerate examples of normal modal logics are characterized by frames that are such that every world can access only itself or no other world. As shown in [Marcos, 2005e], inside any non-degenerate normal modal logic, a paraconsistent negation $\smile$ may be defined by setting $\smile\alpha \overset{\text{def}}{=} \Diamond\neg\alpha$, and a consistency connective may be defined by setting $\circ\alpha \overset{\text{def}}{=} \alpha \rightarrow \Box\alpha$.

Conversely, take the signature Σ°, and interpret the primitive negation $\neg$ now over Kripke structures so as to make it behave exactly like the above connective $\smile$, that is, an interpretation such that, for worlds x and y of a model $\mathcal{M}$ with an accessibility relation R:

$$\models_x^{\mathcal{M}} \neg\alpha \ \text{ iff } \ (\exists y)(x\mathsf{R}y \text{ and } \not\models_y^{\mathcal{M}} \alpha).$$

Moreover, let the consistency connective be interpreted in such a way that:

$$\models_x^{\mathcal{M}} \circ\alpha \ \text{ iff } \ \models_x^{\mathcal{M}} \alpha \text{ implies } (\forall y)(\text{if } x\mathsf{R}y \text{ then } \models_y^{\mathcal{M}} \alpha).$$

Then, in the present case, one can still redefine the previous connectives of $\Sigma^{\Diamond\Box}$. Indeed, one can define a bottom $\bot$ by setting $\bot \stackrel{\text{def}}{=} \alpha\wedge(\neg\alpha\wedge\circ\alpha)$, for an arbitrary formula α, and then define a classical negation $\sim$ by setting ${\sim}\alpha \stackrel{\text{def}}{=} \alpha \to \bot$. The original modal connectives can finally be defined by setting $\Diamond\alpha \stackrel{\text{def}}{=} \neg{\sim}\alpha$ and $\Box\alpha \stackrel{\text{def}}{=} {\sim}\neg\alpha$.

The above arguments show that any non-degenerate normal modal logic may be naturally reformulated in the signature of an **LFI**. In that sense, modal logics are typically paraconsistent, and could be recast as the study of paraconsistent negations (instead of operators such as $\Box$ and $\Diamond$). ■

THEOREM 35.
(i) **LFI1** (see Example 18) is a **C**-system (based either on $\mathbf{CPL}^+$ or on **CPL**), but not a **dC**-system.

(ii) $\mathbf{P}^1$ (see Example 19) is a direct **dC**-system.

(iii) The logics $C_n, 1 \leq n < \omega$, (see Definition 28) are all direct **dC**-systems.

(iv) Jaśkowski's Discussive Logic **D2** (see Example 24) is a direct **dC**-system.

(v) The normal modal logics from Example 34 are all direct **dC**-systems.

Proof. For item (i), observe first that **LFI1** is a **C**-system based on classical logic. Indeed, the binary connectives of **LFI1** all behave classically: All axioms of $\mathbf{CPL}^+$ are validated by the 3-valued truth-tables of **LFI1**, and (MP) preserves validity. Second, as we already know, the classical negation $\sim$ can be defined in **LFI1**. Third, the connective $\circ$ expresses consistency in **LFI1**, and the latter logic is indeed a conservative extension of *Pac* obtained exactly by the addition of that connective. Similarly, the non-explosive negation $\neg$ of **LFI1** can easily be seen not to be definable, in **LFI1**, from the truth-tables of the classical connectives. Finally, recall from Theorem 25 that *Pac* is not an **LFI**, and observe that $\circ$ is not definable from the other connectives of **LFI1**. Items (ii)–(v) were already explained when the corresponding logics were introduced. ■

The first examples of indirect **dC**-systems will appear only in Theorems 106 and 110, as well as Remark 111.

All **LFI**s studied from the next subsection on, unless explicit mention to the contrary, are **C**-systems based on classical logic, and can therefore be axiomatized starting from $\mathbf{CPL}^+$.

3.2 Towards ***mbC****, a fundamental* ***LFI***

Before introducing our weakest **LFI** based on classical logic, we will introduce a very weak non-gently explosive paraconsistent logic.

Do bear in mind, from Remark 29, that $\neg$ in **CPL** was axiomatized by the addition to $\mathbf{CPL}^+$ of two dual clauses, (Ax10) and (exp).

DEFINITION 36. The paraconsistent logic *PI*, investigated in [Batens, 1980], extends $\mathbf{CPL}^+$ in the signature Σ (see Remark 29) by the addition of (Ax10). In other words, *PI* is axiomatized by (Ax1)–(Ax10) and (MP) (recall Definition 28). ■

It is worth noting that, due to (Ax8), (Ax10) and to the fact that *PI* has a deductive implication (recall Definition 6), one can count on the classical proof strategy known as *proof-by-cases*:

THEOREM 37. If $(\Gamma, \alpha \vdash_{PI} \beta)$ and $(\Delta, \neg\alpha \vdash_{PI} \beta)$ then $(\Gamma, \Delta \vdash_{PI} \beta)$. ■

Here are some other important properties of *PI*:

THEOREM 38. (i) *PI* is boldly paraconsistent.

Moreover, for any boldly paraconsistent extension $\mathbf{L}$ of *PI*:

(ii) *Reductio ad absurdum* is not a valid rule, i.e. rules such as:
$(\Delta, \beta \vdash_{\mathbf{L}} \alpha)$ and $(\Pi, \beta \vdash_{\mathbf{L}} \neg\alpha)$ implies $(\Delta, \Pi \vdash_{\mathbf{L}} \neg\beta)$, and
$(\Delta, \neg\beta \vdash_{\mathbf{L}} \alpha)$ and $(\Pi, \neg\beta \vdash_{\mathbf{L}} \neg\alpha)$ implies $(\Delta, \Pi \vdash_{\mathbf{L}} \beta)$
cannot obtain.

(iii) If the implication $\to$ is still a deductive implication (recall Definition 6), contraposition is not a valid rule, i.e. rules such as:
$\Gamma, \alpha \to \beta \vdash_{\mathbf{L}} \neg\beta \to \neg\alpha$
$\Gamma, \alpha \to \neg\beta \vdash_{\mathbf{L}} \beta \to \neg\alpha$
$\Gamma, \neg\alpha \to \beta \vdash_{\mathbf{L}} \neg\beta \to \alpha$
$\Gamma, \neg\alpha \to \neg\beta \vdash_{\mathbf{L}} \beta \to \alpha$
cannot obtain.

Proof. For item (i), note that *PI* has a deductive implication and is a fragment of both *Pac* and $\mathbf{P}^1$. Indeed, the axioms of *PI* are all validated by the truth-tables of *Pac* and by the truth-tables of $\mathbf{P}^1$, and (MP) preserves validity. Recall that those 3-valued extensions of *PI* were already proven to be boldly paraconsistent in Theorem 20.
For item (ii), let $\Delta = \Pi = \{\alpha, \neg\alpha\}$. Then, by *reductio*, the logic would be partially explosive.
For item (iii), using the properties of the deductive implication, we have that $\gamma \vdash_{\mathbf{L}} \alpha \to \gamma$. Then again, by contraposition, the logic would turn out to be partially explosive. ■

As we will soon see (check Theorem 48), the upgrade of non-gently explosive logics into **LFI**s will help remedy the above mentioned deductive weaknesses, so typical of paraconsistent logics in general.

Here again, using the fact that *PI* is a deductive fragment of *Pac*, it can also be easily checked that:

THEOREM 39. The logic *PI*:
(i) does not have a supplementing negation, nor a bottom particle;
(ii) is not finitely trivializable;
(iii) is not an **LFI**. ■

Before proceeding, this seems to be a convenient place to mention some logics that live very close to *PI*:

DEFINITION 40. The logic C_{min} (cf. [Carnielli and Marcos, 1999]) is obtained from *PI* by the addition of $\neg\neg\alpha \to \alpha$ as a new axiom. The logic C_ω (cf. [da Costa, 1963]) is obtained from C_{min} by dropping (Ax9). Finally, the logic *CAR* (cf. [da Costa and Béziau, 1993]) is obtained from *PI* by adding $\alpha \to (\neg\alpha \to \neg\beta)$ as a new axiom. ■

Finally, here are some other important facts about *PI*:

THEOREM 41.
(i) *PI* does not prove any negated formula (that is, any formula of the form $\neg\delta$).
(ii) No two different negated formulas of *PI* are equivalent, that is, if $\neg\alpha \dashv\vdash_{PI} \neg\beta$ then $\alpha = \beta$.

Proof. Item (i) was already proven in [Carnielli and Marcos, 1999] for C_{min}. Item (ii) was proven in [Urbas, 1989] for C_ω, and the proof may be easily adapted for *PI*. ■

As we saw in Theorem 39(iii), *PI* is not an **LFI**. We will now make the most obvious upgrade of *PI* that will turn it into an **LFI**, endowing it with the most straightforward axiomatic version of the principle (10), the so-called Finite Gentle Principle of Explosion:

DEFINITION 42. Recall the signature Σ° from Remark 15 and the logic *PI* from Definition 36. The logic **mbC** is obtained from *PI*, over Σ°, by the addition of the following axiom schema:

(bc1) $\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$

In other words, **mbC** is axiomatized by (Ax1)–(Ax10) plus (MP) (recall Definition 28), but now over the signature Σ°, together with the extra axiom (bc1), above. ■

Notice that a particular form of axiom (bc1) had already been considered in Definition 28, but there $\circ\alpha$ was considered as an abbreviation for $\neg(\alpha \wedge \neg\alpha)$, instead of a primitive connective. We recall that the intended reading of $\circ\alpha$ is 'α is consistent'. As we shall see, in general, $\circ\alpha$ is logically independent from $\neg(\alpha \wedge \neg\alpha)$.

If $\vdash_{\mathbf{mbC}}$ denotes the consequence relation of **mbC**, then we obtain, by (MP), the following:

$$\circ\alpha,\ \alpha,\ \neg\alpha \vdash_{\mathbf{mbC}} \beta \tag{11}$$

Rule (11) may be read as saying that 'if α is consistent and contradictory, then it explodes'. Clearly, this rule amounts to a realization of the Finite Gentle Principle of Explosion (10), as in our formulation of da Costa's C_n (Definition 28), with the difference that now $\circ$ is a primitive unary connective and *not* an abbreviation depending on conjunction and negation.

REMARK 43. It is easy to define supplementing negations in **mbC**. Consider first a negation $\wr$ set by $\wr\alpha \stackrel{\text{def}}{=} (\neg\alpha \wedge \circ\alpha)$. Notice that, as a particular instance of Theorem 13(i), $\bot_\beta \stackrel{\text{def}}{=} (\beta \wedge \wr\beta)$ defines a bottom particle, for every β. Consider next a negation $\sim_\beta$ set by $\sim_\beta\alpha \stackrel{\text{def}}{=} \alpha \rightarrow \bot_\beta$. Clearly, $\forall\alpha\forall\gamma(\alpha, \wr\alpha \vdash_{\mathbf{mbC}} \gamma)$ and $\forall\beta\forall\alpha\forall\gamma(\alpha, \sim_\beta\alpha \vdash_{\mathbf{mbC}} \gamma)$. In Remark 70, the semantic tools of Subsection 3.4, granting sound and complete possible-translations interpretations for **mbC**, will help us showing that neither $\sim_\beta\alpha$ nor $\wr\alpha$ are always bottom particles. Moreover, these supplementing negations will in fact be seen to be inequivalent: though $\wr\alpha$ derives $\sim_\beta\alpha$, the converse is not true. While $\sim_\beta$ defines a classical negation, $\wr$ fails to be complementing (the latter facts will be proven in Remark 70).

From now on, we will simply write $\bot$ and $\sim$ to refer to any of the connectives $\bot_\beta$ and $\sim_\beta$ defined above. Despite $\bot_\beta$ and $\bot_\gamma$, as well as $\sim_\beta\alpha$ and $\sim_\gamma\alpha$, being equivalent for every β, γ and α, they cannot be freely intersubstituted (check the end of Remark 29). It will be often useful, in this paper, to consider our **C**-systems to be written from the start in an extended signature containing both the non-explosive negation $\neg$ and the classical negation $\sim$, to be set as in the above definition. ■

THEOREM 44. **mbC** is an **LFI**. In fact, it is a **C**-system based on **CPL**.

Proof. Note that **mbC** is indeed a fragment of **LFI1** and of $\mathbf{P}^1$, and in Theorem 25 the latter were shown to be **LFI**s. Moreover, we now know from rule (11) that the principle (9) holds in **mbC** (in fact its finite form (10) already holds). By design, we also know that **mbC** contains $\mathbf{CPL}^+$, and $\neg$ cannot be defined in the latter logic. Thus, **mbC** is a **C**-system based on $\mathbf{CPL}^+$ such that $\bigcirc(p) = \{\circ p\}$. To check that **mbC** can also be seen as a **C**-system based on full **CPL** one might notice that **mbC** extends **CPL** in a signature with two negations (as in the preceding remark). This extension must be conservative, given that **CPL** is well-known to be maximal with respect to the trivial logic. ■

So, **mbC** may be considered as a deductive fragment of **CPL**, provided that **CPL** is presented as **eCPL** in the signature Σ°. On the other hand, taking into account the signature $\Sigma^{\circ\sim}$ obtained from Σ° by adding a symbol $\sim$ for the classical negation $\sim\alpha = \alpha \rightarrow \bot$ of **mbC** (recall Remark 43), and where $\neg$ denotes the paraconsistent negation, **CPL** is a deductive fragment of **mbC** such that **mbC** is a **C**-system based on **CPL**, provided that the obvious axioms defining $\sim$ in terms of the other connectives of Σ° are added to **mbC**.

REMARK 45. In spite of the term 'Logics of Formal *In*consistency', we have mentioned but a *consistency* connective $\circ$ this far. But **mbC** can also count on the dual *inconsistency* connective $\bullet$. To define it, in general, one might make use of a classical negation, such as the negation $\sim$ defined in the above remark, and set $\bullet\alpha \stackrel{\text{def}}{=} \sim\circ\alpha$. ■

The logic **mbC** inherits the main properties of the positive fragment of *PI* (such as those properties of the standard conjunction, the standard disjunction and the deductive implication), but above we have seen that the former logic is much richer than the latter. As another illustration of this fact, from Theorem 44 and Remark 43 we can immediately see that none of the claims from Theorem 39 are any longer valid in **mbC**. Furthermore, the claims of Theorem 41 also do not hold good for **mbC**:

THEOREM 46.
(i) There are in **mbC** theorems of the form $\neg\delta$, for some formula δ.
(ii) There are formulas α and β in **mbC** such that $\alpha \neq \beta$, α and β are equivalent, and $\neg\alpha$ and $\neg\beta$ are also equivalent.

Proof. (i) Consider any bottom particle $\bot$ of **mbC**. Then $(\bot \vdash_{\mathbf{mbC}} \neg\bot)$ and $(\neg\bot \vdash_{\mathbf{mbC}} \neg\bot)$, thus $\vdash_{\mathbf{mbC}} \neg\bot$, by Theorem 37.
(ii) Take α and β to be any two syntactically distinct bottom particles. ■

Even if, differently from *PI*, **mbC** *does* have negated theorems, it does *not* have consistent theorems:

THEOREM 47. There are in **mbC** no theorems of the form $\circ\delta$.

Proof. Use the classical truth-tables over $\{0, 1\}$ for $\wedge, \vee, \rightarrow$ and $\neg$, and pick for $\circ$ a truth-table with value constant and equal to 0. ■

The price to pay for paraconsistency is that we necessarily lose some theorems and inferences dependent on the 'consistency presupposition'. This has been illustrated, for instance, in Theorem 38, where *PI* and its extensions (satisfying certain assumptions) were shown to lack some usual classical proof strategies such as *reductio* and contraposition. This loss in inferential power can be remedied in the **LFI**s exactly by adding convenient consistency assumptions at the object-language level, as advanced in Remark 26. Indeed, some restricted forms of those rules may be proven in **mbC**:

THEOREM 48. The following *reductio* rules hold good in **mbC**:

(i) $(\Gamma \vdash_{\mathbf{mbC}} \circ\alpha)$ and $(\Delta, \beta \vdash_{\mathbf{mbC}} \alpha)$ and $(\Lambda, \beta \vdash_{\mathbf{mbC}} \neg\alpha)$ implies $(\Gamma, \Delta, \Lambda \vdash_{\mathbf{mbC}} \neg\beta)$

(ii) $(\Gamma \vdash_{\mathbf{mbC}} \circ\alpha)$ and $(\Delta, \neg\beta \vdash_{\mathbf{mbC}} \alpha)$ and $(\Lambda, \neg\beta \vdash_{\mathbf{mbC}} \neg\alpha)$ implies $(\Gamma, \Delta, \Lambda \vdash_{\mathbf{mbC}} \beta)$

The following contraposition rules hold in **mbC**:

(iii) $\circ\beta, (\alpha \rightarrow \beta) \vdash_{\mathbf{mbC}} (\neg\beta \rightarrow \neg\alpha)$

(iv) $\circ\beta, (\alpha \rightarrow \neg\beta) \vdash_{\mathbf{mbC}} (\beta \rightarrow \neg\alpha)$

(v) $\circ\beta, (\neg\alpha \rightarrow \beta) \vdash_{\mathbf{mbC}} (\neg\beta \rightarrow \alpha)$

(vi) $\circ\beta, (\neg\alpha \rightarrow \neg\beta) \vdash_{\mathbf{mbC}} (\beta \rightarrow \alpha)$ ■

The last theorem is an instance of a more general phenomenon: Any classical rule may be recovered within our **C**-systems based on classical logic (check the discussion about that at Subsection 3.6).

Intuitively, a contradiction might be seen as a sufficient condition for inconsistency. Indeed, here are some properties that relate the new connective of consistency to the more familiar connectives of $\mathbf{CPL}^+$:

THEOREM 49. In **mbC** the following hold good:

(i) $\alpha, \neg\alpha \vdash_{\mathbf{mbC}} \neg{\circ}\alpha$

(ii) $\alpha \wedge \neg\alpha \vdash_{\mathbf{mbC}} \neg{\circ}\alpha$

(iii) ${\circ}\alpha \vdash_{\mathbf{mbC}} \neg(\alpha \wedge \neg\alpha)$

(iv) ${\circ}\alpha \vdash_{\mathbf{mbC}} \neg(\neg\alpha \wedge \alpha)$

The converses of these rules are all failed by **mbC**.

Proof. Items (i)–(iv) are easy consequences of the restricted forms of *reductio* from Theorem 48.

In order to prove the second half of the theorem, consider the truth-tables of $\mathbf{P}^1$ (Example 19), but substitute the truth-table for negation, $\neg$, by the 3-valued truth-table for classical negation, $\sim$, to be found in Example 17. Then, **mbC** is sound for this set of truth-tables, and so it is enough to prove the failure of the converse rules using these same truth-tables. For instance, the rule $\neg(\neg\alpha \wedge \alpha) \vdash {\circ}\alpha$, converse to rule (iv), is failed if we put an atom p in the place of the schema α and pick a valuation v such that $v(p) = \frac{1}{2}$. Indeed, observe that the above described set of truth-tables will make $v(\neg p) = 0$, thus $v(p \wedge \neg p) = 0$ and $v(\neg(p \wedge \neg p)) = 1$, while they will also make $v({\circ}p) = 0$, providing a counter-model for this inference that is nevertheless sound for **mbC**. (Alternative counter-models, in terms of possible-translations semantics, will be offered in Example 69.) ■

The last result hints to the fact that paraconsistent logics may easily have certain unexpected asymmetries. That's what happens, for instance, with da Costa's C_1. As we shall see, the converse of (iii) holds good in C_1, while the converse of (iv) fails, so that $\neg(\alpha\wedge\neg\alpha)$ and $\neg(\neg\alpha\wedge\alpha)$ are not equivalent formulas in C_1. Other even more shocking examples of asymmetries are the following:

THEOREM 50. In **mbC**:

(i) $(\alpha \wedge \beta) \dashv\vdash_{\mathbf{mbC}} (\beta \wedge \alpha)$ holds good,
but $\neg(\alpha \wedge \beta) \dashv\vdash_{\mathbf{mbC}} \neg(\beta \wedge \alpha)$ does not hold.

(ii) $(\alpha \vee \beta) \dashv\vdash_{\mathbf{mbC}} (\beta \vee \alpha)$ holds good,
but $\neg(\alpha \vee \beta) \dashv\vdash_{\mathbf{mbC}} \neg(\beta \vee \alpha)$ does not hold.

(iii) $(\alpha \wedge \neg\alpha) \dashv\vdash_{\mathbf{mbC}} (\neg\alpha \wedge \alpha)$ holds good,
but $\neg(\alpha \wedge \neg\alpha) \dashv\vdash_{\mathbf{mbC}} \neg(\neg\alpha \wedge \alpha)$ does not hold.

(iv) $\gamma \vee \neg\gamma$ is a top particle, thus $(\alpha \vee \neg\alpha) \dashv\vdash_{\mathbf{mbC}} (\beta \vee \neg\beta)$ holds good. But $\neg(\alpha \vee \neg\alpha) \dashv\vdash_{\mathbf{mbC}} \neg(\beta \vee \neg\beta)$ does not hold.

(v) The equivalence $\alpha \dashv\vdash_{PI} (\neg\alpha \to \alpha)$ holds good, but $\neg\alpha \dashv\vdash_{\mathbf{mbC}} \neg(\neg\alpha \to \alpha)$ does not hold.

Proof. Using *PI* it is easy to prove the first halves of each item.

Items (i) to (iii). In order to check that none of the other halves hold, we can use again the truth-tables of $\mathbf{P}^1$ (Example 19), but redefining $(1 \wedge \frac{1}{2}) = (1 \vee \frac{1}{2}) = \frac{1}{2}$.

For item (iv), use the truth-tables of **LFI1** (Example 18), and take a valuation v such that $v(p) \neq v(q)$ and $v(p), v(q) \in \{1, \frac{1}{2}\}$. For item (v), use again the truth-tables of $\mathbf{P}^1$, and consider $v(p) = \frac{1}{2}$. ■

REMARK 51. The last theorem illustrates the failure of the so-called *replacement property*. This property states that, for any choice of formulas $\alpha_0, \ldots, \alpha_n$, $\beta_0, \ldots, \beta_n$ and of formula $\varphi(p_0, \ldots, p_n)$:

(RP) $(\alpha_0 \dashv\vdash \beta_0)$ and ... and $(\alpha_n \dashv\vdash \beta_n)$ implies
$$\varphi(\alpha_0, \ldots, \alpha_n) \dashv\vdash \varphi(\beta_0, \ldots, \beta_n)$$

For example, from $\alpha \dashv\vdash \beta$ one would immediately derive $\neg\alpha \dashv\vdash \neg\beta$, using (RP). But this does not hold for **mbC**. Recall, by the way, that $\alpha \dashv\vdash_{\mathbf{mbC}} \beta$ amounts to $\vdash_{\mathbf{mbC}} \alpha \leftrightarrow \beta$, given the definition of bi-implication and the presence of a deductive implication in **mbC**. Logics enjoying (RP) are called *self-extensional* in [Wójcicki, 1988]. Paradigmatic examples of such logics are provided by normal modal logics. ■

We will show below that various other classes of **LFI**s fail the replacement property (see Theorems 52, 81 and 133).

A natural question here is whether our logics can be upgraded so as to restore the interesting property (RP) inside the paraconsistent territory. To ensure that (RP) is obtainable in extensions of *PI* in the signature Σ, it is enough to add the rule:

(EC) $\forall\alpha\forall\beta((\alpha \dashv\vdash \beta)$ implies $(\neg\alpha \dashv\vdash \neg\beta))$

In [Urbas, 1989] paraconsistent extensions of C_ω (see Definition 40) enjoying the rule (EC) are shown to exist. The argument may be easily adapted to several extensions of *PI*, but it does not follow for many other such extensions, as it will be shown below. In [da Costa and Béziau, 1993], the logic *CAR* (see Definition 40) was introduced as an extension of *PI* where (RP) holds good. But *CAR* is not an **LFI**, and it is not boldly paraconsistent, being partially explosive exactly as the Minimal Intuitionistic Logic *MIL* from Example 10. To obtain the replacement property in extensions of **mbC**, in the signature Σ°, a further rule is needed, namely:

(EO) $\forall\alpha\forall\beta((\alpha \dashv\vdash \beta)$ implies $(\circ\alpha \dashv\vdash \circ\beta))$

Before ending this subsection, let us quickly survey some results on the possible validity of (RP) in paraconsistent extensions of **mbC**, or in some of its fragments:

THEOREM 52. The replacement property (RP) cannot hold in any paraconsistent extension of **mbC** in which:
(i) $\circ\daleth\daleth\alpha$ holds, for some given classical negation $\daleth$; or
(ii) $\neg(\alpha \to \beta) \vdash (\alpha \wedge \neg\beta)$ holds.
The replacement property (RP) cannot hold in any left-adjunctive paraconsistent extension of *PI* in which:
(iii) $(\alpha \wedge \beta) \dashv\vdash \neg(\neg\alpha \vee \neg\beta)$ holds.
The replacement property (RP) cannot hold in any left-adjunctive paraconsistent logic in which:
(iv) $\neg(\alpha \wedge \neg\alpha)$ holds and $(\alpha \wedge \neg\alpha) \dashv\vdash \neg\neg(\alpha \wedge \neg\alpha)$.

Proof. Assume that (i) holds good. Since $\daleth$ is a classical negation, $\alpha \dashv\vdash \daleth\daleth\alpha$ and then, by (RP), we infer that $\circ\alpha \dashv\vdash \circ\daleth\daleth\alpha$. But $\circ\daleth\daleth\alpha$ is a theorem of the given logic, by hypothesis, then $\circ\alpha$ is a theorem. From (bc1), the logic turns out to be explosive with respect to the original primitive negation $\neg$.
Now, assume that (ii) holds good. Consider the supplementing negation $\sim\alpha = (\alpha \to \bot)$ for **mbC**, where $\bot = (p_0 \wedge (\neg p_0 \wedge \circ p_0))$, proposed in Remark 43. This negation was shown to be classical. Then, $\neg\sim\alpha \vdash (\alpha \wedge \neg\bot)$, by hypothesis, and so $\neg\sim\alpha \vdash \alpha$, using (Ax4). Since $\alpha, \sim\alpha \vdash \beta$ for every α and β, then $\neg\sim\alpha, \sim\alpha \vdash \beta$ for every α and β, that is, the logic is controllably explosive in contact with $\sim p$. In particular, $\neg\sim\sim\alpha, \sim\sim\alpha \vdash \beta$ for every α and β. But $\alpha \dashv\vdash \sim\sim\alpha$ for a classical negation and so, using (RP), we may conclude that $\neg\alpha \dashv\vdash \neg\sim\sim\alpha$ and then $\neg\alpha, \alpha \vdash \beta$ for every β. In other words, the logic will be explosive, not paraconsistent (with respect to the original negation $\neg$).
Assume next that (iii) holds good. Since $(\neg\alpha \vee \neg\neg\alpha)$ is a theorem of *PI*, then $\neg(\neg\alpha \vee \neg\neg\alpha) \dashv\vdash \neg(\neg\beta \vee \neg\neg\beta)$, for every α and β, by (RP). By hypothesis we infer that $(\alpha \wedge \neg\alpha) \dashv\vdash (\beta \wedge \neg\beta)$. So, by the rules of a standard conjunction, we conclude in particular that $\alpha, \neg\alpha \vdash \beta$.
Finally, assume that (iv) holds good. Since $\neg(\alpha \wedge \neg\alpha)$ is a theorem, by hypothesis, then $\neg\neg(\alpha \wedge \neg\alpha) \dashv\vdash \neg\neg(\beta \wedge \neg\beta)$ for every α and β, by (RP). Then, again by hypothesis, we have that $(\alpha \wedge \neg\alpha) \dashv\vdash (\beta \wedge \neg\beta)$. The result follows now as in item (iii). ■

With the help of Theorem 52(ii) it is easy to see, for instance, that Jaśkowski's **D2** (recall Example 24) fails the replacement property. This feature was used in [Marcos, 2005b] to show that this logic is not 'modal' in the current usual sense of the word, in spite of its very definition in terms of a double translation into the modal logic $S5$.

REMARK 53. To obtain paraconsistent extensions of **mbC** validating both (EC) and (EO) is a perfectly feasible task. Examples of such logics were already offered in Example 34: Notice indeed that axiom (bc1) and rules (EC) and (EO) are all satisfied by the minimal normal modal logic K, thus also by any of its normal modal extensions. ■

3.3 Bivaluation semantics for ***mbC***

At the beginning of their historical trajectory, most **C**-systems were introduced exclusively in proof-theoretical terms (see, for a survey, [Carnielli and Marcos, 2002]). Later on, many of them were proven not to be characterizable by finite-valued truth-tables (such results are generalized here in Theorems 121 and 125). If we add to this the frequent failure of the replacement property and the consequent difficulty in characterizing those same logics by way of usual Kripke-like modal semantics, it will seem clear that semantic presentations for many of our present **C**-systems will have to rely upon some alternative kinds of semantics.

There are of course many examples of paraconsistent logics with adequate *finite-valued semantics*. Several 3-valued samples of such logics were already mentioned above in Examples 17, 18 and 19), and many more will be presented below in Section 5.3. Additionally, many examples of paraconsistent logics with a *modal semantics* were also mentioned above, in Example 34. However, we have already seen that a logic such as **mbC**, our weakest **LFI** based on classical logic, fails the replacement property. Moreover, as a particular consequence of Theorem 121, **mbC** will also be seen not to be finite-valued. What kind of semantics can we attach to such a logic, thus?

The first examples of adequate *non-truth-functional bivalued semantics* were proposed in [da Costa and Alves, 1977] in order to provide interpretations for some historically distinguished **C**-systems, those in the hierarchy C_n, $1 \leq n < \omega$ (check Definition 28). Such decidable semantics are now known to be a particular case of a more general semantic presentation, called 'dyadic' (check Subsection 3.5 and [Caleiro *et al.*, 2005a]). We will show in the following how a simple characteristic (non-truth-functional) adequate bivaluation semantics may be attached to the logic **mbC**. This example will help in clarifying the connections with other semantic presentations, as well as in devising relevant open problems towards obtaining a theoretical framework for further investigation in the foundations of paraconsistent logic. In the next subsection, we will endow **mbC** with the much richer semantics of possible-translations. This new semantics, as we shall see, not only gives an interpretation to contradictory situations, but it also offers an explanation for the existence of conflicting scenarios.

DEFINITION 54. Let $\mathbf{2} \stackrel{\text{def}}{=} \{0, 1\}$ be the set of truth-values, where 1 denotes the 'true' value and 0 denotes the 'false' value. An **mbC**-valuation is any function $v\colon For^{\circ} \longrightarrow \mathbf{2}$ subject to the following clauses:

$(v1)$ $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$;

$(v2)$ $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$;
$(v3)$ $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$;
$(v4)$ $v(\neg\alpha) = 0$ implies $v(\alpha) = 1$;
$(v5)$ $v(\circ\alpha) = 1$ implies $v(\alpha) = 0$ or $v(\neg\alpha) = 0$. ■

For a collection $\Gamma \cup \{\alpha\}$ of formulas of **mbC**, $\Gamma \vDash_{\mathbf{mbC}} \alpha$ means, as usual (recall Definition 16), that α is assigned the value 1 for every **mbC**-valuation that assigns value 1 to the elements of Γ.

REMARK 55. Given clause $(v5)$ in the above definition of a bivaluation semantics for **mbC**, it is clear that this logic does not admit of a trivial model, that is, that there is no v such that $v(\alpha) = 1$ for every formula α. In particular, given a trivial theory Γ of **mbC**, for every **mbC**-valuation v, then there must be some $\gamma \in \Gamma$ such that $v(\gamma) = 0$ (and thus $v(\neg\gamma) = 1$, by clause $(v4)$). This observation reveals a typical semantical feature of **LFI**s. Indeed, other non-gently explosive paraconsistent logics might well allow for such trivial models. For instance, the logic *Pac* (Example 17), despite being maximal relative to classical logic (cf. [Batens, 1980]), does admit of such a model: Consider indeed $v(\alpha) = \frac{1}{2}$, and recall that $\frac{1}{2}$ is a designated value. ■

The soundness proof for **mbC** with respect to **mbC**-valuations is immediate:

THEOREM 56. [Soundness] Let $\Gamma \cup \{\alpha\}$ be a set of formulas in For°. Then: $\Gamma \vdash_{\mathbf{mbC}} \alpha$ implies $\Gamma \vDash_{\mathbf{mbC}} \alpha$.

Proof. Just check that all axioms of **mbC** assume only the value 1 in any **mbC**-valuation, and that (MP) preserves validity. ■

In order to prove completeness it is convenient to prove first some auxiliary lemmas. Let $\Delta \cup \{\alpha\}$ be a set of formulas in For°. We say that a theory Δ is *relatively maximal with respect to* α *in* **mbC** if $\Delta \nvdash_{\mathbf{mbC}} \alpha$ and for any formula β in For° such that $\beta \notin \Delta$ we have $\Delta, \beta \vdash_{\mathbf{mbC}} \alpha$. The usual Lindenbaum-Asser argument (cf. [Béziau, 1999]) shows that inside any compact S-logic — such as **mbC** — every non-trivial theory may be extended into a relatively maximal theory:

LEMMA 57. Let $\mathbf{L}$ be a compact S-logic over a signature $\widehat{\Sigma}$. Given some set of formulas Γ and a formula α such that $\Gamma \nVdash_{\mathbf{L}} \alpha$, then there is a set $\Delta \supseteq \Gamma$ that is relatively maximal with respect to α in $\mathbf{L}$.

Proof. Consider an enumeration $\{\varphi_n\}_{n\in\mathbb{N}}$ of the formulas in $For_{\mathbf{L}}$, and a chain Δ_n, $n \in \mathbb{N}$, of theories built as follows:

$$\Delta_0 = \Gamma$$
$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\}, & \text{if } \Delta_n, \varphi_n \nVdash_{\mathbf{L}} \alpha \\ \Delta_n, & \text{otherwise} \end{cases}$$

Let $\Delta = \bigcup_{n\in\mathbb{N}} \Delta_n$. We will show that Δ is relatively maximal with respect to α in **L**. First of all, notice that, by an easy induction over the above chain, we can conclude that $\Delta_n \not\Vdash_{\mathbf{L}} \alpha$, for every $n \in \mathbb{N}$. Moreover, $\Delta \not\Vdash_{\mathbf{L}} \alpha$. Indeed, if that was not the case, by compactness there would be some finite $\Delta^{\mathsf{fin}} \subseteq \Delta$ such that $\Delta^{\mathsf{fin}} \Vdash_{\mathbf{L}} \alpha$. But then, using cut, there would be some $\Delta_m \supseteq \Delta^{\mathsf{fin}}$ such that $\Delta_m \Vdash_{\mathbf{L}} \alpha$, and that is impossible. Now, consider some $\beta \notin \Delta$. That β must be such that $\beta = \varphi_n$, for some n. Thus $\beta \notin \Delta_{n+1}$, given reflexivity and $\Delta_{n+1} \subseteq \Delta$. So, $\Delta_{n+1} = \Delta_n$ and $\Delta_n, \beta \Vdash_{\mathbf{L}} \alpha$, by construction. Once $\Delta_n \subseteq \Delta$, we are bound to conclude by monotonicity that $\Delta, \beta \Vdash_{\mathbf{L}} \alpha$. ■

We can also prove that:

LEMMA 58. Any relatively maximal set of formulas is a closed theory.

Proof. Given a set of formulas Δ that is relatively maximal with respect to a formula α, we have to check that $\Delta \vdash_{\mathbf{mbC}} \beta$ iff $\beta \in \Delta$. From right to left is obvious by reflexivity. From left to right, given some $\beta \notin \Delta$ we have that (a) $\Delta \nvdash_{\mathbf{mbC}} \alpha$ and (b) $\Delta, \beta \vdash_{\mathbf{mbC}} \alpha$, since Δ is relatively maximal with respect to α. But then, from (a) and (b) we conclude, using cut, that $\Delta \nvdash_{\mathbf{mbC}} \beta$. ■

LEMMA 59. Let $\Delta \cup \{\alpha\}$ be a set of formulas in For° such that Δ is relatively maximal with respect to α in **mbC**. Then:

(i) $(\beta \wedge \gamma) \in \Delta$ iff $\beta \in \Delta$ and $\gamma \in \Delta$.

(ii) $(\beta \vee \gamma) \in \Delta$ iff $\beta \in \Delta$ or $\gamma \in \Delta$.

(iii) $(\beta \to \gamma) \in \Delta$ iff $\beta \notin \Delta$ or $\gamma \in \Delta$.

(iv) $\beta \notin \Delta$ implies $\neg\beta \in \Delta$.

(v) $\circ\beta \in \Delta$ implies $\beta \notin \Delta$ or $\neg\beta \notin \Delta$.

Proof. The closure guaranteed by Lemma 58 will be used to prove each of the above items.
Item (i) is proven from closure, axioms (Ax3), (Ax4), (Ax5) and (MP).
Item (ii) follows from closure, axioms (Ax6), (Ax7), (Ax8) and (MP).
Item (iii) from closure, (ii), axioms (Ax1), (Ax9) and (MP).
Item (iv) from closure, axiom (Ax10) and (MP).
For item (v), suppose $\beta \in \Delta$ and $\neg\beta \in \Delta$. Then, from closure, (bc1) and relative maximality, we conclude that $\circ\beta \notin \Delta$. ■

COROLLARY 60. The characteristic function of a relatively maximal set of formulas in **mbC** defines an **mbC**-valuation.

Proof. Let Δ be a set of formulas relatively maximal with respect to α and define a function $v\colon For^{\circ} \longrightarrow \mathbf{2}$ such that, for any formula β in For°, $v(\beta) = 1$ iff $\beta \in \Delta$. Using the previous lemma it is easy to see that v satisfies clauses $(v1)$ to $(v5)$ of Definition 54. ■

THEOREM 61. [Completeness] Let $\Gamma \cup \{\alpha\}$ be a set of formulas in For°. Then: $\Gamma \vDash_{\mathbf{mbC}} \alpha$ implies $\Gamma \vdash_{\mathbf{mbC}} \alpha$.

Proof. Given a formula α in For° such that $\Gamma \nvdash_{\mathbf{mbC}} \alpha$ one may, by the Lindenbaum-Asser argument, extend Γ to a set Δ that is relatively maximal with respect to α. As $\Delta \nvdash_{\mathbf{mbC}} \alpha$, then $\alpha \notin \Delta$, because of (Con1). By Corollary 60, the characteristic function v of Δ is an **mbC**-valuation such that, for any $\beta \in \Delta$, $v(\beta) = 1$, while $v(\alpha) = 0$. So, $\Delta \nvDash_{\mathbf{mbC}} \alpha$, and in particular $\Gamma \nvDash_{\mathbf{mbC}} \alpha$. ■

Using the bivaluation semantics for **mbC**, we obtain easy semantical proofs of several remarkable features of **mbC** (see Theorem 64 below). Previous to do this, we need to show how it is possible to construct an **mbC**-valuation satisfying a given set of requirements.

DEFINITION 62. Let the mapping $\ell\colon For^{\circ} \longrightarrow \mathbb{N}$ denote the *complexity measure* defined over the signature Σ°, by: $\ell(p) = 0$, for $p \in \mathcal{P}$; $\ell(\varphi \# \psi) = \ell(\varphi) + \ell(\psi) + 1$, for $\# \in \{\wedge, \vee, \rightarrow\}$; $\ell(\neg\varphi) = \ell(\varphi) + 1$; and $\ell(\circ\varphi) = \ell(\varphi) + 2$. ■

LEMMA 63. Let $v_0\colon \mathcal{P} \cup \{\neg p : p \in \mathcal{P}\} \longrightarrow \mathbf{2}$ be a mapping such that $v_0(\neg p) = 1$ whenever $v_0(p) = 0$ (for $p \in \mathcal{P}$). Then, there exists an **mbC**-valuation $v\colon For^{\circ} \longrightarrow \mathbf{2}$ extending v_0, that is, such that $v(\varphi) = v_0(\varphi)$ for every $\varphi \in \mathcal{P} \cup \{\neg p : p \in \mathcal{P}\}$.

Proof. We will define the value of $v(\varphi)$ while doing an induction on the complexity $\ell(\varphi)$ of a formula $\varphi \in For^{\circ}$. Thus, we begin by setting $v(\varphi) = v_0(\varphi)$ for every $\varphi \in \mathcal{P} \cup \{\neg p : p \in \mathcal{P}\}$, and $v(p \# q)$ is defined according to clauses $(v1)$–$(v3)$ of Definition 54, for $\# \in \{\wedge, \vee, \rightarrow\}$ and $p, q \in \mathcal{P}$. This completes the definition of $v(\varphi)$ for every $\varphi \in For^{\circ}$ such that $\ell(\varphi) \leq 1$. Suppose now that $v(\varphi)$ has been defined for every $\varphi \in For^{\circ}$ such that $\ell(\varphi) \leq n$ (for $n \geq 1$) and let $\varphi \in For^{\circ}$ such that $\ell(\varphi) = n + 1$. If $\varphi = (\psi_1 \# \psi_2)$ for $\# \in \{\wedge, \vee, \rightarrow\}$ then $v(\varphi)$ is defined according to $(v1)$–$(v3)$. If $\varphi = \neg\psi$ then we define $v(\varphi) = 1$, if $v(\psi) = 0$, and $v(\varphi)$ is defined arbitrarily, otherwise. Finally, if $\varphi = \circ\psi$ then $v(\varphi) = 0$, if $v(\psi) = v(\neg\psi) = 1$, and $v(\varphi)$ is defined arbitrarily otherwise. It is clear that v is an **mbC**-valuation that extends the mapping v_0. ■

THEOREM 64. The connectives $\wedge, \vee$ and $\rightarrow$ are not interdefinable as in the classical case. Indeed, the following rule holds good in **mbC**:

(i) $(\neg\alpha \rightarrow \beta) \Vdash (\alpha \vee \beta)$,

but none of the following rules hold in **mbC**:

(ii) $(\alpha \vee \beta) \Vdash (\neg\alpha \rightarrow \beta)$;
(iii) $\neg(\neg\alpha \rightarrow \beta) \Vdash \neg(\alpha \vee \beta)$;
(iv) $\neg(\alpha \vee \beta) \Vdash \neg(\neg\alpha \rightarrow \beta)$;

(v) $(\alpha \to \beta) \Vdash \neg(\alpha \wedge \neg\beta)$;
(vi) $\neg(\alpha \wedge \neg\beta) \Vdash (\alpha \to \beta)$;
(vii) $\neg(\alpha \to \beta) \Vdash (\alpha \wedge \neg\beta)$;
(viii) $(\alpha \wedge \neg\beta) \Vdash \neg(\alpha \to \beta)$;
(ix) $\neg(\neg\alpha \wedge \neg\beta) \Vdash (\alpha \vee \beta)$;
(x) $(\alpha \vee \beta) \Vdash \neg(\neg\alpha \wedge \neg\beta)$;
(xi) $\neg(\neg\alpha \vee \neg\beta) \Vdash (\alpha \wedge \beta)$;
(xii) $(\alpha \wedge \beta) \Vdash \neg(\neg\alpha \vee \neg\beta)$.

Proof. (i) Let v be an **mbC**-valuation such that $v(\alpha \vee \beta) = 0$. Then $v(\alpha) = 0 = v(\beta)$ and so $v(\neg\alpha) = 1$. Therefore $v(\neg\alpha) = 1$ and $v(\beta) = 0$, that is, $v(\neg\alpha \to \beta) = 0$. This shows that $(\neg\alpha \to \beta) \vDash_{\mathbf{mbC}} (\alpha \vee \beta)$. The result for $\vdash_{\mathbf{mbC}}$ follows from Theorem 61.
(ii) Consider a mapping $v_0\colon \mathcal{P} \cup \{\neg p : p \in \mathcal{P}\} \longrightarrow \mathbf{2}$ such that $v_0(p_0) = 1 = v_0(\neg p_0)$, $v_0(p_1) = 0$ and $v_0(\varphi)$ is defined arbitrarily otherwise. Let v be an **mbC**-valuation extending v_0 (check the Lemma 63). Then $v(p_0 \vee p_1) = 1$ but $v(\neg p_0 \to p_1) = 0$. This shows that $(p_0 \vee p_1) \nvDash_{\mathbf{mbC}} (\neg p_0 \to p_1)$. The result for $\vdash_{\mathbf{mbC}}$ follows from Theorem 56.
The remainder of the proof is analogous. ∎

EXAMPLE 65. The first **LFI** ever to receive an interpretation in terms of bivaluation semantics was the logic C_1 of Example 28 (cf. [da Costa and Alves, 1977]). The original set of clauses characterizing the C_1-valuations is the following:

$(vC1)$ $v(\alpha_1 \wedge \alpha_2) = 1$ iff $v(\alpha_1) = 1$ and $v(\alpha_2) = 1$;

$(vC2)$ $v(\alpha_1 \vee \alpha_2) = 1$ iff $v(\alpha_1) = 1$ or $v(\alpha_2) = 1$;

$(vC3)$ $v(\alpha_1 \to \alpha_2) = 1$ iff $v(\alpha_1) = 0$ or $v(\alpha_2) = 1$;

$(vC4)$ $v(\neg\alpha) = 0$ implies $v(\alpha) = 1$;

$(vC5)$ $v(\neg\neg\alpha) = 1$ implies $v(\alpha) = 1$;

$(vC6)$ $v(\circ\beta) = v(\alpha \to \beta) = v(\alpha \to \neg\beta) = 1$ implies $v(\alpha) = 0$;

$(vC7)$ $v(\circ(\alpha \# \beta)) = 0$ implies $v(\circ\alpha) = 0$ or $v(\circ\beta) = 0$, for $\# \in \{\wedge, \vee, \to\}$,

where, as usual, $\circ\alpha$ abbreviates the formula $\neg(\alpha \wedge \neg\alpha)$. ∎

3.4 Possible-translations semantics for ***LFI****s*

Notwithstanding the fact that the completeness proof by means of bivaluations for **LFI**s is simple to obtain, this semantics does not do a good job in explaining intrinsic singularities of such logics. In particular, it is not obvious right from the definition of the bivaluation semantics for **mbC** (Definition 54) that this logic is *decidable*. A decision procedure can be obtained with some further effort, however, by adapting the well-known procedure of truth-tables, or 'matrices', into a procedure of 'quasi matrices' (check for instance [da Costa and Alves, 1977] and [da Costa *et al.*, 1995]). At any rate, bivaluation semantics may be very useful as a technical device that helps in simplifying the completeness proof with respect to possible-translations semantics that we present in this subsection, as well as in defining two-signed tableaux for our logics, as it will be illustrated in the next section. Possible-translations semantics were introduced in [Carnielli, 1990]; for a study of their scope and for formal definitions related to them check [Marcos, 2004]. Of course, the notion of *translation* between a logic **L**1 and a logic **L**2 is essential here (recall Definition 31).

Consider now the following 3-valued truth-tables, where T and t are the designated values:

$\wedge$	T	t	F
T	t	t	F
t	t	t	F
F	F	F	F

$\vee$	T	t	F
T	t	t	t
t	t	t	t
F	t	t	F

$\to$	T	t	F
T	t	t	F
t	t	t	F
F	t	t	t

	$\neg_1$	$\neg_2$	$\circ_1$	$\circ_2$
T	F	F	t	F
t	F	t	F	F
F	T	t	t	F

In order to provide interpretations to the connectives of **mbC** by means of possible-translations semantics one should first understand these truth-tables. The truth-value t may be interpreted as 'true by default', or 'true by lack of evidence to the contrary', and T and F are, as usual, 'true' and 'false'. The truth-tables for conjunction, disjunction and implication never return the value T, so, in principle, one is never absolutely sure about the truth-status of some compound sentences. There are two distinct interpretations for negation, $\neg$, and for the consistency operator, $\circ$. The basic intuition is the idea of *multiple scenarios* concerning the dynamics of evaluation of propositions: One may think that there are two kinds of situations concerning non-true propositions with respect to successive moments of time. In the first situation, a true-by-default proposition is treated as a true proposition with respect to the negation $\neg_1$. In the other situation, one can consider the case in which the negation of any other value than 'true' becomes true-by-default — this is expressed by the negation $\neg_2$. On what concerns the consistency operator $\circ$, the first interpretation $\circ_1$ only considers as true-by-default the 'classical' values T and F, while $\circ_2$ assigns falsehood to every truth-value.

The above collection of truth-tables, which we call $\mathcal{M}_0$, will be used to provide the desired semantics for **mbC**. Now, considering the algebra

$For_{\mathcal{M}_0}$ of formulas generated by $\mathcal{P}$ over the signature of $\mathcal{M}_0$, let's define the set $\textsc{Tr}_0$ of all mappings $*$: $For^\circ \longrightarrow For_{\mathcal{M}_0}$ subjected to the following restrictive clauses:

$(tr0)$ $p^* = p$, if $p \in \mathcal{P}$;

$(tr1)$ $(\alpha\#\beta)^* = (\alpha^*\#\beta^*)$, for all $\# \in \{\wedge, \vee, \rightarrow\}$;

$(tr2)$ $(\neg\alpha)^* \in \{\neg_1\alpha^*, \neg_2\alpha^*\}$;

$(tr3)$ $(\circ\alpha)^* \in \{\circ_1\alpha^*, \circ_2\alpha^*, \circ_1(\neg\alpha)^*\}$.

We say the pair $\mathrm{PT}_0 = \langle \mathcal{M}_0, \textsc{Tr}_0 \rangle$ is a *possible-translations semantical structure for* **mbC**. If $\vDash_{\mathcal{M}_0}$ denotes the consequence relation in $\mathcal{M}_0$, and $\Gamma \cup \{\alpha\}$ is a set of formulas of **mbC**, the associated PT-consequence relation, $\models_{\mathrm{PT}_0}$, is defined by setting:

$$\Gamma \models_{\mathrm{PT}_0} \alpha \;\text{ iff }\; \Gamma^* \vDash_{\mathcal{M}_0} \alpha^* \;\text{ for all translations } * \text{ in } \textsc{Tr}_0.$$

We will call *possible translation* of a formula α any image of it through some mapping in $\textsc{Tr}_0$. One can immediately check the following:

THEOREM 66. [Soundness] Let $\Gamma \cup \{\alpha\}$ be a set of formulas of **mbC**. Then $\Gamma \vdash_{\mathbf{mbC}} \alpha$ implies $\Gamma \models_{\mathrm{PT}_0} \alpha$.

Proof. It is sufficient to check that the (finite) collection of all possible translations of each axiom produces tautologies in the truth-tables of $\mathcal{M}_0$ and that all possible translations of the rule (MP) preserve validity. The verification is immediate, and we leave it as exercise to the reader. ■

As a corollary of the above result, we see that each mapping in $\textsc{Tr}_0$ defines in fact a translation (recall Definition 31) from **mbC** to the logic defined by $\mathcal{M}_0$.

In order to prove completeness, now, our strategy will be to show that each **mbC**-valuation v determines a translation $*$ and a 3-valued valuation w defined in the usual way over the truth-tables of $\mathcal{M}_0$ such that, for every formula α of **mbC**,

$$w(\alpha^*) \in \{T, t\} \;\text{ iff }\; v(\alpha) = 1$$

and thus rely on the completeness proof for the bivaluation semantics of **mbC**.

Recall the definition of complexity $\ell(\alpha)$ of a formula $\alpha \in For^\circ$ introduced in Definition 62. The following result comes from [Marcos, 2005f]:

THEOREM 67. [Representability] Given an **mbC**-valuation v there is a translation $*$ in $\textsc{Tr}_0$ and a valuation w in $\mathcal{M}_0$ such that, for every formula α in **mbC**:

$$w(\alpha^*) = T \;\text{ implies }\; v(\neg\alpha) = 0; \;\text{ and}$$

$$w(\alpha^*) = F \;\text{ iff }\; v(\alpha) = 0.$$

Proof. For $p \in \mathcal{P}$ define the valuation w as follows:

$$\begin{array}{lll} w(p) = F & \text{if} & v(p) = 0; \\ w(p) = T & \text{if} & v(p) = 1 \text{ and } v(\neg p) = 0; \\ w(p) = t & \text{if} & v(p) = 1 \text{ and } v(\neg p) = 1. \end{array}$$

Such w can be homomorphically extended to the algebra $For_{\mathcal{M}_0}$. We define the translation mapping $*$ as follows:

1. $p^* = p$, if $p \in \mathcal{P}$;
2. $(\alpha \# \beta)^* = (\alpha^* \# \beta^*)$, for $\# \in \{\wedge, \vee, \rightarrow\}$;
3. $(\neg\alpha)^* = \neg_1 \alpha^*$, if $v(\neg\alpha) = 0$ or $v(\alpha) = v(\neg\neg\alpha) = 0$;
4. $(\neg\alpha)^* = \neg_2 \alpha^*$, otherwise;
5. $(\circ\alpha)^* = \circ_2 \alpha^*$, if $v(\circ\alpha) = 0$;
6. $(\circ\alpha)^* = \circ_1 (\neg\alpha)^*$, if $v(\circ\alpha) = 1$ and $v(\neg\alpha) = 0$;
7. $(\circ\alpha)^* = \circ_1 \alpha^*$, otherwise.

Note that the mapping $*$ is well-defined, given the definition of **mbC** (see Definition 54). The proof is now done by induction on the complexity measure $\ell(\alpha)$ of a formula α. Details are left to the reader. ■

THEOREM 68. [Completeness] Let $\Gamma \cup \{\alpha\}$ be a set of formulas in **mbC**. Then $\Gamma \models_{\mathrm{PT}_0} \alpha$ implies $\Gamma \vdash_{\mathbf{mbC}} \alpha$.

Proof. Suppose that $\Gamma \models_{\mathrm{PT}_0} \alpha$, and suppose that v is an **mbC**-valuation such that $v(\Gamma) \subseteq \{1\}$. By Theorem 67, there is a translation $*$ and a 3-valued valuation w such that, for every formula β, $w(\beta^*) \in \{T, t\}$ iff $v(\beta) = 1$. From this, $w(\Gamma^*) \subseteq \{T, t\}$ and so $w(\alpha^*) \in \{T, t\}$, because $\Gamma \models_{\mathrm{PT}_0} \alpha$. Then $v(\alpha) = 1$. To wit: For every **mbC**-valuation v, $v(\Gamma) \subseteq \{1\}$ implies $v(\alpha) = 1$. Using the completeness of **mbC** with respect to **mbC**-valuations we obtain that $\Gamma \vdash_{\mathbf{mbC}} \alpha$ as desired. ■

It is now easy to check validity for inferences in **mbC**, as shown in the following example.

EXAMPLE 69. We will prove that $\circ\alpha \vdash_{\mathbf{mbC}} \neg(\neg\alpha \wedge \alpha)$ using possible-translations semantics. We have that, for any translation $*$ in $\mathrm{T}\text{R}_0$,

$$(\circ\alpha)^* \in \{\circ_1(\alpha^*), \circ_2(\alpha^*), \circ_1\neg_1(\alpha^*), \circ_1\neg_2(\alpha^*)\},$$

$$(\neg(\neg\alpha \wedge \alpha))^* \in \{\neg_i(\neg_j(\alpha^*) \wedge \alpha^*) : i, j \in \{1, 2\}\}.$$

Let $*$ be a translation in $\mathrm{T}\text{R}_0$, w be a valuation in $\mathcal{M}_0$, and $D = \{T, t\}$. Let $x = w(\alpha^*)$, $y = w((\circ\alpha)^*)$ and $z = w((\neg(\neg\alpha \wedge \alpha))^*)$, and suppose that $y \in D$; this rules out the translation $(\circ\alpha)^* = \circ_2(\alpha^*)$ because $\circ_2(x) \notin D$. In order to prove that $z \in D$ we have the following cases:

1. $(\circ\alpha)^* = \circ_1(\alpha^*)$. Then $\circ_1(x) \in D$, thus $x \in \{T, F\}$.

(a) $x = T$. Then $\neg_j(x) = F$ ($j \in \{1, 2\}$) and so $\neg_i(\neg_j(x) \wedge x) \in D$ for $i, j \in \{1, 2\}$.

(b) $x = F$. Then $(\neg_j(x) \wedge x) = F$ ($j \in \{1, 2\}$) and so $\neg_i(\neg_j(x) \wedge x) \in D$ for $i, j \in \{1, 2\}$.

2. $(\circ\alpha)^* = \circ_1\neg_1(\alpha^*)$. Then $\circ_1\neg_1(x) \in D$, thus $\neg_1(x) \in \{T, F\}$ and $z = \neg_i(\neg_1(x) \wedge x)$.

   (a) $\neg_1(x) = T$. Then $x = F$ and the proof is as in (1b).

   (b) $\neg_1(x) = F$. In this case the proof is as in (1a).

3. $(\circ\alpha)^* = \circ_1\neg_2(\alpha^*)$. Then, given $\circ_1\neg_2(x) \in D$, we have $\neg_2(x) \in \{T, F\}$ and $z = \neg_i(\neg_2(x) \wedge x)$. From the truth-table for $\neg_2$ we obtain that $\neg_2(x) = F$, and the proof is as in (1a).

This proves the desired result. On the other hand, we may prove that the converse $\neg(\neg\alpha \wedge \alpha) \vdash_{\mathbf{mbC}} \circ\alpha$ is not true in **mbC**, as announced in Theorem 49. Using the same notation as above for a given translation $*$ in $\text{T}\textsc{r}_0$ and a valuation w in $\mathcal{M}_0$, it is enough to consider α as a propositional variable p, and choose $*$ and w such that $x = F$, and $(\circ\alpha)^* = \circ_2(\alpha^*)$. Then $z \in D$ and $y = F$. For yet some other counter-models to that inference, take $x = t$, $(\neg(\neg\alpha \wedge \alpha))^* = \neg_2(\neg_2(\alpha^*) \wedge \alpha^*)$ and $(\circ\alpha)^* \in \{\circ_1(\alpha)^*, \circ_1(\neg_2\alpha)^*\}$. ■

Possible-translations semantics offer an immediate decision procedure for any logic **L** that is complete with respect to a possible-translations semantical structure $\text{PT} = \langle \mathcal{M}, \text{T}\textsc{r} \rangle$ where $\mathcal{M}$ is decidable (and this is the case here, where $\mathcal{M}$ is a finite-valued logic) and $\text{T}\textsc{r}$ is recursive. Indeed, given a formula α, if we wish to decide whether it is a theorem of **L**, it is sufficient to consider the (in this case finitely many) possible translations of α, and to check each translated formula using the corresponding semantics of the target logics (in the present case, defined by sets of 3-valued truth-tables). Questions on the complexity of such decision procedures could be readily answered by taking into account the complexity of translations and of the semantics of the target logics. This is a problem of independent interest, since it is immediate to see that the decision procedure of **mbC** is NP-complete, as one might expect: Indeed, there exists a polynomial-time conservative translation from **CPL** into **mbC**, as illustrated in Theorem 74 below.

One can also use possible-translations semantics to help proving important properties about the logics in question.

REMARK 70. Recall from Remark 43 the two explosive negations represented by $\wr\alpha \stackrel{\text{def}}{=} (\neg\alpha \wedge \circ\alpha)$ and $\sim\!\alpha \stackrel{\text{def}}{=} \alpha \to (p_0 \wedge (\neg p_0 \wedge \circ p_0))$. Recall again, also, the notion of a classical negation from Definition 8. Now, while it is easy to check that $\sim$ defines a classical negation in **mbC** (the reader can, as an exercise, check that both $(\alpha \vee \sim\!\alpha)$ and $(\alpha \to (\sim\!\alpha \to \beta))$ are provable / validated by **mbC**), it is also straightforward to check that $\wr$ is not a complementing negation. Indeed, to see that α and $\wr\alpha$ can be simultaneously false, take some bottom particle $\bot = p \wedge \wr p$ and notice that $w(\bot^*) = F$, for any valuation w in $\mathcal{M}_0$ and any translation $*$ in Tr_0. Consider now some translation such that $(\circ p) = \circ_2 p$. In that case, $w((\wr \bot)^*) = F$, for any w. Then, while $\bot \models_{\text{PT}_0} \wr \bot$ certainly holds good, it is not the case that $\models_{\text{PT}_0} \wr \bot$. Notice moreover that, while $\wr \alpha \models_{\text{PT}_0} {\sim}\alpha$, we have that ${\sim}\alpha \not\models_{\text{PT}_0} \wr \alpha$. ■

We trust the above features to confirm the importance of possible-translations semantics as a philosophically apt and computationally useful semantical tool for treating not only Logics of Formal Inconsistency but also many other logics in the literature. An remarkable particular case of possible-translations semantics is the so-called *non-deterministic semantics* (cf. [Avron and Lev, 2005]), proposed as an immediate generalization of the notion of a truth-functional semantics (for comparisons between possible-translations semantics and non-deterministic semantics see [Carnielli and Coniglio, 2005]). A 3-valued non-deterministic semantics for the logic **mbC** may be found in [Avron, 2005a] (where this logic is called **B**).

*3.5 Tableau proof systems for **LFIs***

In this section we will use a very general method to obtain adequate tableau systems for **mbC** and for C_1. The method introduced in [Caleiro *et al.*, 2005b] (check also [Caleiro *et al.*, 2005a]) permits one to obtain an adequate tableau system for any propositional logic which has an adequate semantics given through the so-called 'dyadic valuations'. Such bivaluations have, as usual, values in $\mathbf{2} = \{0, 1\}$ (or, equivalently, in $\{T, F\}$), and are axiomatized by first-order clauses of a certain specific form, explained below.

Briefly, suppose that there is a set of clauses governing a class of bivaluation mappings v: $For \longrightarrow \mathbf{2}$ of the form

$$(v(\varphi_1) = Q_1, \ldots, v(\varphi_n) = Q_n) \Rrightarrow (S_1 | \cdots | S_k)$$

where $n \geq 0$ and $k \geq 0$ and, for every $1 \leq i \leq k$,

$$S_i = (v(\varphi_1^i) = Q_1^i, \ldots, v(\varphi_{r_i}^i) = Q_{r_i}^i),$$

with $Q_i, Q_j^i \in \{T, F\}$ $(1 \leq j \leq r_i)$ and $r_i \geq 1$. If $n = 0$ then $(v(\varphi_1) = Q_1, \ldots, v(\varphi_n) = Q_n)$ is just $\top$; if $k = 0$ then $(S_1 | \cdots | S_k)$ is $\bot$. Commas ',' and bars '|' denote conjunctions and disjunctions, respectively, and '$\Rrightarrow$' denotes implication. Examples of axioms for bivaluations that may be put in this format are provided by the clauses that characterize **mbC**-valuations (cf. Definition 54) and also by those provided by the characteristic bivaluation semantics of da Costa's C_1 (cf. Example 65). For instance, clause $(v5)$ of Definition 54 clearly has the required form:

$(v5)$ $v(\circ\alpha) = T \Rrightarrow (v(\alpha) = F \mid v(\neg\alpha) = F)$

whereas clause $(v3)$ may be split into three clauses of the required form:

$(v3.1)$ $v(\alpha \to \beta) = T \Rrightarrow (v(\alpha) = F \mid v(\beta) = T)$;
$(v3.2)$ $v(\alpha) = F \Rrightarrow v(\alpha \to \beta) = T$;
$(v3.3)$ $v(\beta) = T \Rrightarrow v(\alpha \to \beta) = T$.

It will be convenient in what follows to keep the more complex formulas on the left-hand side of the implication; we thus substitute $(v3.2)$ and $(v3.3)$ by:

$(v3.4)$ $v(\alpha \to \beta) = F \Rrightarrow (v(\alpha) = T, v(\beta) = F)$

The next step in the algorithm described in [Caleiro *et al.*, 2005b] is to 'translate' every clause of the dyadic semantics into a tableau rule by interpreting an equation '$v(\varphi) = Q$' as a signed formula $Q(\varphi)$ (recalling that $Q \in \{T, F\}$). Thus, a clause as above is transformed in a (two-signed) tableau rule of the form:

$$
\begin{array}{ccc}
 & Q_1(\varphi_1), \ldots, Q_n(\varphi_n) & \\
\swarrow & \cdots & \searrow \\
Q_1^1(\varphi_1^1) & & Q_1^k(\varphi_1^k) \\
\vdots & & \vdots \\
Q_{r_1}^1(\varphi_{r_1}^1) & & Q_{r_k}^k(\varphi_{r_k}^k)
\end{array}
$$

By transforming each clause of the dyadic semantic valuation into a tableau rule, we obtain a tableau system for the given logic. In order to ensure completeness of the tableau system, it is necessary to consider two extra axioms for the bivaluation semantics:

(DV1) $(v(\varphi) = T,\ v(\varphi) = F) \Rrightarrow \bot$;
(DV2) $\top \Rrightarrow (v(\varphi) = T \mid v(\varphi) = F)$.

Axioms (DV1) and (DV2) guarantee that the mapping respecting them is a bivaluation $v\colon For \longrightarrow \mathbf{2}$. The translation of axiom (DV1) gives us the usual closure condition for a branch in a given tableau. On the other hand, (DV2) gives us the following branching tableau rule, R_b:

$$\frac{}{T(\varphi) \mid F(\varphi)}$$

As a consequence, the resulting tableau system loses the 'analytic' character. Fortunately, in many important cases this branching rule can be eliminated or at least it can have its scope of application restricted to formulas of a certain format.

We apply next the above technique to obtain an adequate tableau system for the logic **mbC**, based on the bivaluation semantics presented in Definition 54.

EXAMPLE 71. We define an adequate tableau system for **mbC** as follows:

$$\frac{F(\neg X)}{T(X)} \qquad \frac{T(\circ X)}{F(X) \mid F(\neg X)} \qquad \frac{}{T(X) \mid F(X)}$$

$$\frac{T(X_1 \wedge X_2)}{T(X_1),\ T(X_2)} \qquad \frac{F(X_1 \wedge X_2)}{F(X_1) \mid F(X_2)}$$

$$\frac{T(X_1 \vee X_2)}{T(X_1) \mid T(X_2)} \qquad \frac{F(X_1 \vee X_2)}{F(X_1),\ F(X_2)}$$

$$\frac{T(X_1 \to X_2)}{F(X_1) \mid T(X_2)} \qquad \frac{F(X_1 \to X_2)}{T(X_1),\ F(X_2)}$$

■

Observe that, except for the branching rule R_b, all other rules are analytic in the sense that the consequences are always less complex than the premises (recall that, as in Definition 62, $\ell(\circ\alpha) = \ell(\alpha) + 2$ and $\ell(\neg\alpha) = \ell(\alpha) + 1$), and they contain in each case only subformulas of the premise. The results proven in [Caleiro *et al.*, 2005b] guarantee that the tableau system defined above is sound and complete for **mbC**.

Another nice application of the techniques described above is the definition of a tableau system for the historical **dC**-system C_1 (see Definition 28).

EXAMPLE 72. Recall from Example 65 the characteristic bivaluation semantics for the logic C_1. Those clauses of course may be formally rewritten as axioms of a dyadic semantics, using '|', '⇛' and ','. Using the above described method it is immediate to define a complete tableau system associated to these axioms. Consider indeed all the rules of the tableau system for **mbC** in Example 71 — except for the rule concerning ∘, since it does not correspond to any axiom of a C_1-valuation — and add the following further rules:

$$\frac{T(\neg\neg X)}{T(X)} \qquad \frac{F(\circ(X_1 \# X_2))}{F(\circ X_1) \mid F(\circ X_2)} \qquad \frac{T(\circ X_2),\ T(X_1 \to X_2),\ T(X_1 \to \neg X_2)}{F(X_1)}$$

where $\# \in \{\wedge, \vee, \to\}$ and $\circ X$ abbreviates the formula $\neg(X \wedge \neg X)$. Comparing this tableau system with the one defined in [Carnielli and Lima-Marques, 1992], we notice that the present system does not allow for loops. Although the looping rules proposed in the latter paper often permit one to obtain somewhat conciser tableau proofs, what we have here is a generic method that *automatically generates* a complete set of tableau rules (though not necessarily the most convenient one). ■

It is worth reinforcing that the branching rule R_b is essential, above, in order to obtain completeness. This rule is not strictly analytic, but it can be bounded in certain cases so as to guarantee the termination of the decidable tableau procedure. In particular, the variables occurring in the formula X must be contained in the finite collection of variables in the tableau branch.

EXAMPLE 73. Consider the tableau system for C_1 given in Example 72 and let γ be the formula $\neg(p \land (\neg p \land \circ p))$, where p is a propositional variable. The formula γ is a thesis of C_1. However, it is easy to see that no C_1-tableau for the set $\{F(\gamma)\}$ can close without using the rule R_b. We show below a closed tableau for $\{F(\gamma)\}$ that uses R_b twice.

$$F(\gamma)$$
$$T(p \land \neg p \land \circ p)$$
$$T(p)$$
$$T(\neg p)$$
$$T(\circ p)$$

$$\begin{array}{ccc} T(p \to p) & & F(p \to p) \\ & & T(p) \\ & & F(p) \end{array}$$

$$\begin{array}{ccc} T(p \to \neg p) & & F(p \to \neg p) \\ F(p) & & T(p) \\ & & F(\neg p) \end{array}$$

■

This example suggests that, in general, it is not possible to eliminate R_b if one wishes to obtain completeness. This holds even in case the tableau system satisfies the subformula property, as in Example 73. In certain cases R_b can be eliminated if we have, for instance, looping rules as in [Carnielli and Lima-Marques, 1992]. For the case of C_1 the tableau system treated in the latter paper uses the looping rule:

$$\frac{T(\neg X)}{F(X) \mid F(\circ X)},$$

while our present formulation has no rule for analyzing $T(\neg X)$.

3.6 Talking about classical logic

When attempting to compare the inferential power of two logics, one often finds difficulties because those logics might not be 'talking about the same thing'. For instance, **mbC** is written in a richer signature than that of

CPL, and so these two logics might seem hard to compare. However, as we have seen in Remark 30, it is possible to linguistically extend **CPL** by the addition of a consistency-like connective. The 'classical truth-tables' for this connective, however, will be such that $\circ(x) = 1$ for every x. Clearly, despite being gently explosive, the resulting logic **eCPL** does not define an **LFI**, given that it is not paraconsistent. It is, indeed, a *consistent* logic (recall Definition 4). Now, **mbC** may be characterized as a deductive fragment of **eCPL**, because all axioms of **mbC** are validated by the truth-tables of **eCPL**. Since **mbC** is a fragment thus of (an alternative formulation of) classical logic, we can conclude that **mbC** is a non-contradictory and non-trivial logic. On the other hand, however, we will show in this subsection that there are several ways of encoding each inference of **CPL** within **mbC**.

First of all, recall the **DAT**s from Remark 26, the Derivability Adjustment Theorems that described how the **LFI**s could be used to recover consistent reasoning by the addition in each case of a convenient number of consistency assumptions. In particular, in logics such as **mbC**, **C**-systems based on classical logic, it should be clear how classical reasoning may be recovered. For each classical rule that is lost by paraconsistency, such as *reductio* and contraposition in items (ii) and (iii) of Theorem 38, there is an adjusted version of the same rule that is gained, as illustrated in Theorem 48. Indeed, it is now easy to give a semantical proof that:

$$\forall\Gamma\forall\gamma\exists\Delta(\Gamma \Vdash_{\mathbf{eCPL}} \gamma \;\text{ iff }\; \circ(\Delta), \Gamma \Vdash_{\mathbf{mbC}} \gamma).$$

Now, besides the **DAT**s, there might well be other more direct ways of recovering consistent reasoning from inside a given **LFI**. We will in the following show how this can be done through the use of a conservative translation (recall Definition 31), without the addition of further assumptions to the set of premises of a given inference.

Except for negation and for the consistency connective, all other connectives of **mbC** have a classic-like behavior. The key for the next result will be, thus, to make use of the classical negation $\sim$ that can be defined within **mbC** (cf. Remark 70) by setting $\sim\alpha \stackrel{\text{def}}{=} \alpha \to (p_0 \wedge (\neg p_0 \wedge \circ p_0))$, in order to recover all classical inferences.

THEOREM 74. Let For° be the algebra of formulas for the signature Σ° of **mbC**. There is a mapping $t_1\colon For \longrightarrow For^\circ$ that conservatively translates **CPL** inside of **mbC**, that is, for every $\Gamma \cup \{\alpha\} \subseteq For$:

$$\Gamma \vdash_{\mathbf{CPL}} \alpha \;\text{ iff }\; t_1(\Gamma) \vdash_{\mathbf{mbC}} t_1(\alpha).$$

Proof. Define the mapping t_1 recursively as follows:

1. $t_1(p) = p$, for every $p \in \mathcal{P}$;
2. $t_1(\gamma \# \delta) = t_1(\gamma) \# t_1(\delta)$, if $\# \in \{\wedge, \vee, \to\}$;
3. $t_1(\neg\gamma) = \sim t_1(\gamma)$.

Since both **CPL** and **mbC** are compact and have a deductive implication, and considering that t_1 preserves implications, it suffices to prove that:

$$\vdash_{\mathbf{CPL}} \alpha \text{ iff } \vdash_{\mathbf{mbC}} t_1(\alpha)$$

for every $\alpha \in For$.

That $\vdash_{\mathbf{CPL}} \alpha$ implies $\vdash_{\mathbf{mbC}} t_1(\alpha)$ is an easy consequence of the fact that $\sim$ is a classical negation within **mbC** and from the definition of the translation mapping t_1. Let's now check that $\vdash_{\mathbf{mbC}} t_1(\alpha)$ implies $\vdash_{\mathbf{CPL}} \alpha$. Consider the classical truth-tables for the classical connectives, and define $\circ(x) = 1$ for all x. Then $\neg\alpha$ and $\sim\alpha$ take the same value and so $t_1(\alpha)$ and α take the same value in this semantics. Therefore, if $t_1(\alpha)$ is a theorem of **mbC** then $t_1(\alpha)$ is valid for the above truth-tables and so α is valid using classical truth-tables. Thus, α is a theorem of **CPL**, by the completeness of classical logic. ■

In view of the last theorem, and as it was already mentioned, it is clear that **mbC** (originally defined as a *deductive fragment* of **eCPL**) can also be seen as an *extension* of **CPL**, if we employ an appropriate signature which contains two symbols for negation: $\sim$ for the classical one, and provided $\neg$ for the paraconsistent one, provided that the axioms defining $\sim$ in terms of the other connectives are added to the new version of **mbC**.

In what follows, and in stronger logics than **mbC**, we will see yet other ways of recovering classical inferences inside our **LFI**s (check Theorems 96, 112 and 113).

4 A RICHER **LFI**

*4.1 The system **mCi**, and its significance*

In Remark 45 we have mentioned the possibility of defining in **mbC** an inconsistency connective that is dual to its native consistency connective. This could be done by setting $\bullet\alpha \stackrel{\text{def}}{=} \sim\circ\alpha$, where $\sim\alpha \stackrel{\text{def}}{=} \alpha \to (\beta \wedge (\neg\beta \wedge \circ\beta))$ (for an arbitrary β) is a classical negation. Now, how could we enrich **mbC** so as to be able to define the inconsistency connective by using the paraconsistent negation instead of the classical $\sim$, that is, by setting $\bullet\alpha \stackrel{\text{def}}{=} \neg\circ\alpha$? This is exactly what will be done in this subsection by extending **mbC** into the logic **mCi**. In fact, **mCi** will reveal to be a logic that can be presented in terms of either $\circ$ or $\bullet$ as primitive connectives. Moreover, $\bullet\alpha$ and $\neg\circ\alpha$ will be inter-translatable, and the same will happen with $\circ\alpha$ and $\neg\bullet\alpha$, as proven in Theorem 98.

From Theorem 49(i) we know that $\alpha \wedge \neg\alpha \vdash_{\mathbf{mbC}} \neg\circ\alpha$. The converse property (which does not hold in **mbC**) will be the first additional axiom we will add to **mbC** in upgrading this logic. On the other hand, we wish that formulas of the form $\neg\circ\alpha$ 'behave classically', and we wish to obtain in fact a logic that is controllably explosive in contact with formulas of the form $\neg^n\circ\alpha$, where $\neg^0\alpha \stackrel{\text{def}}{=} \alpha$ and $\neg^{n+1}\alpha \stackrel{\text{def}}{=} \neg\neg^n\alpha$. Any formula of the form $\neg^n\circ\alpha$ would thus be assumed to 'behave classically', and $\{\neg^n\circ\alpha, \neg^{n+1}\circ\alpha\}$ would be an explosive theory. This desideratum leads us into considering the following (cf. [Marcos, 2005f]):

DEFINITION 75. The logic **mCi** is obtained from **mbC** (recall Definition 42) by the addition of the following axiom schemas:
(ci) $\neg\circ\alpha \to (\alpha \wedge \neg\alpha)$
$\mathbf{(cc)}_n$ $\circ\neg^n\circ\alpha$ $(n \geq 0)$
To the above axiomatization we add the definition by abbreviation of an inconsistency connective $\bullet$ by setting $\bullet\alpha \stackrel{\text{def}}{=} \neg\circ\alpha$. ■

Notice that $\neg\circ\alpha$ and $(\alpha \wedge \neg\alpha)$ are equivalent in **mCi**. Clearly every set $\{\neg^n\circ\alpha, \neg^{n+1}\circ\alpha\}$ is explosive in **mCi**, in view of (bc1) and $(\text{cc})_n$. This expresses the 'classical behavior' of formulas of the form $\circ\alpha$ (with respect to the paraconsistent negation). In other words, a formula α in general needs the extra assumption $\circ\alpha$ to 'behave classically', but the formula $\circ\alpha$ and its iterated negations will always 'behave classically'. In Theorem 78 below we will see that $\neg\bullet\alpha$ is equivalent to $\circ\alpha$, and in Definition 97, further on, we will introduce a new formulation of **mCi** that introduces $\bullet$ as a primitive connective. Notice in that case how close is the bond that is established here in between inconsistency and contradictoriness by way of the paraconsistent negation.

We can immediately check that the equivalence in **mCi** between $\neg\circ\alpha$ and $(\alpha \wedge \neg\alpha)$ is in fact logically weaker than the identification of $\circ\alpha$ and $\neg(\alpha \wedge \neg\alpha)$ assumed in C_1 (recall also Theorem 49(iii)–(iv)) since the latter formula implies the former, in **mCi**, but the converse is not true.

THEOREM 76. This rule holds good in **mCi**:
(i) $\neg\circ\alpha \vdash_{\mathbf{mCi}} (\alpha \wedge \neg\alpha)$,
but the following rules do not hold:
(ii) $\neg(\alpha \wedge \neg\alpha) \vdash_{\mathbf{mCi}} \circ\alpha$;
(iii) $\neg(\neg\alpha \wedge \alpha) \vdash_{\mathbf{mCi}} \circ\alpha$.

Proof. Item (i) is obvious. In order to prove that (ii) and (iii) do not hold in **mCi**, observe that **mCi** is sound for the truth-tables of **LFI1** (see Examples 17 and 18), where 0 is the only non-designated value. Then it is enough to check that (ii) and (iii) have counter-models in such a truth-functional semantics. ■

It should be clear that, even though in **mCi** there is a formula in the classical language *For* (namely, the formula $(\alpha \wedge \neg\alpha)$) that is equivalent to a formula that expresses inconsistency (the formula $\bullet\alpha$), there is no formula in the classical language that can express consistency in **mCi**. We also have the following:

THEOREM 77. (i) $\neg(\alpha \wedge \neg\alpha)$ and $\neg(\neg\alpha \wedge \alpha)$ are not top particles in **mCi**.
(ii) $\circ\alpha$ and $\neg\circ\alpha$ are not bottom particles.
(iii) The schemas $(\alpha \to \neg\neg\alpha)$ and $(\neg\neg\alpha \to \alpha)$ are not provable in **mCi**.

Proof. Items (i), (ii) and the first part of item (iii) can be checked using again the truth-tables of $\mathbf{P}^1$, enriched with the (definable) truth-table for $\circ$ (Example 19), and using the fact that **mCi** is sound for such a semantics. For the second part of item (iii) one could use for instance the bivaluation semantics of **mCi** (see Example 90). ■

It is straightforward to check the following properties of **mCi**:

THEOREM 78. The following rules hold good in **mCi**:
(i) $\neg\neg\circ\alpha \vdash_{\mathbf{mCi}} \circ\alpha$;
(ii) $\circ\alpha \vdash_{\mathbf{mCi}} \neg\neg\circ\alpha$;
(iii) $\circ\alpha, \neg\circ\alpha \vdash_{\mathbf{mCi}} \beta$;
(iv) $(\Gamma, \beta \vdash_{\mathbf{mCi}} \circ\alpha)$ and $(\Delta, \beta \vdash_{\mathbf{mCi}} \neg\circ\alpha)$ implies $(\Gamma, \Delta \vdash_{\mathbf{mCi}} \neg\beta)$.

Proof. For item (i), from $\neg\neg\circ\alpha$ and $\circ\alpha$ we obviously prove $\circ\alpha$ in **mCi**. On the other hand, from $\neg\neg\circ\alpha$ and $\neg\circ\alpha$ we also prove $\circ\alpha$ in **mCi**, because $\circ\neg\circ\alpha$ and (bc1) are axioms of **mCi**. Using proof-by-cases we conclude that $\neg\neg\circ\alpha \vdash_{\mathbf{mCi}} \circ\alpha$. The other items are proven similarly. Notice in particular how items (i) and (ii) together show that $\neg\bullet\alpha \dashv\vdash_{\mathbf{mCi}} \circ\alpha$ holds good. ■

Item (ii) of Theorem 77 and item (iii) of Theorem 78 guarantee that **mCi** is controllably explosive in contact with $\circ p_0$ (recall Definition 9(iii)). In fact, the following relation between consistency and controllable explosion can be checked:

THEOREM 79. Let **L** be a non-trivial extension of **mCi** such that the implication (involving the axioms of **mCi**) is deductive (recall Definition 6). A schema $\sigma(p_0, \ldots, p_n)$ is provably consistent in **L** if, and only if, **L** is controllably explosive in contact with $\sigma(p_0, \ldots, p_n)$.

Proof. If $\vdash_{\mathbf{L}} \circ\sigma(\alpha_0, \ldots, \alpha_n)$ then, by axiom (bc1),

$$\Gamma, \sigma(\alpha_0, \ldots, \alpha_n), \neg\sigma(\alpha_0, \ldots, \alpha_n) \vdash_{\mathbf{L}} \beta$$

for any choice of Γ and β.

Conversely, assume that $\Gamma, \sigma(\alpha_0, \ldots, \alpha_n), \neg\sigma(\alpha_0, \ldots, \alpha_n) \vdash_{\mathbf{L}} \beta$ for any Γ and β. Since, from (ci), we have that $\neg\circ\sigma(\alpha_0, \ldots, \alpha_n) \vdash_{\mathbf{L}} (\sigma(\alpha_0, \ldots, \alpha_n) \wedge \neg\sigma(\alpha_0, \ldots, \alpha_n))$, then it follows that $\neg\circ\sigma(\alpha_0, \ldots, \alpha_n)$ is a bottom particle. As in the proof of Theorem 46(i) (using here the fact that the original implication of **mCi** is still deductive in **L**), we get $\vdash_{\mathbf{L}} \neg\neg\circ\sigma(\alpha_0, \ldots, \alpha_n)$. By Theorem 78(i), we conclude that $\vdash_{\mathbf{L}} \circ\sigma(\alpha_0, \ldots, \alpha_n)$. ■

Complementing the versions of contraposition mentioned in Theorem 48, we have:

THEOREM 80. Here are some restricted forms of contraposition introduced by **mCi**:
(i) $(\alpha \to \circ\beta) \vdash_{\mathbf{mCi}} (\neg\circ\beta \to \neg\alpha)$;
(ii) $(\alpha \to \neg\circ\beta) \vdash_{\mathbf{mCi}} (\circ\beta \to \neg\alpha)$;
(iii) $(\neg\alpha \to \circ\beta) \vdash_{\mathbf{mCi}} (\neg\circ\beta \to \alpha)$;
(iv) $(\neg\alpha \to \neg\circ\beta) \vdash_{\mathbf{mCi}} (\circ\beta \to \alpha)$.

Proof. Item (i). By axiom $(cc)_0$, $\circ\circ\beta$ is a theorem of **mCi**. The result now follows from Theorem 48(iii). The other items are proven similarly. ■

On the other hand, properties such as $(\circ\alpha \to \beta) \vdash_{\mathbf{mCi}} (\neg\beta \to \neg\circ\alpha)$ do not hold; this can easily be checked after Corollary 93, to be established below. Notice how the above theorem opens yet another way for the internalization of classical inferences, as discussed in Subsection 3.6.

Recall now the replacement property (RP) discussed in Remark 51. We had already proven in Theorem 52 that (RP) cannot hold in certain paraconsistent extensions of **mbC**. On what concerns its possible validity in paraconsistent extensions of **mCi**, we can now prove that:

THEOREM 81.
(i) The replacement property (RP) is not enjoyed by **mCi**.
The replacement property (RP) cannot hold in any paraconsistent extension of **mCi** in which:
(ii) $\neg(\neg\alpha \wedge \neg\beta) \vdash_{\mathbf{mbC}} (\alpha \vee \beta)$ holds; or
(iii) $(\neg\alpha \vee \neg\beta) \vdash_{\mathbf{mbC}} \neg(\alpha \wedge \beta)$ holds.

Proof. (i) Consider again the first set of truth-tables (with the same set of designated values) used in the proof of Theorem 50.
(ii) Consider the supplementing negation $\wr\alpha = (\neg\alpha \wedge \circ\alpha)$ for **mCi** proposed in Remark 43. By Theorem 78 this last formula is equivalent to $(\neg\alpha\wedge\neg\neg\circ\alpha)$. In Theorem 94, this negation will be shown to behave classically inside this logic. But then, $\neg\wr\alpha \vdash \alpha\vee\neg\circ\alpha$, by hypothesis, and so $\neg\wr\alpha \vdash \alpha$, using axiom (ci), proof-by-cases and conjunction elimination. The rest of the proof now follows exactly like in Theorem 52(ii).
Finally, for item (iii), recall that, from (Ax10), $(\neg\alpha \vee \neg\neg\alpha)$ is a theorem of **mCi**. But then, by hypothesis, $\neg(\alpha \wedge \neg\alpha)$ would also be a theorem. From Theorem 49(ii) and replacement it follows that $\neg\neg\circ\alpha$ is provable, and by Theorem 78(i) this results in $\circ\alpha$ being provable. Thus, the resulting logic would be explosive. ■

In the case of the logic **mbC**, we have called the reader's attention to the fact that the validity of (RP) required the validity of rules (EC) and (EO) (see the end of Subsection 3.2). Interestingly, now in **mCi** we can check that (EC) is enough:

THEOREM 82. In extensions of **mCi** the validity of:

(EC) $\forall\alpha\forall\beta((\alpha \dashv\vdash \beta)$ implies $(\neg\alpha \dashv\vdash \neg\beta))$

guarantees the validity of:

(EO) $\forall\alpha\forall\beta((\alpha \dashv\vdash \beta)$ implies $(\circ\alpha \dashv\vdash \circ\beta))$.

Proof. Suppose $(\alpha \dashv\vdash \beta)$. By (EC) we have that $(\neg\alpha \dashv\vdash \neg\beta)$, and from these two equivalences we conclude that $(\alpha \wedge \neg\alpha) \dashv\vdash (\beta \wedge \neg\beta)$. But from Theorems 49(ii) and 76(i) we have that $\neg\circ\gamma \dashv\vdash_{\mathbf{mCi}} (\gamma \wedge \neg\gamma)$, so we have that $\neg\circ\alpha \dashv\vdash \neg\circ\beta$. By Theorem 80(iv) we conclude then that $\circ\alpha \dashv\vdash \circ\beta$. ■

Suppose now we considered the addition to **mCi** of a stronger rule than (EC), in order to ensure replacement:

THEOREM 83. Consider the following rule:

(RC) $\forall\alpha\forall\beta((\alpha \Vdash \beta)$ implies $(\neg\beta \Vdash \neg\alpha))$.

Then, the least extension **L** of **mCi** that satisfies (RC) and proof-by-cases collapses into classical logic.

Proof. From the axioms of **mCi** we first obtain $\neg\circ\alpha \vdash_{\mathbf{L}} \alpha$, and $\neg\circ\alpha \vdash_{\mathbf{L}} \neg\alpha$. By (RC) and Theorem 78(i) we then get $\neg\alpha \vdash_{\mathbf{L}} \circ\alpha$ and $\neg\neg\alpha \vdash_{\mathbf{L}} \circ\alpha$. But then, using proof-by-cases, we conclude that $\vdash_{\mathbf{L}} \circ\alpha$, that is, all formulas are consistent. The result now follows, as usual, from Remark 29. ■

Notice that our paraconsistent formulations of the normal modal logics from Example 34 do *not* extend the logic **mCi** (contrast this with Remark 53 about **mbC**). As we said at the beginning of this subsection, an inconsistency connective $\bullet$ can often be defined from a consistency connective $\circ$ by taking $\sim\circ$, where $\sim$ is a classical negation. The definition of an inconsistency connective by taking $\neg\circ$ is an innovation of **mCi** over **mbC**, and it is typical in fact of most **LFI**s from the literature, as the ones we will be studying in the rest of this chapter. The reader should not think though that the latter class of **C**-systems has any intrinsic advantage over the former. This far, it only seems to have more often met the intuitions of the working paraconsistentists, for some reason or another — or maybe by pure coincidence. At any rate, the distinction between the two classes is only made clear in a framework such as the one set in the present study, where consistency and inconsistency are taken as (primitive or defined) connectives in their own right.

4.2 Other features of ***mCi***

In this subsection we will extend to **mCi** the results obtained for **mbC** in Subsections 3.3, 3.4, 3.5 and 3.6, that is, we will introduce a bivaluation semantics, a possible-translations semantics and a tableau system for **mCi**. Finally, we will exhibit some novel conservative translations from classical logic into **mCi**.

We begin by a brief description of a bivaluation semantics for **mCi**, in the same manner as it was done in Subsection 3.3 with **mbC**. The plan of action is similar to that of **mbC**, and we just outline the main points of departure. First of all, we should remark that, as a consequence of Theorem 121, to be proven at Subsection 5.2, the logic **mCi** cannot be characterized by any finite-valued set of truth-tables, and that gives an extra motivation for the semantics presented in the following.

DEFINITION 84. An **mCi**-valuation is an **mbC**-valuation $v\colon For^{\circ} \longrightarrow \mathbf{2}$ (see Definition 54) respecting, additionally, the following clauses:

$(v6)$ $v(\neg\circ\alpha) = 1$ implies $v(\alpha) = 1$ and $v(\neg\alpha) = 1$;
$(v7.n)$ $v(\circ\neg^n\circ\alpha) = 1$ (for $n \geq 0$). ■

The semantic consequence relation obtained from **mCi**-valuations will be denoted by $\vDash_{\mathbf{mCi}}$. It is easy to prove soundness for **mCi** with respect to **mCi**-valuations.

THEOREM 85. [Soundness] Let $\Gamma\cup\{\alpha\}$ be a set of formulas in For°. Then: $\Gamma \vdash_{\mathbf{mCi}} \alpha$ implies $\Gamma \vDash_{\mathbf{mCi}} \alpha$. ■

The completeness proof is similar to that of **mbC**, but obviously substituting $\vdash_{\mathbf{mCi}}$ for $\vdash_{\mathbf{mbC}}$. Analogously, given a set of formulas $\Delta\cup\{\alpha\}$ in For° we say that Δ is *relatively maximal with respect to* α *in* **mCi** if $\Delta \nvdash_{\mathbf{mCi}} \alpha$ and for any formula β in For° such that $\beta \notin \Delta$ we have $\Delta, \beta \vdash_{\mathbf{mCi}} \alpha$. As in Lemma 58, relatively maximal theories are closed. An analogue to Lemma 59 can immediately be checked:

LEMMA 86. Let $\Delta\cup\{\alpha\}$ be a set of formulas in For° such that Δ is relatively maximal with respect to α in **mCi**. Then Δ satisfies properties (i)–(v) of Lemma 59, plus the following:

(vi) $\neg\circ\beta \in \Delta$ implies $\beta \in \Delta$ and $\neg\beta \in \Delta$.

(vii) $\circ\neg^n\circ\beta \in \Delta$. ■

COROLLARY 87. The characteristic function of a relatively maximal theory of **mCi** defines an **mCi**-valuation. ■

THEOREM 88. [Completeness w.r.t. bivaluation semantics] Let $\Gamma\cup\{\alpha\}$ be a set of formulas in For°. Then $\Gamma \vDash_{\mathbf{mCi}} \alpha$ implies $\Gamma \vdash_{\mathbf{mCi}} \alpha$. ■

We can obtain a version of Lemma 63 for **mCi**, that is, it is always possible to define an **mCi**-valuation from a given specification of the values of the literals.

LEMMA 89. Let $v_0\colon \mathcal{P}\cup\{\neg p : p \in \mathcal{P}\} \longrightarrow \mathbf{2}$ be a mapping such that $v_0(\neg p) = 1$ whenever $v_0(p) = 0$ (for $p \in \mathcal{P}$). Then, there exists an **mCi**-valuation $v\colon For^{\circ} \longrightarrow \mathbf{2}$ extending v_0, that is, such that $v(\varphi) = v_0(\varphi)$ for every $\varphi \in \mathcal{P}\cup\{\neg p : p \in \mathcal{P}\}$.

Proof. The proof is analogous to that of Lemma 63. Thus, we will define

the value of $v(\varphi)$ while doing an induction on the complexity $\ell(\varphi)$ of $\varphi \in For^{\circ}$. Let $v(\varphi) = v_0(\varphi)$ for every $\varphi \in \mathcal{P} \cup \{\neg p : p \in \mathcal{P}\}$, and define $v(p\#q)$ according to clauses $(v1)$–$(v3)$ of Definition 54, for $\# \in \{\wedge, \vee, \rightarrow\}$ and $p, q \in \mathcal{P}$. So, $v(\varphi)$ is defined for every $\varphi \in For^{\circ}$ such that $\ell(\varphi) \leq 1$. Assume that $v(\varphi)$ was defined for every $\varphi \in For^{\circ}$ such that $\ell(\varphi) \leq n$ (for $n \geq 1$) and let $\varphi \in For^{\circ}$ such that $\ell(\varphi) = n + 1$. If $\varphi = (\psi_1 \# \psi_2)$ for $\# \in \{\wedge, \vee, \rightarrow\}$ then $v(\varphi)$ is defined using the corresponding clause from $(v1)$–$(v3)$. If $\varphi = \neg\psi$ then there are two cases to analyze:
(a) $\psi = \neg^k \circ\alpha$, for some $\alpha \in For^{\circ}$ and $k \geq 0$. Then we define $v(\varphi) = 1$ iff $v(\psi) = 0$.
(b) $\psi \neq \neg^k \circ\alpha$, for every $\alpha \in For^{\circ}$ and every $k \geq 0$. Then we define $v(\varphi) = 1$, if $v(\psi) = 0$, and $v(\varphi)$ is defined arbitrarily, otherwise.
Finally, if $\varphi = \circ\psi$ then we set $v(\varphi) = 0$ iff $v(\psi) = v(\neg\psi) = 1$.

It is easy to see that v is an **mCi**-valuation that extends v_0. ■

EXAMPLE 90. With the help of Lemma 89, the bivaluation semantics for **mCi** may be used to show, for instance, that $\neg\neg\alpha \rightarrow \alpha$ is not a thesis of this logic. Indeed, fix $p \in \mathcal{P}$ and consider the mapping

$$v_0\colon \mathcal{P} \cup \{\neg q : q \in \mathcal{P}\} \longrightarrow \mathbf{2}$$

such that $v(q) = 0$ and $v(\neg q) = 1$ for every $q \in \mathcal{P}$. From the proof of Lemma 89 we know that there exists an **mCi**-valuation $v\colon For^{\circ} \longrightarrow \mathbf{2}$ extending v_0 such that $v(\neg\neg p) = 1$. Then $v(\neg\neg p \rightarrow p) = 0$ and so $\nvDash_{\mathbf{mCi}} (\neg\neg p \rightarrow p)$. By Theorem 85, it follows that $\nvdash_{\mathbf{mCi}} (\neg\neg p \rightarrow p)$. ■

Next, as it was done in Subsection 3.4 with the logic **mbC**, we can also provide an alternative semantics for **mCi** in terms of possible-translations semantics.

Consider the collection $\mathcal{M}_1$ of 3-valued truth-tables formed by the truth-tables of $\mathcal{M}_0$, introduced in Subsection 3.4, but now considering just one consistency operator called $\circ_3$ *instead of* $\circ_1$ and $\circ_2$, presented by the truth-table:

	$\circ_3$
T	T
t	F
F	T

Again, T and t are the designated values. In $\mathcal{M}_1$, the only truth-value that is not consistent is t. If $For_{\mathcal{M}_1}$ denotes the algebra of formulas generated by $\mathcal{P}$ over the signature of $\mathcal{M}_1$, let's consider the set Tr_1 of all functions $*\colon For^{\circ} \longrightarrow For_{\mathcal{M}_1}$ respecting the clauses $(tr0)$–$(tr2)$ on translations introduced in Subsection 3.4, plus the following clauses:

$(tr3)_1$ $(\circ\alpha)^* \in \{\circ_3\alpha^*, \circ_3(\neg\alpha)^*\}$;

$(tr3)_2$ if $(\neg\alpha)^* = \neg_1\alpha^*$ then $(\circ\alpha)^* = \circ_3(\neg\alpha)^*$;

$(tr4)_1$ $(\neg^{n+1}\circ\alpha)^* = \neg_1(\neg^n\circ\alpha)^*$.

We say the pair $\mathrm{PT}_1 = \langle \mathcal{M}_1, \mathrm{T}\textsc{r}_1 \rangle$ is a *possible-translations semantical structure for* **mCi**. If $\vDash_{\mathcal{M}_1}$ denotes the consequence relation in $\mathcal{M}_1$, and $\Gamma \cup \{\alpha\}$ is a set of formulas of **mCi**, the PT_1-consequence relation, $\models_{\mathrm{PT}_1}$, is defined as:

$$\Gamma \models_{\mathrm{PT}_1} \alpha \ \text{ iff } \ \Gamma^* \vDash_{\mathcal{M}_1} \alpha^* \ \text{ for all } * \in \mathrm{T}\textsc{r}_1.$$

We leave to the reader the proof of the following easy result:

THEOREM 91. [Soundness] Let $\Gamma \cup \{\alpha\}$ be a set of formulas of **mCi**. Then $\Gamma \vdash_{\mathbf{mCi}} \alpha$ implies $\Gamma \models_{\mathrm{PT}_1} \alpha$. ■

The completeness proof follows the same lines than the one obtained for **mbC** (cf. [Marcos, 2005f]).

THEOREM 92. [Representability] Given an **mCi**-valuation v there is a translation $*$ in $\mathrm{T}\textsc{r}_1$ and a valuation w in $\mathcal{M}_1$ such that, for every formula α in **mCi**:

$$w(\alpha^*) = T \ \text{ implies } \ v(\neg\alpha) = 0; \ \text{ and}$$

$$w(\alpha^*) = F \ \text{ iff } \ v(\alpha) = 0.$$

Proof. The proof is similar to that of Theorem 67, but now defining $(\circ\alpha)^* = \circ_3(\neg\alpha)^*$ if $v(\neg\alpha) = 0$, and $(\circ\alpha)^* = \circ_3\alpha^*$ otherwise. Finally, set $(\neg^{n+1}\circ\alpha)^* = \neg_1(\neg^n\circ\alpha)^*$. Details are left to the reader. ■

COROLLARY 93. [Completeness w.r.t. possible-translations semantics] Let $\Gamma \cup \{\alpha\}$ be a set of formulas in **mCi**. Then $\Gamma \models_{\mathrm{PT}_1} \alpha$ implies $\Gamma \vdash_{\mathbf{mCi}} \alpha$.

In Remark 43 we have defined two supplementing negations for **mbC**, $\wr$ and $\sim$, and in Remark 70 we have shown that only one of them, namely $\sim$, was classical in **mbC**. Now we can use the possible-translations semantics of **mCi** to check that in this logic the two negations produce equivalent formulas:

THEOREM 94. Given a formula α, the formulas $\wr\alpha$ and $\sim\alpha$ are equivalent in **mCi**. As a result, $\wr$ defines a classical negation in **mCi**.

Proof. Notice that, using the above possible-translations semantics for **mCi**, the formulas $\wr p$ and $\sim p$ produce exactly the same truth-tables. ■

Now, using the general techniques introduced in [Caleiro *et al.*, 2005b] we can easily obtain an adequate tableau system for **mCi**, in the same way that was done for **mbC** in Subsection 3.5. Thus, in view of the bivaluation semantics for **mCi** stated in Definition 84 from the bivaluation semantics for **mbC**, it is enough to define the following:

DEFINITION 95. We define a tableau system for **mCi** by adding to the tableau system for **mbC** introduced in Example 71 the following rules:

$$\frac{T(\neg\circ X)}{T(X),\ T(\neg X)} \qquad \frac{}{T(\circ\neg^{n}\circ X)} \quad (\text{for } n \geq 0)$$ ■

Finally, let's talk again about classical logic. In Theorem 74 of Subsection 3.6 we have seen how **CPL** can be encoded inside **mbC** through a conservative translation. Clearly, that same translation works for **mCi**. We will now show how it is possible to encode **eCPL** inside **mCi**, in a similar fashion.

THEOREM 96. Let For° be the algebra of formulas for the signature Σ° of **mCi**. Consider any mapping t_2: $For^{\circ} \longrightarrow For^{\circ}$ such that:

1. $t_2(p) = p$, for every $p \in \mathcal{P}$;
2. $t_2(\gamma \# \delta) = t_2(\gamma) \# t_2(\delta)$, if $\# \in \{\wedge, \vee, \rightarrow\}$;
3. $t_2(\neg\gamma) \in \{\sim t_2(\gamma),\ \wr\, t_2(\gamma)\}$;
4. $t_2(\circ\gamma) = \circ\circ t_2(\gamma)$.

Then, t_2 is a conservative translation from **eCPL** to **mCi**.

Proof. The proof is almost identical to that of Theorem 74. The only novel clause is 4, but it is clear how it works (recall axiom $(\text{cc})_0$). ■

4.3 Inconsistency operator as primitive

Up to now we have concentrated almost exclusively on the formal notion of consistency; formal inconsistency has appeared only derivatively, defined with the help of a classical or of a paraconsistent negation. It is equally natural, however, to provide alternative axiomatizations for the logic **mCi** or its close relatives starting from a primitive inconsistency connective. We will now show how to do this in two different ways, one in terms of $\circ$ and $\bullet$, and the other in terms of $\bullet$ alone.

Let $\Sigma^{\bullet}$ and $\Sigma^{\circ\bullet}$ be the extensions of the signature Σ (recall Remark 15) obtained by the addition, respectively, of a new unary connective $\bullet$ and of two unary connectives $\circ$ and $\bullet$. Let $For^{\bullet}$ and $For^{\circ\bullet}$ be the respective algebras of formulas. The idea of axiomatizing **mCi** just in terms of $\bullet$ involves the assumption that $\bullet\alpha$ means $\neg\circ\alpha$ while $\neg\bullet\alpha$ means $\circ\alpha$. As a consequence, axiom schemas (bc1), (ci) and $(\text{ci})_n$ should adopt the following forms:

(bc1)′ $\neg\bullet\alpha \to (\alpha \to (\neg\alpha \to \beta))$

(ci)′ $\bullet\alpha \to (\alpha \wedge \neg\alpha)$

(cc)$'_n$ $\neg\bullet\neg^n\bullet\alpha \qquad (n \geq 0)$

This leads to the following definition:

DEFINITION 97. The logic $\mathbf{mCi}^\bullet$ defined over signature $\Sigma^\bullet$ is defined by the axiom schemas (Ax1)–(Ax10) (recall Definition 28) plus the axiom schemas (bc1)′, (ci)′ and (cc)$'_n$ (for $n \geq 0$) introduced above, together with (MP). ■

The next result will demonstrate to which extent **mCi** and $\mathbf{mCi}^\bullet$ are 'the same logic'. Since these logics are written in distinct signatures, an appropriate way of comparing them is by way of (some very strict and specific) translations.

THEOREM 98.
(i) Let $+ : For^\circ \longrightarrow For^\bullet$ be a mapping defined as follows:

1. $p^+ = p$ if $p \in \mathcal{P}$;
2. $(\alpha\#\beta)^+ = (\alpha^+\#\beta^+)$ where $\# \in \{\wedge, \vee, \to\}$;
3. $(\neg\circ\alpha)^+ = \bullet\alpha^+$;
4. $(\neg\alpha)^+ = \neg\alpha^+$ if $\alpha \neq \circ\beta$ for every β;
5. $(\circ\alpha)^+ = \neg\bullet\alpha^+$.

Then, the mapping $+$ is a translation from **mCi** to $\mathbf{mCi}^\bullet$, that is, for every $\Gamma \cup \{\alpha\} \subseteq For^\circ$:

$$\Gamma \vdash_{\mathbf{mCi}} \alpha \text{ implies } \Gamma^+ \vdash_{\mathbf{mCi}^\bullet} \alpha^+.$$

(ii) Let $- : For^\bullet \longrightarrow For^\circ$ be a mapping defined as follows:

1. $p^- = p$ if $p \in \mathcal{P}$;
2. $(\alpha\#\beta)^- = (\alpha^-\#\beta^-)$ where $\# \in \{\wedge, \vee, \to\}$;
3. $(\neg\bullet\alpha)^- = \circ\alpha^-$;
4. $(\neg\alpha)^- = \neg\alpha^-$ if $\alpha \neq \bullet\beta$ for every β;
5. $(\bullet\alpha)^- = \neg\circ\alpha^-$.

Then, the mapping $-$ is a translation from $\mathbf{mCi}^\bullet$ to **mCi**, that is, for every $\Gamma \cup \{\alpha\} \subseteq For^\bullet$:

$$\Gamma \vdash_{\mathbf{mCi}^\bullet} \alpha \text{ implies } \Gamma^- \vdash_{\mathbf{mCi}} \alpha^-.$$

Proof. Item (i). Suppose that $\Gamma \vdash_{\mathbf{mCi}} \alpha$. By induction on the length of a derivation in **mCi** of α from Γ, it can be easily proven that $\Gamma^{+} \vdash_{\mathbf{mCi}^{\bullet}} \alpha^{+}$. There are just three cases that deserve some attention. These cases occur when $\alpha \in For^{\circ}$ is an instance of an axiom in **mCi** but α^{+} is not an instance of an axiom in $\mathbf{mCi}^{\bullet}$. These three cases are: (1) $\alpha = \circ\circ\gamma \to (\circ\gamma \to (\neg\circ\gamma \to \delta))$ for $\gamma, \delta \in For^{\circ}$; (2) $\alpha = (\circ\gamma \vee \neg\circ\gamma)$; and (3) $\alpha = \neg\circ\circ\gamma \to (\circ\gamma \wedge \neg\circ\gamma)$. In case (1), $\alpha^{+} = \neg\bullet\neg\bullet\gamma^{+} \to (\neg\bullet\gamma^{+} \to (\bullet\gamma^{+} \to \delta^{+}))$, which is not an instance of an axiom in $\mathbf{mCi}^{\bullet}$. However, it is immediate to see that α^{+} is a theorem of $\mathbf{mCi}^{\bullet}$, because of axioms (bc1)$'$, (ci)$'$ and by the deduction theorem. Indeed, $\bullet\gamma^{+} \vdash_{\mathbf{mCi}^{\bullet}} (\gamma^{+} \wedge \neg\gamma^{+})$ and $\neg\bullet\gamma^{+}, (\gamma^{+} \wedge \neg\gamma^{+}) \vdash_{\mathbf{mCi}^{\bullet}} \delta^{+}$, therefore $\neg\bullet\neg\bullet\gamma^{+}, \neg\bullet\gamma^{+}, \bullet\gamma^{+} \vdash_{\mathbf{mCi}^{\bullet}} \delta^{+}$. Using the deduction theorem it then follows that $\vdash_{\mathbf{mCi}^{\bullet}} \alpha^{+}$. In case (2), $\alpha^{+} = (\neg\bullet\gamma^{+} \vee \bullet\gamma^{+})$, which is not an axiom of $\mathbf{mCi}^{\bullet}$, yet it can be easily checked to be a theorem of $\mathbf{mCi}^{\bullet}$. In case (3), $\alpha^{+} = \bullet\neg\bullet\gamma^{+} \to (\neg\bullet\gamma^{+} \wedge \bullet\gamma^{+})$. This is not an axiom, but it can be easily proven in $\mathbf{mCi}^{\bullet}$. Indeed, by (cc)$'_1$, (bc1)$'$, the deduction theorem and proof-by-cases it follows that $\neg\neg\bullet\delta \vdash_{\mathbf{mCi}^{\bullet}} \bullet\delta$ holds in $\mathbf{mCi}^{\bullet}$, for every δ. Using this, (ci)$'$, properties of the standard conjunction and the deduction theorem, it follows that α^{+} is a theorem of $\mathbf{mCi}^{\bullet}$.
Item (ii). The proof is entirely analogous to that of item (i). ■

The fact that both logics are inter-translatable means that **mCi** encodes $\mathbf{mCi}^{\bullet}$ and vice-versa. Moreover, we could take the combined logic $\mathbf{mCi}^{\circ\bullet}$ defined over $\Sigma^{\circ\bullet}$ by putting together all the axiom schemas of **mCi** and $\mathbf{mCi}^{\bullet}$, plus (MP) (technically, $\mathbf{mCi}^{\circ\bullet}$ can be obtained as the fibring of **mCi** and $\mathbf{mCi}^{\bullet}$; see, for instance, the entry on fibring [Caleiro *et al.*, 2005] in this Handbook). It is also possible to show that the logic $\mathbf{mCi}^{\circ\bullet}$ is a conservative extension of both **mCi** and $\mathbf{mCi}^{\bullet}$. The following result is easy to check:

THEOREM 99. Let α be a formula in $For^{\circ\bullet}$. Then

$$\circ\alpha \dashv\vdash_{\mathbf{mCi}^{\circ\bullet}} \neg\bullet\alpha \quad \text{and} \quad \neg\circ\alpha \dashv\vdash_{\mathbf{mCi}^{\circ\bullet}} \bullet\alpha.$$ ■

However, as yet another witness to the fact that the replacement property (RP) (see Remark 51) is not enjoyed by these logics, it is not difficult to see (say, by means of bivaluations) that, in general, the following is true, for $\alpha \in For^{\circ}$ and $\beta \in For^{\bullet}$:

$$\alpha \nvdash_{\mathbf{mCi}} (\alpha^{+})^{-}, \quad (\alpha^{+})^{-} \nvdash_{\mathbf{mCi}} \alpha, \quad \alpha \nvdash_{\mathbf{mCi}^{\circ\bullet}} \alpha^{+}, \quad \alpha^{+} \nvdash_{\mathbf{mCi}^{\circ\bullet}} \alpha$$

$$\beta \nvdash_{\mathbf{mCi}^{\bullet}} (\beta^{-})^{+}, \quad (\beta^{-})^{+} \nvdash_{\mathbf{mCi}^{\bullet}} \beta, \quad \beta \nvdash_{\mathbf{mCi}^{\circ\bullet}} \beta^{-}, \quad \beta^{-} \nvdash_{\mathbf{mCi}^{\circ\bullet}} \beta.$$

The corresponding bivaluation semantics, possible-translations semantics and tableau procedures for the versions of **mCi** in the above signatures can be easily implemented and we will not annoy the reader with details.

4.4 *Enhancing* ***mCi*** *in dealing with double negations*

In this subsection we will see what happens when the logics **mbC** and **mCi** are further extended with axioms dealing with doubly negated formulas, namely:

(cf) $\neg\neg\alpha \to \alpha$

(ce) $\alpha \to \neg\neg\alpha$

Note that (cf) has already appeared as (Ax11) in Definition 28. From item (iii) of Theorem 77 we know that neither (ce) nor (cf) is provable in **mCi**. Adding such axioms makes the negation of this logic a bit closer to classical negation. Moreover, we will see that adding them helps in simplifying the axiomatic presentations of the resulting logics, and it also has a nice consequence for the interaction of negation with the connectives for consistency and inconsistency.

DEFINITION 100. Consider the signature Σ°. Recall the axiomatizations of **mbC** and **mCi** from Definitions 42 and 75. Then:

1. **bC** is axiomatized as **mbC** plus (cf).
2. **Ci** is axiomatized as **mCi** plus (cf).
3. **mbCe** is axiomatized as **mbC** plus (ce).
4. **mCie** is axiomatized as **mCi** plus (ce).
5. **bCe** is axiomatized as **bC** plus (ce).
6. **Cie** is axiomatized as **Ci** plus (ce). ■

It is easy to check that:

THEOREM 101.
(i) $\circ\alpha \vdash_{\mathbf{Ci}} \circ\neg\alpha$;
(ii) $\bullet\neg\alpha \vdash_{\mathbf{Ci}} \bullet\alpha$;
(iii) $\circ\neg\alpha \vdash_{\mathbf{mCie}} \circ\alpha$;
(iv) $\bullet\alpha \vdash_{\mathbf{mCie}} \bullet\neg\alpha$. ■

Using the latter result one might provide a simpler and finitary axiomatization for the logic **Ci** (thus also for **Cie**):

THEOREM 102. The logic **Ci** may be obtained from **mbC** by adding the axiom schemas (ci) (see Definition 75) and (cf) (Subsection 4.4), to wit:

(ci) $\neg\circ\alpha \to (\alpha \wedge \neg\alpha)$

(cf) $\neg\neg\alpha \to \alpha$

Proof. Let $\Vdash$ be the consequence relation of the logic obtained from **mbC** by adding the axiom schemas (ci) and (cf). Of course $\Vdash\, \subseteq\, \vdash_{\mathbf{Ci}}$. In order to prove the converse, it is enough to prove that $\Vdash \circ\neg^n\circ\,\alpha$ (that is, axiom schema $(\mathrm{cc})_n$) holds good for every formula α and every natural number n.

For $n = 0$ note that $\circ\alpha, \alpha, \neg\alpha \Vdash \circ\circ\alpha$, by (bc1), and $\neg\circ\alpha \Vdash \alpha \wedge \neg\alpha$, by (ci). In particular $\neg\circ\circ\alpha \Vdash \circ\alpha \wedge \neg\circ\alpha$, thus $\neg\circ\circ\alpha \Vdash \circ\circ\alpha$. But $\circ\circ\alpha \Vdash \circ\circ\alpha$ and then proof-by-cases gives us

$$(*) \qquad \Vdash \circ\circ\alpha.$$

Now, by (ci) again, we have that $\neg\circ\neg\alpha \Vdash \neg\alpha\wedge\neg\neg\alpha$ and then $\neg\circ\neg\alpha \Vdash \neg\alpha\wedge\alpha$, by (cf). Using (bc1) we obtain $\circ\alpha, \alpha, \neg\alpha \Vdash \circ\neg\alpha$ and so $\circ\alpha, \neg\circ\neg\alpha \Vdash \circ\neg\alpha$. Since $\circ\alpha, \circ\neg\alpha \Vdash \circ\neg\alpha$ then proof-by-cases gives us $\circ\alpha \Vdash \circ\neg\alpha$ for every α, as in Theorem 101(i). In particular,

$$(**) \qquad \circ\neg^n\circ\alpha \Vdash \circ\neg^{n+1}\circ\alpha$$

for every $n \geq 0$ and every α. Using $(*)$ and $(**)$, it is now immediate to obtain $(\text{cc})_n$ by induction on n. ■

As regards semantic presentations, in view of Theorem 121 (see Subsection 5.2), we know that the logics from Definition 100 are not characterizable by a collection of finite-valued truth-tables. However, it is straightforward to endow these new systems with adequate bivaluation semantics, using the methods from previous sections. Indeed:

THEOREM 103. Axiom (cf) corresponds to the following clause on the definition of a bivaluation semantics:

($v8$) $v(\neg\neg\alpha) = 1$ implies $v(\alpha) = 1$.

Similarly, axiom (ce) corresponds to:

($v9$) $v(\alpha) = 1$ implies $v(\neg\neg\alpha) = 1$. ■

Accordingly, one can now prove, for instance, that **Ci** is sound and complete for the class of bivaluations $v\colon For^\circ \longrightarrow \mathbf{2}$ satisfying clauses ($v1$)–($v5$) of Definition 54 plus clause ($v6$) of Definition 84 and clause ($v8$) of Theorem 103.

The next useful result concerning the definability of bivaluations for the systems introduced in Definition 100 can be obtained. The proof is done by appropriately adapting the proofs of Lemmas 63 and 89.

LEMMA 104. Let $v_0\colon \mathcal{P} \cup \{\neg p : p \in \mathcal{P}\} \longrightarrow \mathbf{2}$ be a mapping such that $v_0(\neg p) = 1$ whenever $v_0(p) = 0$ (for $p \in \mathcal{P}$). Then, there exist bivaluations extending v_0, for each one of the logics introduced in Definition 100.

Proof. We only prove the case for **Ci**. Thus, given v_0, define $v(\varphi) = v_0(\varphi)$ for every $\varphi \in \mathcal{P} \cup \{\neg p : p \in \mathcal{P}\}$, and $v(p\#q)$ is defined according to clauses ($v1$)–($v3$) of Definition 54, for $\# \in \{\wedge, \vee, \rightarrow\}$ and $p, q \in \mathcal{P}$. Suppose that $v(\varphi)$ was defined for every $\varphi \in For^\circ$ such that $\ell(\varphi) \leq n$ (for $n \geq 1$) and let $\varphi \in For^\circ$ such that $\ell(\varphi) = n + 1$. If $\varphi = (\psi_1\#\psi_2)$ for $\# \in \{\wedge, \vee, \rightarrow\}$ then we use clauses ($v1$)–($v3$) to define $v(\varphi)$. If $\varphi = \circ\psi$ then define $v(\varphi) = 0$ iff $v(\psi) = v(\neg\psi) = 1$.

Finally, suppose that $\varphi = \neg\psi$. If $v(\psi) = 0$ then define $v(\varphi) = 1$. On the

other hand, if $v(\psi) = 1$ then there are three cases to analyze:
(a) $\psi = \circ\alpha$, for some $\alpha \in For^{\circ}$. Then, define $v(\varphi) = 0$.
(b) $\psi = \neg\alpha$, for some $\alpha \in For^{\circ}$, such that $v(\alpha) = 0$. Then we define $v(\varphi) = 0$.
(c) In any other case, $v(\varphi)$ is defined arbitrarily.
It is straightforward to check that v is a **Ci**-valuation extending v_0. The proof for the other systems is entirely analogous, and we leave the details to the reader. ∎

We can also obtain adequate tableaux for these systems, as in previous sections. Possible-translations semantics for **bC**, **Ci**, **bCe** and **Cie** may be found in [Marcos, 2005f]. These four logics were exhaustively studied in [Carnielli and Marcos, 2002]. Non-deterministic semantics for these logics can be found in [Avron, 2005a].

5 ADDITIONAL TOPICS ON **LFI**S

*5.1 The **dC**-systems*

As we have seen in Theorem 98, the formulas $\bullet\alpha$ and $\neg\circ\alpha$ have the same meaning (up to translations) in **mCi**. Moreover, we also know from Theorem 49(i) and axiom (ci) that the formulas $\bullet\alpha$ and $(\alpha \wedge \neg\alpha)$ are equivalent in **mCi**. However, as we know from Theorem 76, the formulas $\neg\bullet\alpha$ and $\neg(\alpha \wedge \neg\alpha)$ are not equivalent, nor are the formulas $\neg\neg\bullet\alpha$ and $\neg\neg(\alpha \wedge \neg\alpha)$, and so on.

It seemed only natural, thus, to consider extensions of **mCi** in which the meaning of statements involving $\bullet$ (and also $\circ$) may be recast in terms of the other connectives, by means of translations or of explicit definitions. This maneuver led us to the class of **LFI**s known as **dC**-systems, in which the new connective of consistency may be dismissed from the beginning, and replaced by a formula built from the other connectives already present in the signature (recall Definition 32).[8] The logic **Cil**, to be defined below, is an example of this strategy.

DEFINITION 105. The logic **Cil**, defined over the signature Σ°, is obtained from **Ci** by the addition of the following axiom schema:

(cl) $\neg(\alpha \wedge \neg\alpha) \to \circ\alpha$

Other logics may be obtained in a similar fashion, such as the logic **Cile**, defined by the addition of (ce) to **Cil** (recall Subsection 4.4). ∎

[8] The reader is invited to adapt Definition 32 to deal also with the inconsistency operator, and to logics defined over signatures $\Sigma^{\bullet}$ and $\Sigma^{\circ\bullet}$.

By the very definition of **Cil**, it is clear there cannot be a paraconsistent extension of **Cil** in which the schema $\neg(\alpha\wedge\neg\alpha)$ is provable. There are, however, other paraconsistent extensions of **Ci**, such as **LFI1** (see Example 18 and Theorem 127) or extensions of **bC** such as all non-degenerate normal modal logics extending the system KT (recall Example 34), in which the schema $\neg(\alpha \wedge \neg\alpha)$ is indeed provable.

It can be checked that **Cil** is in fact an indirect a **dC**-system based on classical logic:

THEOREM 106. The logic **Cil** may be defined over Σ by identifying $\circ\alpha$ with $\neg(\alpha \wedge \neg\alpha)$. More precisely: Let $\mathbf{Cil}^{©}$ be the logic over Σ defined by axiom schemas (Ax1)–(Ax11) (see Definition 28), rule (MP), plus the following axiom schema:

(bc1)″ $\neg(\alpha \wedge \neg\alpha) \to (\alpha \to (\neg\alpha \to \beta))$

Let $\star : For^{\circ} \longrightarrow For$ be a mapping defined as follows:

1. $p^{\star} = p$ if $p \in \mathcal{P}$;
2. $(\alpha\#\beta)^{\star} = (\alpha^{\star}\#\beta^{\star})$ where $\# \in \{\wedge, \vee, \to\}$;
3. $(\neg\alpha)^{\star} = \neg\alpha^{\star}$;
4. $(\circ\alpha)^{\star} = \neg(\alpha^{\star} \wedge \neg\alpha^{\star})$.

Then, the mapping $\star$ is a translation from **Cil** to $\mathbf{Cil}^{©}$, that is, for every $\Gamma \cup \{\alpha\} \subseteq For^{\circ}$:

$$\Gamma \vdash_{\mathbf{Cil}} \alpha \text{ implies } \Gamma^{\star} \vdash_{\mathbf{Cil}^{©}} \alpha^{\star}.$$

On the other hand, **Cil** is a conservative extension of $\mathbf{Cil}^{©}$, that is, for every $\Gamma \cup \{\alpha\} \subseteq For$:

$$\Gamma \vdash_{\mathbf{Cil}^{©}} \alpha \text{ iff } \Gamma \vdash_{\mathbf{Cil}} \alpha.$$

As a consequence of the above, the following holds good, for every $\Gamma \cup \{\alpha\} \subseteq For^{\circ}$:

$$\Gamma \vdash_{\mathbf{Cil}} \alpha \text{ implies } \Gamma^{\star} \vdash_{\mathbf{Cil}} \alpha^{\star}.$$

Proof. The proof follows the lines of the proof of Theorem 98, and there is just one further critical case to analyze: Any axiom of **Cil** of the form $\alpha = \neg(\gamma \wedge \neg\gamma) \to \circ\gamma$ is translated as $\alpha^{\star} = \neg(\gamma^{\star} \wedge \neg\gamma^{\star}) \to \neg(\gamma^{\star} \wedge \neg\gamma^{\star})$, which is not an axiom of $\mathbf{Cil}^{©}$, but it is obviously a theorem of $\mathbf{Cil}^{©}$. This shows that $\star$ is a translation from **Cil** to $\mathbf{Cil}^{©}$.

Consider now a set $\Gamma \cup \{\alpha\} \subseteq For$. Observe that every axiom of $\mathbf{Cil}^{©}$ different from (bc1)″ is an axiom of **Cil**. On the other hand, it is easy to see (using the deduction theorem) that (bc1)″ is a theorem of **Cil**. Hence, by induction on the length of a derivation in $\mathbf{Cil}^{©}$ of α from Γ it follows that, if $\Gamma \vdash_{\mathbf{Cil}^{©}} \alpha$ then $\Gamma \vdash_{\mathbf{Cil}} \alpha$. Conversely, if $\Gamma \vdash_{\mathbf{Cil}} \alpha$ then $\Gamma^{\star} \vdash_{\mathbf{Cil}^{©}} \alpha^{\star}$, since $\star$ is a translation from **Cil** to **Cil**$^{©}$. But, if $\beta \in For$ then $\beta^{\star} = \beta$, so $\Gamma \vdash_{\mathbf{Cil}^{©}} \alpha$. This shows that **Cil** is a conservative extension of **Cil**$^{©}$. The rest of the proof is straightforward. ■

The above theorem shows that **Cil**$^{©}$ is a direct **dC**-system, just as much as **Cil** is an indirect one (recall Definitions 32 and 33). Observe that (bc1)$''$ was already introduced in Definition 28 as axiom (bc1). Thus, the logic **Cil**$^{©}$ is obtained from C_1 by the elimination of axioms (ca1)–(ca3) (or, equivalently, C_1 is obtained from **Cil**$^{©}$ by adding axiom schemas (ca1)–(ca3); see Definition 108 and Remark 109). The formula schema $\neg(\alpha \wedge \neg\alpha)$ played an important role in the original construction of the logics C_n, and it has often been identified with the so-called 'Principle of Non-Contradiction'. Notice, however, that such an identification is not possible with our present definition of this principle (Principle (1) in Subsection 2.1).

There is no consensus in the literature on what concerns the status of the schema $\neg(\alpha \wedge \neg\alpha)$ inside paraconsistent logics. Its validity has been criticized by some (see, for instance, [Béziau, 2002a]). A good technical reason for expecting this schema to fail is connected to the possible consequent failure of the replacement property, as predicted in Theorem 52(iv). On the other hand, the proposal of paraconsistent logics in which this schema does not hold has *also* been criticized, as for instance in [Routley and Meyer, 1976], where the authors claim that, for dialectical logics (i.e. for logics disrespecting our version of the Principle of Non-Contradiction), not only do we usually have that $\neg(\alpha \wedge \neg\alpha)$ is a theorem, but that feature does not conflict with other logical truths of such logics. On our approach, the whole controversy seems artificial and ill-advised. It might well be just a sterile offspring of the misidentification of the Principle of Explosion and the Principle of Non-Contradiction: In general, only the former should worry a paraconsistent logician, the latter being a much less demanding and a very often strictly observed principle (check the ensuing discussion in section 3.8 of [Carnielli and Marcos, 2002]).

Using (bc1) and (cl), every theorem of the form $\circ(\alpha \wedge \neg\alpha)$ can be proven. In the presence of axiom (cf), as in Theorem 101(i), this allows one to prove, in **Cil**, every theorem of the form $\circ\neg^{n}(\alpha \wedge \neg\alpha)$. This feature was to raise protests by some authors (see for instance [Sylvan, 1990]), according to whom it makes no sense to declare contradictions (case $n = 0$ in the above formula) to be provably consistent.

With respect to semantics, Theorem 125 (see Subsection 5.2) proves that the logics **Cil** and **Cil**$^{©}$ are not characterizable by a collection of finite-valued truth-tables. Of course, we can obtain a bivaluation semantics for **Cil**$^{©}$ by considering mappings $v\colon For \longrightarrow \mathbf{2}$ satisfying axioms $(v1)$–$(v4)$ of Definition 54, plus the following:

$(v10)$ $v(\neg(\alpha \wedge \neg\alpha)) = 1$ implies $v(\alpha) = 0$ or $v(\neg\alpha) = 0$;

$(v11)$ $v(\neg\neg\alpha) = 1$ implies $v(\alpha) = 1$.

In the case of **Cil**, one may consider bivaluations $v: For^{\circ} \longrightarrow \mathbf{2}$ that satisfy axioms $(v1)$–$(v5)$ of Definition 54, plus $(v6)$ (see Definition 84) and $(v11)$. Of course, a result analogous to Lemma 104 can be stated and proven for the logics **Cil** and **Cil**$^{©}$. At this point it should be obvious to the reader how the tableaux for these logics would look like.

If the reader has still not gotten used to the frequent failure of the replacement property, he might be surprised with the following asymmetry allowed by the logic **Cil**. The consistency operator $\circ\alpha$ is equivalent in **Cil** to the formula $\neg(\alpha \wedge \neg\alpha)$ (cf. Definition 105) and consequently the logic resulting from the addition of $\neg(\alpha \wedge \neg\alpha)$ to **Cil** is no longer paraconsistent. On the other hand:

THEOREM 107. The logic resulting from the addition of $\neg(\neg\alpha \wedge \alpha)$ to **Cil** is still paraconsistent, and so the operator $\circ$ cannot be alternatively expressed by the formula $\neg(\neg\alpha \wedge \alpha)$.

Proof. The first collection of truth-tables from the proof of Theorem 50 provides a model of **Cil** plus $\neg(\neg\alpha \wedge \alpha)$. The same collection of truth-tables show that there are atomic formulas p and q such that $\neg(\neg p \wedge p), p, \neg p$ take designated values, while q does not: Just assign the value $\frac{1}{2}$ to p and 0 to q. ■

The above asymmetry has been sharply pointed out in Theorem 4 of [Urbas, 1989] for the case of the logic C_1 which is, as we mentioned before, an extension of **Cil**$^{©}$ (see also Remark 109). This asymmetry remained hidden for a long time within the realm of the logics C_n. Indeed, the first decision procedure offered for the logic C_1 in terms of quasi matrices, in [da Costa and Alves, 1977], was mistaken exactly in assuming $\neg(\alpha \wedge \neg\alpha)$ and $\neg(\neg\alpha \wedge \alpha)$ to be equivalent formulas.

Some natural alternatives to (cl) can immediately be considered:

(cd) $\neg(\neg\alpha \wedge \alpha) \to \circ\alpha$;

(cb) $(\neg(\alpha \wedge \neg\alpha) \vee \neg(\neg\alpha \wedge \alpha)) \to \circ\alpha$.

(RG) $\beta \dashv\vdash \alpha \wedge \neg\alpha$ implies $\neg\beta \dashv\vdash \neg(\alpha \wedge \neg\alpha)$

Clearly, the addition to **Ci** of the axiom (cd) instead of the axiom (cl), would produce a logic in which the asymmetry pointed out in Theorem 107 is inverted. That inconvenient can be solved if the axiom (cb) is added instead, as that move produces a logic in which both $\neg(\alpha\wedge\neg\alpha)$ and $\neg(\neg\alpha\wedge\alpha)$ express consistency. However, that will not make the difficulties about the replacement property, (RP), go away. In fact, the equivalence of similar more complex formulas would not be guaranteed by (cb): It can be shown for instance that formulas such as $\neg(\alpha \wedge (\alpha \wedge \neg\alpha))$ and $\neg((\alpha \wedge \neg\alpha) \wedge \alpha)$ are not automatically equivalent, even though $(\alpha \wedge (\alpha \wedge \neg\alpha))$ and $((\alpha \wedge \neg\alpha) \wedge \alpha)$ are equivalent on any **C**-system based on (positive) classical logic. As pointed out in [Carnielli and Marcos, 2002], a way of solving that specific predicament without necessarily going as far as establishing the validity of (RP) is simply by adding the rule (RG). Of course, **dC**-systems with full (RP) are clearly available, as it has been illustrated by the modal logics proposed in Example 34, all of which extend the fundamental **C**-system **mbC** (recall Remark 53).

It should be clear that each **dC**-system can in principle generate an infinite number of other **dC**-systems, if one applies to it the same strategy as that of the C_n logics, for $1 \leq n < \omega$ (cf. Definition 28), namely, if one simply requires stronger and stronger conditions to be met in order to establish the consistency of a formula.

5.2 Adding modularity: Letting consistency propagate

Given a class of consistent formulas, an important issue is to understand how this consistency propagates towards simpler or more complex formulas. As we have seen in Theorem 101, the addition to **mCi** of axioms or rules controlling the behavior of doubly negated formulas reflects directly on the propagation of consistency through negation. As we will see in this subsection, one can in fact produce interesting variations on the recipe that constructs **LFI**s by directly controlling the way consistency propagates.

DEFINITION 108.
(i) The logic **Cia** is obtained by the addition of the following axiom schemas to **Ci** (see Definition 100):

(ca1) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \wedge \beta)$;
(ca2) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \vee \beta)$;
(ca3) $(\circ\alpha \wedge \circ\beta) \to \circ(\alpha \to \beta)$.

(ii) The logic **Cila** is obtained by the addition of the axiom schema (cl) to **Cia** or, equivalently, of the axioms (ca1)–(ca3) to **Cil** (see Definition 105). Using axioms (cd) or (cb) instead of (cl) one might similarly define the logics **Cida** or **Ciba**. Adding axiom (ce) to those systems one might define the logics **Cilae**, **Cidae** and **Cibae**. ■

REMARK 109. It is worth insisting that the only difference between **Cila** and the original formulation of C_1 (recall Definition 28) is that the connective $\circ$ in C_1 was not taken as primitive, but $\circ\alpha$, originally denoted as α°, was assumed from the start to be an abbreviation of the formula $\neg(\alpha \wedge \neg\alpha)$. A transformation to that same effect is done by the translation $\star$ from Theorem 106. However, it should be noted that there are formulas $\alpha \in For^\circ$ such that α and $\alpha^\star$ are not equivalent in **Cila**. On the other hand, C_1 coincides with **Cila**$^{©}$, the logic obtained from **Cil**$^{©}$ (see again Theorem 106) by adding axioms (ca1)–(ca3). In other words, **Cila** is obtained from C_1 by adding the consistency operator $\circ$ to the signature as well as the obvious axioms stating the equivalence between the formulas $\circ\alpha$ and $\neg(\alpha \wedge \neg\alpha)$. In the terminology of Definition 33, we may say that **Cila** *corresponds* to C_1. For the other logics in the hierarchy C_n, $1 \leq n < \omega$, the formula $\circ\alpha$ abbreviates more and more complex formulas, or sets of formulas, as it can be seen in Definition 28. ■

The logics **Cila** and C_1 are not exactly coincident since they are defined over distinct signatures. However, they are related by means of translations in the same way as **Cil** and **Cil**© were so related (recall Theorem 106). In other terms, **Cila** is an indirect **dC**-system, while C_1 is a direct **dC**-system, as the theorem below shows.

THEOREM 110. Let $\star : For^\circ \longrightarrow For$ be the translation mapping defined as in Theorem 106. Then $\star$ is a translation from **Cila** to C_1, that is, for every $\Gamma \cup \{\alpha\} \subseteq For^\circ$,

$$\Gamma \vdash_{\mathbf{Cila}} \alpha \;\text{ implies }\; \Gamma^\star \vdash_{C_1} \alpha^\star.$$

On the other hand, **Cila** is a conservative extension of C_1, that is, for every $\Gamma \cup \{\alpha\} \subseteq For$,

$$\Gamma \vdash_{C_1} \alpha \;\text{ iff }\; \Gamma \vdash_{\mathbf{Cila}} \alpha.$$

As a consequence of this, the following holds, for every $\Gamma \cup \{\alpha\} \subseteq For^\circ$:

$$\Gamma \vdash_{\mathbf{Cila}} \alpha \;\text{ implies }\; \Gamma^\star \vdash_{\mathbf{Cila}} \alpha^\star.$$

Proof. An easy extension of the proof of Theorem 106. In fact, taking into account that C_1 coincides with **Cila**© (recall Remark 109) and also the fact that axioms (ca1)–(ca3) of Definition 108 are translated by $\star$ in terms of the homonymous axioms of Definition 28, the proof is immediate. ■

REMARK 111. Consider the logic **Cl** obtained from **Cil** by removing axiom (ci). In other words, **Cl** is defined by axiom schemas (Ax1)–(Ax11) (see Definition 28), (cl) (see Definition 105), plus (MP). Let **Cil**© be the logic defined in Theorem 106. It is easy to check, though, that the results in Theorem 106 are still valid if we uniformly substitute **Cl** for **Cil**. The logic **Cla**, studied in [Avron, 2005b], may now be obtained from **Cl** by adding axiom schemas (ca1)–(ca3) of Definition 108, and the proof of Theorem 110 is still valid if if we uniformly substitute **Cla** for **Cila**.[9] However, according to Definition 33, we can say that **Cila** corresponds to C_1, but we cannot say the same about the **C**-system **Cla**. ■

Taking into account the new axioms from Definition 108, it is easy to prove in **Cia** the following particular version of a Derivability Adjustment

[9]We thank Arnon Avron for pointing this fact to us.

Theorem (recall Remark 26 and compare the following with what was said at the beginning of Subsection 3.6):

THEOREM 112.
Let Π denote the set of atomic formulas occurring in $\Gamma \cup \{\alpha\}$.
Then, $\Gamma \vdash_{\mathbf{CPL}} \alpha$ iff there is some $\Delta \subseteq \Pi$ such that $\circ(\Delta), \Gamma \vdash_{\mathbf{Cia}} \alpha$.

Proof. Recall that **CPL** may be axiomatized by (Ax1)–(Ax11), (MP) and the 'explosion law': (exp) $\alpha \to (\neg\alpha \to \beta)$. Consider some $\Gamma \cup \{\alpha\} \subseteq For$ such that $\Gamma \vdash_{\mathbf{CPL}} \alpha$. By induction on the length n of a derivation in **CPL** of α from Γ it will be proven that $\circ(\Delta), \Gamma \vdash_{\mathbf{Cia}} \alpha$ for some $\Delta \subseteq \Pi$. If $n = 1$ then either $\alpha \in \Gamma$ or α is an instance of an axiom of **CPL**. In the first case the proof is trivial. In the second case, there is just one case in which α is not an axiom of **Cia**, namely, when α is an instance $\delta \to (\neg\delta \to \beta)$ of (exp). Let Δ be the set of propositional variables occurring in δ. Then, by Theorem 101(i) and by (ca1)–(ca3), it is easy to prove (by induction on the complexity of δ) that $\circ(\Delta) \vdash_{\mathbf{Cia}} \circ\delta$. On the other hand, from (bc1) and (MP) it follows that $\circ\delta \vdash_{\mathbf{Cia}} \alpha$. Thus $\circ(\Delta), \Gamma \vdash_{\mathbf{Cia}} \alpha$, where $\Delta \subseteq \Pi$. Suppose now that α follows from β and $\beta \to \alpha$ by (MP), in the last step of a given derivation in **CPL** of α from Γ. By induction hypothesis, $\circ(\Delta_1), \Gamma \vdash_{\mathbf{Cia}} \beta$ and $\circ(\Delta_2), \Gamma \vdash_{\mathbf{Cia}} \beta \to \alpha$ for some $\Delta_1, \Delta_2 \subseteq \Pi$. Thus $\circ(\Delta_1), \circ(\Delta_2), \Gamma \vdash_{\mathbf{Cia}} \alpha$, by (MP). But of course $\circ(\Delta_1) \cup \circ(\Delta_2) = \circ(\Delta_1 \cup \Delta_2)$, so we have that $\circ(\Delta_1 \cup \Delta_2), \Gamma \vdash_{\mathbf{Cia}} \alpha$, and that concludes the first half of the proof.

Conversely, suppose now that $\Gamma \cup \{\alpha\} \subseteq For$ is such that $\circ(\Delta), \Gamma \vdash_{\mathbf{Cia}} \alpha$ for some $\Delta \subseteq \Pi$. If $\Gamma \nvdash_{\mathbf{CPL}} \alpha$ then there exists a classical valuation v such that $v(\Gamma) \subseteq \{1\}$ and $v(\alpha) = 0$. Extend v to For° by putting $v(\circ\beta) = 1$ for every $\beta \in For^\circ$. Then v is a model for **Cia** such that $v(\circ(\Delta) \cup \Gamma) \subseteq \{1\}$, therefore $v(\alpha) = 1$, a contradiction. Thus $\Gamma \vdash_{\mathbf{CPL}} \alpha$. ■

As pointed out already in [da Costa, 1963] and [da Costa, 1974], the same result holds good for any logic C_n, assuming in each case the appropriate definition of $\circ\alpha$.

Recalling that **eCPL** is just the classical propositional logic **CPL** plus the axiom schema $\circ\alpha$, we may also propose the following alternative way of recovering classical reasoning inside our present **LFI**s:

THEOREM 113. Consider the mapping t_3: $For^\circ \longrightarrow For^\circ$, recursively defined as follows:

1. $t_3(p) = \circ p$, for every $p \in \mathcal{P}$;

2. $t_3(\gamma\#\delta) = (t_3(\gamma)\#t_3(\delta))$, if $\# \in \{\wedge, \vee, \to\}$;

3. $t_3(\circledast\gamma) = \circledast t_3(\gamma)$, if $\circledast \in \{\neg, \circ\}$.

Then t_3 conservatively translates **eCPL** inside of **Cia**.

Proof. Using compactness, the deduction theorem, and the definition of t_3, it is enough to prove that $\vdash_{\mathbf{eCPL}} \alpha$ iff $\vdash_{\mathbf{Cia}} t_3(\alpha)$ for every α in For°.

We first prove from left to right. Given a formula $\alpha(p_1, \ldots, p_n)$ in For°, then $t_3(\alpha) = \alpha(\circ p_1, \ldots, \circ p_n)$. From this, using axioms (ca1)–(ca3), axiom $(\mathrm{cc})_n$ (Definition 75) and Theorem 101(i) it is not hard to prove by induction on the complexity $\ell(\alpha)$ of α that $\vdash_{\mathbf{Cia}} \circ t_3(\alpha)$ for every $\alpha \in For^\circ$. Observe that, if β is an axiom of **eCPL** different from (exp) (see Remark 29) then $t_3(\beta)$ is a theorem of **Cia**. On the other hand, if $\beta = \delta \to (\neg\delta \to \gamma)$ is an instance of (exp) then $t_3(\beta) = t_3(\delta) \to (\neg t_3(\delta) \to t_3(\gamma))$, and the latter is provable in **Cia** from (bc1) and $\circ t_3(\delta)$. Thus, $t_3(\beta)$ is a theorem of **Cia**. Note also that any application of *modus ponens* in **eCPL** is transformed into an application of *modus ponens* in **Cia**. Consequently, given a derivation $\alpha_1, \ldots, \alpha_n = \alpha$ of α in **eCPL**, the finite sequence of formulas $t_3(\alpha_1), \ldots, t_3(\alpha_n) = t_3(\alpha)$ may be transformed into a derivation of $t_3(\alpha)$ in **Cia**. This shows that $\vdash_{\mathbf{eCPL}} \alpha$ implies $\vdash_{\mathbf{Cia}} t_3(\alpha)$.

In order to prove the converse, consider the definition of an adequate bivaluation semantics for **Cia**, adding to the clauses of a bivaluation semantics for **Ci** (see Definition 84) the clause $(vC7)$ of Example 65. Now, given an **eCPL**-valuation v, consider the mapping $v' \colon \mathcal{P} \cup \{\neg p : p \in \mathcal{P}\} \longrightarrow \mathbf{2}$ such that $v'(p) = 1$ for every $p \in \mathcal{P}$, and $v'(\neg p) = 1$ iff $v(p) = 0$. Define now $v'(\circ p) = 1$ iff $v'(\neg p) = 0$, and extend v' homomorphically to the remaining formulas in For° using the truth-tables for **eCPL**. That is, for formulas other than p, $\neg p$ and $\circ p$ (for $p \in \mathcal{P}$) the mapping v' is defined as a classical valuation and moreover satisfies $v'(\circ\alpha) = 1$ for every non-atomic α. It is easy to see that this v' is indeed a **Cia**-valuation. An induction on the complexity $\ell(\alpha)$ of α shows that $v(\alpha) = v'(t_3(\alpha))$ for every $\alpha \in For^\circ$. Finally, suppose that $\nvdash_{\mathbf{eCPL}} \alpha$. Then, there is some **eCPL**-valuation v such that $v(\alpha) = 0$. But then, by the above argument, there is some **Cia**-valuation v' such that $v'(t_3(\alpha)) = 0$ and so $\nvdash_{\mathbf{Cia}} t_3(\alpha)$. ■

Straightforward adaptations of the above argument show that the same t_3 acts as a conservative translation between **eCPL** and all logics defined in item (ii) of Definition 108. So, in order to perform 'classical inferences' within such logics (and even within C_1, in view of Theorem 110), it suffices to translate every atomic formula p into $\circ p$.

Axioms (ca1)–(ca3) of Definition 108 describe a certain form of propagation of consistency through conjunction. There are several other sensible ways of allowing consistency or inconsistency to propagate. For instance, it also makes sense to think of propagation of consistency through disjunction:

DEFINITION 114.

(i) The logic **Cio** is obtained by the addition to **Ci** of the axiom schemas:

(co1) $(\circ\alpha \vee \circ\beta) \to \circ(\alpha \wedge \beta)$;

(co2) $(\circ\alpha \vee \circ\beta) \to \circ(\alpha \vee \beta)$;

(co3) $(\circ\alpha \vee \circ\beta) \to \circ(\alpha \to \beta)$.

(ii) The logic **Cilo** is obtained by the addition to **Cio** of the axiom schema (cl) or, equivalently, by the addition of axioms (co1)–(co3) to **Cil** (see Definition 105). ■

The logic **Cilo**$^{©}$, the version of **Cilo** over signature Σ (using **Cil**$^{©}$ instead of **Cil**, see Theorem 106), was introduced in [Béziau, 1990] and was studied under the name C_1^+ in [da Costa *et al.*, 1995]. As in Definition 108, several other logics may be defined extending **Cio** by tinkering with axioms (cf), (cb) and (ce).

Obviously, C_1^+ is a deductive extension of C_1. Its characteristic weaker requirement to obtain consistency of a complex formula, namely, the consistency of at least one of its components, reflects in the following immediate stronger result:

THEOREM 115.
If $\Gamma \vdash_{\mathbf{Cio}} \circ\beta$ for some subformula β of α, then $\Gamma \vdash_{\mathbf{Cio}} \circ\alpha$. ■

An argument similar to the one presented in the proof of Theorem 113 will show again that the same t_3 defines also a conservative translation between **eCPL** and the logics presented in Definition 114.

On what concerns the interdefinability of the binary connectives with the help of our primitive paraconsistent negation (compare with Theorem 64), one can now count on the following extra rules:

THEOREM 116.
In **Cia** the following holds good:
(ix) $\neg(\neg\alpha \wedge \neg\beta) \vdash_{\mathbf{Cia}} (\alpha \vee \beta)$.
In **Cio** the following hold good:
(vi) $\neg(\alpha \wedge \neg\beta) \vdash_{\mathbf{Cio}} (\alpha \to \beta)$;
(vii) $\neg(\alpha \to \beta) \vdash_{\mathbf{Cio}} (\alpha \wedge \neg\beta)$;
(xi) $\neg(\neg\alpha \vee \neg\beta) \vdash_{\mathbf{Cio}} (\alpha \wedge \beta)$. ■

From Theorem 116(vii) and Theorem 52(ii) we can conclude that the replacement property (RP) (recall Remark 51) does not hold for any extension of **Cio**. However, a restricted form of this property may be recovered, in this specific case:

REMARK 117. Say that a logic **L** *allows for replacement with respect to* $\approx$ when $p_1 \approx p_2$ is a formula depending on the variables p_1 and p_2 such that, for every formula $\varphi(p_0, \ldots, p_n)$ and formulas $\alpha_0, \ldots, \alpha_n$, $\beta_0, \ldots, \beta_n$:

(RRP) $(\Vdash_{\mathbf{L}} \alpha_0 \approx \beta_0)$ and $\ldots$ and $(\Vdash_{\mathbf{L}} \alpha_n \approx \beta_n)$ implies
$$\Vdash_{\mathbf{L}} \varphi(\alpha_0, \ldots, \alpha_n) \approx \varphi(\beta_0, \ldots, \beta_n).$$

Any such formula, when it exists, will be called a *congruence* of **L**. Notice that, for our present logics, full replacement holds exactly when $\leftrightarrow$ is a congruence. ■

In the case of C_1 (and, not surprisingly, also of **Cia**), it has been shown in [Mortensen, 1980] that no congruence exists distinct from the 'trivial' one, namely, the identity between formulas. The situation is different though in the case of **Cio** and its deductive extensions:

THEOREM 118. A congruence in **Cio** can be defined by setting $\alpha \approx \beta \stackrel{\text{def}}{=} (\alpha \leftrightarrow \beta) \wedge (\circ\alpha \wedge \circ\beta)$.

Proof. A semantic proof for **Cilo** was offered in Theorem 3.21 of [da Costa *et al.*, 1995]. A similar argument, adapted for **Cio**, can be found in Fact 3.81 of [Carnielli and Marcos, 2002]. ■

On what concerns the semantic presentation of the above logics, the following Theorems 121 and 125 exhibit sufficient conditions for showing that several of the logics mentioned so far fail to be characterizable by finite-valued truth-tables.

The first widely applicable theorem on non-characterizability by finite-valued truth-tables proceeds as follows. Consider the signature Σ°. Recall from Definition 28 that α^1 denotes the formula $\neg(\alpha \wedge \neg\alpha)$ and α^{n+1} abbreviates the formula $\neg(\alpha^n \wedge \neg\alpha^n)$ for $n \geq 1$. Consider, additionally, $\alpha^0 \stackrel{\text{def}}{=} \alpha$ for every α in For°. Finally, set $\delta(m) \stackrel{\text{def}}{=} (\bigwedge_{0 \leq i < m} \delta^i) \to \delta^m$ for $\delta \in For^\circ$ and $m \geq 1$.

LEMMA 119. Any set $\mathcal{M}$ of n-valued truth-tables for which positive classical logic ($\mathbf{CPL}^+$) or some deductive extension thereof is sound must validate all formulas of the form $\delta(m)$, for $m > n$.

Proof. The case $n < 2$ is obvious, for then $\mathcal{M}$ must be an adequate set of truth-tables for the trivial logic. The other cases are easy consequences of the Pigeonhole Principle of finite combinatorics and of the cyclic character of the composition of finite functions. Indeed, if $\mathcal{M}$ is n-valued, for some finite n, the truth-table determined by a formula δ^n must be identical to the truth-table of at least one among the formulas $\delta^0, \ldots, \delta^{n-1}$. But in that case, using classical properties of conjunction and implication, it follows that $\delta(m)$, and consequently $\delta(m)$, is valid according to $\mathcal{M}$. ■

The above lemma can be found at [Avron, 2007b]. The next result comes from [Marcos, 2005f].

LEMMA 120. No formula of the form $\delta(m)$ is derivable in the logic **Ciae**.

Proof. Consider, for $n \in \mathbb{N}$, the following sets $\mathcal{M}_n$ of infinitary truth-tables that take the truth-values from the ordinal $\omega + 1 = \omega \cup \{\omega\}$, where ω (the set of natural numbers) is the only undesignated truth-value:

$$x \wedge y = \begin{cases} 0, & \text{if } x = n \text{ and } y = n+1 \\ \mathsf{max}(x, y), & \text{otherwise} \end{cases}$$

$$x \vee y = \mathsf{min}(x, y)$$

$$x \to y = \begin{cases} \omega, & \text{if } x \in \mathbb{N} \text{ and } y = \omega \\ y, & \text{if } x = \omega \text{ and } y \in \mathbb{N} \\ 0, & \text{if } x = \omega = y \\ \mathsf{max}(x, y), & \text{otherwise} \end{cases}$$

$$\neg x = \begin{cases} \omega, & \text{if } x = 0 \\ 0, & \text{if } x = \omega \\ x + 1, & \text{otherwise} \end{cases} \qquad \circ x = \begin{cases} 0, & \text{if } x \in \{0, \omega\} \\ \omega, & \text{otherwise} \end{cases}$$

It is clear, on the one hand, that **Ciae** is sound for each $\mathcal{M}_n$. On the other hand, $\mathcal{M}_{2m+1}$ falsifies the formula $\delta(m+1)$. Indeed, consider an atomic sentence p in the place of δ and consider a valuation v such that $v(p) = 1$. It follows then that $v(p^i) = 2i + 1$, for $0 \leq i \leq m$, yet $v(p^{m+1}) = \omega$. But in that case $v(\delta(m+1)) = ((2m+1) \to \omega) = \omega$. ■

THEOREM 121. No **LFI** lying in between $\mathbf{CPL}^+$ and **Ciae** is finite-valued.

Proof. Suppose that **L** is a logic defined over Σ° lying in between $\mathbf{CPL}^+$ and **Ciae** such that **L** has an adequate finite-valued truth-functional semantics with, say, m truth-values. By Lemma 119 the formula $\delta(m+1)$ is valid with respect to this semantics and so it is a theorem of **L**. But then $\delta(m+1)$ would be a theorem of **Ciae**, contradicting Lemma 120. ■

The previous result, albeit very general, does not cover cases of uncharacterizability by finite-valued truth-tables for logics satisfying the axiom (cl), for the truth-tables presented in Lemma 120 provide counter-models to this axiom. Here is, however, a similar argument that works fine in the latter case.

DEFINITION 122. Let $\mathbf{Cl}^-$ be the logic defined over the signature Σ° and obtained from **Cl** (see Remark 111) by removing axiom schemas (Ax10)–(Ax11). In other words, $\mathbf{Cl}^-$ is characterized by axiom schemas (Ax1)–(Ax9) (see Definition 28), (bc1) (see Definition 42), (cl) (see Definition 105), and the rule (MP). ■

Let δ_{ij}, for $i, j \neq 0$, denote the formula $\neg(p_i \wedge \neg p_j) \wedge (p_i \wedge \neg p_j)$, and let $\delta^{[n]}$ denote the disjunctive formula $\bigvee_{1 \leq i < j \leq n} (\delta_{ij} \to p_{n+1})$ for $n \geq 1$. Then:

LEMMA 123. Any set of n-valued truth-tables that is sound for the logic $\mathbf{Cl}^-$ must validate all formulas of the form $\delta^{[m]}$ for $m > n$.

Proof. Use the Pigeonhole Principle and the fact that

$$(\neg(\alpha \wedge \neg\alpha) \wedge (\alpha \wedge \neg\alpha)) \to \beta$$

may be derived from axioms (bc1), (cl) and the deduction theorem. ■

LEMMA 124. No formula of the form $\delta^{[n]}$ is derivable in the logic **Cilae**.

Proof. Use again the truth-tables in Lemma 120, but now simplify the table of conjunction as follows:

$$x \wedge y = \begin{cases} 0, & \text{if } y = x+1 \\ \mathsf{max}(x,y), & \text{otherwise} \end{cases}$$

It is routine to check that these truth-tables are sound for **Cilae**. Consider next a valuation v such that $v(p_i) = i$, for $i \leq n$, and $v(p_{n+1}) = \omega$. Then $v(\delta_{ij}) = j+2$ and so $v(\delta_{ij} \to p_{n+1}) = ((j+2) \to \omega) = \omega$ (for $1 \leq i < j \leq n$). Thus $v(\delta^{[n]}) = \omega$. ■

THEOREM 125. No **LFI** lying in between $\mathbf{Cl}^-$ and **Cilae** is finite-valued.

Proof. Analogous to the proof of Theorem 121, but now using formulas $\delta^{[n]}$, Lemma 123 and Lemma 124. ■

REMARK 126. A somewhat stronger version of Theorem 125 has recently been proven in [Avron, 2005b], where all logics in between $\mathbf{Cl}^-$ and **Cilae** are shown not to be characterizable even with the use of finite-valued non-deterministic truth-tables.

The logic **Cibae** (Definitions 108), an obvious extension of **Cila**, received an adequate interpretation in terms of possible-translations semantics in [Carnielli, 2000] and in [Marcos, 1999]. In the latter study, all the other logics from Definitions 108 and 114 have also received adequate possible-translations semantics. In [Avron, 2007a; Avron, 2005c; Avron, 2007b], even larger families of related logics have recently been given interpretations in terms of non-deterministic semantics, in a modular way. ■

We end this subsection with an axiomatization of two important 3-valued **LFI**s through the regulation of their ability to propagate inconsistency.

THEOREM 127. The logic **LFI1** described in Example 18 is axiomatized by adding to **Cie** (check Definition 100) the following axiom schemas:

(cj1) $\bullet(\alpha \wedge \beta) \leftrightarrow ((\bullet\alpha \wedge \beta) \vee (\bullet\beta \wedge \alpha))$

(cj2) $\bullet(\alpha \vee \beta) \leftrightarrow ((\bullet\alpha \wedge \neg\beta) \vee (\bullet\beta \wedge \neg\alpha))$

(cj3) $\bullet(\alpha \to \beta) \leftrightarrow (\alpha \wedge \bullet\beta)$

where, as usual, $\bullet\alpha$ is an abbreviation for $\neg\circ\alpha$. The logic $\mathbf{P}^1$ described in Example 19 is axiomatized by adding to **Ci** (check Definition 100) the following schema:

(cz) $\circ\alpha$ (for α non-atomic) ■

In the last theorem, note that (cz), in fact, consists of five axiom schemas, one for each connective in the signature Σ°, that is, (cz) is equivalent to the conjunction $\circ(\neg\alpha) \wedge \circ(\alpha \wedge \beta) \wedge \circ(\alpha \vee \beta) \wedge \circ(\alpha \to \beta) \wedge \circ(\circ\alpha)$. The logic $\mathbf{P}^1$ describes an extreme case of propagation of consistency into complex formulas, where no premises are needed so as to guarantee their consistency.

*5.3 **LFI**s that are maximal fragments of **CPL***

The paper [da Costa, 1974] suggested a list of 'natural' features that a paraconsistent logic should enjoy. One of these is that a paraconsistent logic should contain the most part of the schemas and rules of the classical propositional logic which do not interfere with paraconsistency. Following [Marcos, 2005d], one way of implementing this feature would be by requiring paraconsistent logics to be, in some specific sense, maximal deductive fragments of classical logic.

The following notion of maximality among logics may be used to analyze how close we are to having 'most of classical logic' inside paraconsistent systems:

DEFINITION 128. Let **L**1 and **L**2 be two logics written in the same signature. Then, **L**2 is said to be *maximal relative to* **L**1 if:
(i) **L**1 is an extension of **L**2;
(ii) if $\vdash_{\mathbf{L1}} \alpha$ but $\nvdash_{\mathbf{L2}} \alpha$, then the logic obtained from **L**2 by adding α as a new axiom schema coincides with **L**1.
When **L**1 is clear from the context, we simply say that a logic **L**2 satisfying conditions (i) and (ii) is *maximal*. ■

This notion of maximality is quite common in the literature.[10] It is well known, for instance, that each Łukasiewicz's logic $Ł_m$, for $m > 2$, is maximal relative to **CPL** if and only if $(m-1)$ is a prime number. Also, **CPL** is maximal relative to the trivial logic, a logic in which all formulas are provable. On the other hand it is also well known that intuitionistic logic is not a maximal fragment of **CPL**, and there exists indeed an infinite number of intermediate logics between them. On what concerns the main **C**-systems presented this far, only the logic **LFI1** and the logic $\mathbf{P}^1$, described in Examples 18 and 19, and Theorem 127, are maximal relative to **CPL**, or relative to **eCPL**, the extended version of **CPL** introduced at the beginning of Subsection 3.6. In particular, the logic C_1 (or, equivalently, **Cila**© — recall Remark 109), despite being the strongest logic introduced by da Costa on his first hierarchy of paraconsistent logics, is properly extended by $\mathbf{P}^1$

[10]Other notions of 'maximality' exist, such as the idea of defining maximal subsets of the classical entailment, considering not only valid formulas but valid inferences. That approach fails monotonicity, though, and the consequent 'maximal fragments' of classical logic do not define thus T-logics nor S-logics. We will make no development in the present paper in that direction, and choose rather to refer to the competent sources, such as [Batens, 1989] and [Batens, 1989].

and fails thus to be maximal with respect to classical logic. Therefore, none of the logics C_n presented in [da Costa, 1974] respects the requirement of containing the most part the schemas of classical logic, a requirement that may be found in that very same paper. Such an observation, in fact, is true also about the stronger logic called C_1^+ (or **Cilo©**), introduced after Definition 114.

Now we explore the intuitions underlying the 3-valued maximal **C**-systems $\mathbf{P}^1$ and **LFI1** showing how to generate a large class of related 3-valued maximal paraconsistent logics. Looking for models for contradictory and non-trivial theories, we start with non-trivial interpretations under which both some formula α and its negation $\neg\alpha$ would be simultaneously satisfied. A natural choice lies in the many-valued domain, more specifically in logics presented in terms of finite-valued truth-tables. Since we want to preserve classical theses as much as possible, the values of the connectives with classical (0 and 1) inputs will have classical outputs. Suppose we just introduce then an intermediate third value $\frac{1}{2}$, besides true (1) and false (0), fixing $D = \{1, \frac{1}{2}\}$ as the set of designated values. Then there are two possible classic-like truth-tables for a negation validating α and $\neg\alpha$ simultaneously, for some α, namely:

	$\neg$
1	0
$^1/_2$	$^1/_2$ or 1
0	1

With respect to the other connectives of the signature Σ (since we try to keep them as classical as possible), we add now the following higher-level classic-like requirements:

(C$\wedge$) $(x \wedge y) \in D$ iff $x \in D$ and $y \in D$;
(C$\vee$) $(x \vee y) \in D$ iff $x \in D$ or $y \in D$;
(C$\to$) $(x \to y) \in D$ iff $x \notin D$ or $y \in D$.

The above constraints leave us with the following options:

$\wedge$	1	$^1/_2$	0
1	1	$^1/_2$ or 1	0
$^1/_2$	$^1/_2$ or 1	$^1/_2$ or 1	0
0	0	0	0

$\vee$	1	$^1/_2$	0
1	1	$^1/_2$ or 1	1
$^1/_2$	$^1/_2$ or 1	$^1/_2$ or 1	$^1/_2$ or 1
0	1	$^1/_2$ or 1	0

$\to$	1	$^1/_2$	0
1	1	$^1/_2$ or 1	0
$^1/_2$	$^1/_2$ or 1	$^1/_2$ or 1	0
0	1	$^1/_2$ or 1	1

This yields 2^3 options for conjunctions, 2^5 options for disjunctions, 2^4 options for implications, and, as stated above, 2^1 options for negations, adding up to 2^{13} ($= 8,192$) possible logics to deal with, in the signature Σ. Of course, not all those logics are necessarily 'interesting'. We can upgrade each of those logics into an **LFI** by considering the signature $\Sigma^{\circ\bullet}$ and adding the following tables for consistency and inconsistency operators:

	$\circ$	$\bullet$
1	1	0
$^1/_2$	0	1
0	1	0

This means that the consistent models are the ones characterized by classical valuations, and only those. Notice that, in the above truth-tables, $\circ$ can be defined by setting $\circ\alpha \stackrel{\text{def}}{=} \neg\bullet\alpha$ or, alternatively, $\bullet$ can be defined by setting $\bullet\alpha \stackrel{\text{def}}{=} \neg\circ\alpha$.

DEFINITION 129. Fix Σ as any one among the signatures Σ°, $\Sigma^{\bullet}$ or $\Sigma^{\circ\bullet}$. The collection of logics over Σ defined by the above truth-tables, with designated values $D = \{1, \frac{1}{2}\}$, will be called $8Kb$. Each logic in this collection makes up a choice as to which truth-table for negation, for conjunction, for disjunction and for implication it will adopt. ■

Clearly, every logic in $8Kb$ is a fragment of **eCPL**, the extended classical propositional logic, if we consider in **eCPL** the usual definition of the inconsistency connective as the negation of the consistency connective. Note also that the logic *Pac* (see Example 17) does not belong to $8Kb$, because it cannot define the connectives $\circ$ and $\bullet$. On the other hand, its conservative extension **LFI1** contains those connectives, and as a matter of fact the latter logic belongs to $8Kb$. The 3-valued logic $\mathbf{P}^1$ also belongs to $8Kb$, and we already know that these two logics are axiomatizable by the addition of suitable axioms to the axiomatization of **Ci** (see Theorem 127). As shown in [Marcos, 2000], this same method may be extended to the whole $8Kb$:

THEOREM 130. (i) Every logic in $8Kb$ is an axiomatic extension of **Cia**.
(ii) All the logics in $8Kb$ are distinct from each other, and they are all maximal relative to **eCPL**.
(iii) All the logics in $8Kb$, and their fragments, are boldly paraconsistent. ■

It is just a combinatorial divertissement to check the following facts:

THEOREM 131. All the $8,192$ logics in $8Kb$ are **C**-systems based on **CPL** and extending **Cia** (cf. Definition 108). Out of these, $7,680$ are in fact **dC**-systems, being able to define $\circ$ and $\bullet$ in terms of the other connectives (all being, therefore, maximal relative to **CPL**, and not only to **eCPL**). Of these, $4,096$ are able to define $\circ\alpha$ as $\neg(\alpha \wedge \neg\alpha)$, and so all of them do extend C_1 (that is, **Cila**$^{©}$). Of the 7,680 logics which are **dC**-systems, 1,680 extend **Cio** (cf. Definition 114), and 980 of the latter are able to define $\circ\alpha$ as $\neg(\alpha \wedge \neg\alpha)$, and so all these 980 logics extend C_1^+ (that is, **Cilo**$^{©}$). ■

REMARK 132. The reader should bear in mind that, in view of Definition 27, if we want to prove that a given logic **L**2 is a **C**-system based on another logic **L**1, we might have to adjust its signature Σ_2 by adding definable connectives so as to guarantee that it will extend the signature Σ_1 of **L**1 (as it was done, for instance, in the proof of Theorem 44). In contrast to this, in view of Definition 128, if we want to prove that **L**2 is maximal relative to a logic **L**3, it might be necessary to adjust the signatures of both logics so that they coincide. Such signature adjustments are tacitly assumed in the statements of Theorems 130 and 131. So, in more practical terms, in order to prove that a given logic **L** in $8Kb$ is a **C**-system based on **CPL** we ought to add to its signature a new symbol for a (definable) classical negation. On the other hand, in order to prove that **L** is maximal relative to classical logic we had better assume in general that the latter logic is presented as **eCPL**, using the signature Σ° of Remark 15. In case **L** is a **dC**-system, then it will suffice to consider classical logic presented as **CPL**, and write **L** in the signature Σ, letting $\circ$ and $\bullet$ be introduced, in each case, by their circumstantial definitions. ■

The replacement property (RP) had already been shown to fail for our foremost logic samples from the $8Kb$. Indeed, the proof of items (iv) and (v) of Theorem 50 showed that both **LFI1** and $\mathbf{P}^1$ fail (RP). This negative feature may be generalized, as shown in [Marcos, 2000]:

THEOREM 133. (RP) cannot hold in any of the logics in $8Kb$.

Proof. This is true in general for any extension of **Cia**, as we may conclude from Theorem 81(ii) and Theorem 116(ix). To complete the proof, recall Theorem 130(i).

You will also be able to check the above result, alternatively, using the classical negation below, whose truth-table could already be found in Example 17 (check also Theorem 134), together with the result in Theorem 52(a)(i). ■

As a consequence of Theorem 133 the logics in $8Kb$ are not suitable to an algebraization by means of a direct Lindenbaum-Tarski-style procedure. However, the following results guarantee that all of them are algebraizable in the sense of Blok-Pigozzi (cf. [Blok and Pigozzi, 1989]).

THEOREM 134. Each one of the logics in $8Kb$ defines the following truth-table for classical negation and at least one of the two congruences below:

	$\sim$
1	0
$^1/_2$	0
0	1

$\equiv$	1	$^1/_2$	0
1	1	0	0
$^1/_2$	0	$^1/_2$ or 1	0
0	0	0	1

Proof. It is possible to define $\bot$ either as $(\alpha \wedge (\neg\alpha \wedge \circ\alpha))$ or as $(\circ\alpha \wedge \neg\circ\alpha)$, for any formula α. Then, we can define $\sim\alpha$ either as $(\neg\alpha \wedge \circ\alpha)$ or as $(\alpha \to \bot)$. One of the above congruences $(\alpha \equiv \beta)$ can always be defined by $((\alpha \leftrightarrow \beta) \wedge (\circ\alpha \leftrightarrow \circ\beta))$. In case we prefer to have $(\frac{1}{2} \equiv \frac{1}{2}) = 1$, we can assure that we define this specific congruence by setting $(\alpha \bowtie \beta) \stackrel{\text{def}}{=} \sim\sim(\alpha \equiv \beta)$. ■

The following theorem generalizes a result obtained in [Lewin *et al.*, 1990] for the logic $\mathbf{P}^1$:

THEOREM 135. All the logics in $8Kb$ are Blok-Pigozzi algebraizable.

Proof. Consider $\Delta(p_0, p_1) = \{(p_0 \equiv p_1)\}$ or $\Delta = \{(p_0 \bowtie p_1)\}$, where $\equiv$ and $\bowtie$ are defined as in the proof of the Theorem 134. Consider the sets

$$\delta(p_0) = \{((p_0 \to p_0) \to p_0)\}, \quad \varepsilon(p_0) = \{(p_0 \to p_0)\}$$

and check that the corresponding algebraizability conditions of [Blok and Pigozzi, 1989] are satisfied. ■

On what concerns the expressibility spectrum of the class $8Kb$ and of the distinguished logics $\mathbf{P}^1$ and **LFI1**, the following results can be checked:

THEOREM 136.
(i) The truth-tables of $\mathbf{P}^1$ can be defined inside of any of the logics in $8Kb$.
(ii) All the truth-tables in $8Kb$ can be defined inside of **LFI1**.

Proof. Item (i). Fix some logic $\mathbf{L}$ belonging to $8Kb$. Let $\wedge, \vee, \to, \neg, \circ$ and $\bullet$ be its primitive connectives, and let $\sim$ be the classical negation defined inside $\mathbf{L}$ as in Theorem 134. Then, the $\mathbf{P}^1$-negation of a formula α may be defined in $\mathbf{L}$ as $\sim\sim\neg\alpha$. The $\mathbf{P}^1$-conjunction of some given formulas α and β may be defined in $\mathbf{L}$ either as $\sim\sim(\alpha \wedge \beta)$ or as $(\sim\sim\alpha \wedge \sim\sim\beta)$. A definition in the same vein applies to both disjunction and implication. Note that the truth-tables in $\mathbf{L}$ for the connectives $\circ$ and $\bullet$ already coincide with those of $\mathbf{P}^1$.
Item (ii). A proof of this property may be found in [Avron, 1999]. A constructive proof may be found in [Marcos, 1999] and [Carnielli *et al.*, 2000]. ■

COROLLARY 137. (i) The logic $\mathbf{P}^1$ can be conservatively translated into any of the logics in $8Kb$. (ii) Any of the logics in $8Kb$ can be conservatively translated into **LFI1**. ∎

As argued in [Avron, 1991], the logic **LFI1** has several properties that justify its role as one of the most 'natural' 3-valued paraconsistent logics. Theorem 136(ii) and Corollary 137(ii) show already how linguistically and deductively expressive this logic is.

A last note on algebraization. We had the chance in several occasions above to witness how replacement fails for many of our **LFI**s. This often makes it difficult to provide algebraic counterparts, in the usual sense, for those logics. However, it is interesting to observe that a kind of algebraic treatment for some wilder **C**-systems has been proposed and studied, for instance, in [Carnielli and de Alcantara, 1984] and [Seoane and de Alcantara, 1991] (for a partial survey, check the section 3.12 of [Carnielli and Marcos, 2002]). Additionally, an approach for algebraizing **LFI**s based on an idea similar to that of a possible-translations structure was presented in [Bueno-Soler *et al.*, 2004] and [Bueno-Soler and Carnielli, 2005].

6 CONCLUSIONS AND FURTHER PERSPECTIVES

In this final part of this chapter we recall some definitions and results obtained and described above, and point to some interesting new problems and research directions connected to what has been presented.

From Section 3 on, some of the possibilities for the formalization and understanding of the relationship between the concepts of consistency, inconsistency, contradictoriness and triviality were explored at a very general and abstract level. Assuming that consistency could be expressed inside some paraconsistent logics, and assuming furthermore that the consistency of a given formula would legitimate its explosive character (that is, assuming (9), a so-called Gentle Principle of Explosion), we have presented in Subsection 3.1 a general definition of a Logic of Formal Inconsistency, **LFI** (Definition 23). To actualize that definition (in a finitary way), we have started our study from the logic **mbC**, a very weak **C**-system based on classical logic (recall Definition 42), constructing all the remaining **C**-systems as extensions of **mbC**. Some specific extensions of **mbC** illustrated a subclass of the **C**-systems in which the connectives '$\circ$' for consistency and '$\bullet$' for inconsistency are expressible by means of other connectives. The members of this class were called **dC**-systems (recall Definition 32).

We briefly recall some consequences of our approach to formal (in)consistency: There are consistent and inconsistent logics. The inconsistent ones may be either paraconsistent or trivial, but not both. Let us say that a theory *has non-trivial models* only if these models do not assign designated values to all formulas. Thus, the theories of a consistent logic have non-trivial models if and only if they are non-contradictory. Paraconsistent logics will tipically have non-trivial models for some of their contradictory theories. Paraconsistent logics may even have some trivial models among those models that satisfy contradictions. Such trivial models, however, cannot exist if the paraconsistent logics we are talking about are gently explosive, that is, if they constitute Logics of Formal Inconsistency. For each formula α of a logic $\mathbf{L}$, the consistency $\circ\alpha$ of α consists in the information that should be added to an α-contradictory theory in order to make it explosive, and consequently trivial. If the answer is 'nothing needs to be added', then α is already consistent in $\mathbf{L}$. This implies that, as expected, a logic is consistent if all of its formulas may be asserted to be consistent.

It will be clear now to the reader that there are many more examples of **C**-systems besides the logics C_n of da Costa and other logics axiomatized in a more or less similar fashion. The general idea is to express consistency and inconsistency inside a logic, at its object-language level. This approach allows us to collect in a single class of **LFI**s logics as diverse as the C_n, $\mathbf{P}^1$, $\mathbf{J}_3$ (renamed **LFI1**), and Jaśkowski's 'discussive' paraconsistent logic **D2** (cf. Example 24). Even normal modal logics in a convenient signature can be very naturally regarded as **dC**-systems. This bears on the relationship between negations and modalities, which reflects upon the possibilities of defining paraconsistent negations in modal environments, as studied by [Vakarelov, 1989], [Došen, 1986], [Béziau, 2002b], [Marcos, 2005e] and [Marcos, 2005b].

The fact that so many logics with diverse motivations and technical features may be recast as a **dC**-systems paves the way for an interesting question: To check whether other logics in the literature on paraconsistent logics could be characterized as **C**-systems, or, in general, as **LFI**s. Another related question is the following: How to enrich a given paraconsistent logic in order to turn it into an **LFI**? This was done by the logic **LFI1** (also known as **CLuNs**, or $\mathbf{J}_3$) with respect to the logic *Pac* (see Example 18). Consider now the 3-valued *closed set logic* studied in [Mortensen, 1995]. This logic consists of **LFI1**'s truth-tables of conjunction and of disjunction, plus the truth-table of negation of $\mathbf{P}^1$, where 0 is the only non-designated value. A consistency connective $\circ$ can then be defined via $\circ\alpha \stackrel{\text{def}}{=} \neg\neg(\alpha \vee \neg\alpha)$. The addition of an appropriate truth-table for implication would enrich the closed-set logic, and the resulting system would most certainly belong to the collection $8Kb$ of 3-valued maximal paraconsistent logics (recall Definition 129). But in that case, what would be the topological or set-theoretical significance of these new connectives?

The question of the duality between intuitionistic-like and paraconsistent logics, not explored in this chapter, is also worth mentioning. The concept of dual-intuitionism was already seized in the 40s by K. Popper, cf. [Popper, 1948], more or less at the same time as paraconsistency was being engendered. More recently, dual-intuitionism and dual-paraconsistency have been studied, for example, in [Sylvan, 1990], [Urbas, 1996] and [Brunner and Carnielli, 2005]. The logics that are dual to paraconsistent are sometimes called 'paracomplete' (cf. [Loparić and da Costa, 1984]). Exploring the issue of duality, a natural question that appears concerns the notions that are dual to consistency and inconsistency, notions that one might dub 'determinedness' and 'undeterminedness'. Some initial explorations in that direction, and the related *Logics of Formal Undeterminedness*, may be found in [Marcos, 2005e].

Apparently, in the 40s, defenders of dual-intuitionism and paraconsistency independently realized that there should be a logic for general reasoning from hypotheses, accepting in certain cases some propositions and their negations as true (in the case of paraconsistency), or retaining some propositions and their negations as unfalsified (in the case of falsificationism). Indeed, there seems to be some common grounds connecting paraconsistency and the falsificationist program in Philosophy of Science, and that line of research seems worth pursuing. Similarly, paracomplete logics could have a contribution to make for the study of verificationism in science. The logical approach to such questions has recently been vindicated by studies such as [Shramko, 2005].

Applications of **LFI**s to yet other fields in philosophy seem promising. In [Costa-Leite, 2003] some possibilities of employing the connectives of consistency and inconsistency for the understanding of (and new regards on) epistemological problems related to the paradox of knowability are investigated. In [Marcos, 2005a] the use of a consistency-like modal connective for the modelling of the metaphysical notion of essence is tackled, and in that environment inconsistency turns out to mean a mere sort of 'accident'.

Another important issue concerns the incompleteness results in Arithmetic. Recall that Gödel's incompleteness theorems are based on the identification of 'consistency' and 'non-contradictoriness'. What would be the consequences if we started instead from the general notion of consistency hereby proposed (recall Definition 4)? Would it still be possible to reproduce Gödel's arguments? Quite possibly, his arguments would be rescued at the cost of assuming consistency (in our sense) of several formulas representing assumptions that would then become more explicit, and consequently open to debate. In the same spirit, it should be interesting to analyze the combination of **LFI**s with Modal Logics of Provability. In [Boolos, 1996], consistency is intended as a kind of opposite to the notion of provability. Using this idea, if the negation of a formula cannot be proven, then it is consistent with whatever else might be proven; a still weaker notion, connected to 'logical independence', would be to consider a formula to be consistent when neither this formula nor its negation can be proven. The insinuated exchange between Logics of Formal Inconsistency and Logics of Provability, in fact, seems attractive and deserves further research.

As it has been noted in the literature, it seems that most interesting problems related to paraconsistency appear already at the propositional level. It is possible though to extend a given propositional paraconsistent logic to higher orders using combination techniques such as fibring, if only we choose the right abstraction level to express our logics. See, for instance, [Caleiro and Marcos, 2001], where the logic C_1 is given a first-order version which coincides with the original one from [da Costa, 1963]. Another interesting possibility that involves first-order versions of paraconsistent logics in general, and especially of first-order **LFI**s, is the investigation of consistent yet ω-inconsistent theories (also related to Gödel's theorems).

Some other items for future research, already hinted at along the present text, are the following. From Theorem 79 we know that, in extensions of **mCi**, the formulas causing controllable explosion (Definition 9(ii)) coincide with the provably consistent formulas, that is, theorems of the form $\circ\alpha$. On the other hand, **mbC** does not have provably consistent formulas (see Theorem 47). So, is the logic **mbC** (see Definition 42) *not* controllably explosive? On another trail, we have seen that there are extensions of **mbC** for which the replacement property holds good (see Remark 53), and we have seen that to find extensions of **mCi** with that same property all one needs to do is to devise logics that respect a certain rule (EC) (see Subsection 3.2 and Theorem 82). Can we circumvent negative results such as Theorems 52 and 81 and find interesting extensions of **mCi** enjoying the replacement property (RP)? At any rate, turning the attention to extensions of **mbC** that do not extend **mCi** but that do enjoy (RP) is a feasible enterprise (recall Remark 53), and it seems indeed to be a very attractive one, still to be further developed. On yet another direction, what other uses could we give to our semantical tools (valuations and possible-translations semantics)? The results about uncharacterizability by finite-valued truth-tables in Theorems 121 and 125 are very powerful and widely applicable, but they cannot help us in proving that logics such as **Cioe** do not have adequate finite-valued truth-tables. Can we find other flexible and wide-ranging similar results to the same effect?[11]

Finally, we have started our work in this chapter from a traditional abstract perspective. We have soon though shown that alternative semantical and proof-theoretical approaches were possible. In particular, we have given a few illustrations of a general method that permits us to deal with **C**-systems in terms of tableaux. The first wide-ranging method to such an effect was sketched in [Carnielli and Marcos, 2001b]. A more general method to obtain tableau procedures for logics endowed with a certain type of two-valued (even non-truth-functional) semantics was introduced in [Caleiro *et al.*, 2005b]. These techniques have been used here in Subsections 3.5 and 4.2 so as to obtain new adequate tableau systems for the logic C_1, as well as for

[11] It came to our notice that the problem concerning **Cioe** has recently been solved in [Avron, 2007b], where in fact all logics in between $\mathbf{Cl}^-$ and **Ciboe** are shown not to be characterizable with the use of finite-valued non-deterministic truth-tables.

mbC and **mCi**. The possibility of further exploring and refining this kind of approach seems promising for applications of **LFI**s in database theory (see Example 18), an area of research critically sensible to the presence of contradictions.

7 LIST OF AXIOMS AND SYSTEMS

We list here all the main principles, axioms and systems studied throughout the chapter, indicating the place where they were introduced in the text.

Principles

(1) Principle of Non-Contradiction : Subsection 2.1
(2) Principle of Non-Triviality : Subsection 2.1
(3) Principle of Explosion, or *Pseudo-Scotus*, or *Ex Contradictione Sequitur Quodlibet* : Subsection 2.1
(4) Paraconsistent logic (first definition) : Subsection 2.2
(5) Paraconsistent logic (second definition) : Subsection 2.2
(6) Paraconsistent logic (third definition) : Subsection 2.2
(7) Principle of *Ex Falso Sequitur Quodlibet* : Subsection 2.2
(8) Supplementing Principle of Explosion : Subsection 2.2
(9) Gentle Principle of Explosion : Subsection 3.1
(10) Finite Gentle Principle of Explosion : Subsection 3.1

Axioms, Rules and Metaproperties

(Ax1)–(Ax11) : Definition 28
(bc1) : Definition 28, Definition 42
(bc1)$'$: Subsection 4.3
(bc1)$''$: Theorem 106
(ca1)–(ca3) : Definition 28, Definition 108
(cb) : Subsection 5.1
$(\mathrm{cc})_n$: Definition 75
$(\mathrm{cc})'_n$: Subsection 4.3
(cd) : Subsection 5.1
(ce) : Subsection 4.4
(cf) (= (Ax11)) : Subsection 4.4
(ci) : Definition 75
(ci)$'$: Subsection 4.3
(cj1)–(cj3) : Theorem 127
(cl) : Definition 105
(co1)–(co3): Definition 114
(Con1)–(Con6) : Subsection 2.1
(cz) : Theorem 127
(EC) : Subsection 3.2

Systems

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