Possible-translations semantics for some weak classically-based paraconsistent logics

João Marcos

Department of Informatics and Applied Mathematics, UFRN, BR

Security and Quantum Information Group, IT, PT

jmarcos@dimap.ufrn.br

ABSTRACT. *In many real-life applications of logic it is useful to interpret a particular sentence as true together with its negation. If we are talking about classical logic, this situation would force all other sentences to be equally interpreted as true. Paraconsistent logics are exactly those logics that escape this explosive effect of the presence of inconsistencies and allow for sensible reasoning still to take effect. To provide reasonably intuitive semantics for paraconsistent logics has traditionally proven to be a challenge. Possible-translations semantics can meet that challenge by allowing for each interpretation to be composed of multiple scenarios. Using that idea, a logic with a complex semantic behavior can be understood as an appropriate combination of ingredient logics with simpler semantic behaviors into which the original logic is given a collection of translations preserving its soundness. Completeness is then achieved through the judicious choice of the admissible translating mappings. The present note provides interpretation by way of possible-translations semantics for a group of fundamental paraconsistent logics extending the positive fragment of classical propositional logic. The logics PI, C_{min}, $\mathbf{mbC}$, $\mathbf{bC}$, $\mathbf{mCi}$ and $\mathbf{Ci}$, among others, are all initially presented through their non-truth-functional bivaluation semantics and sequent versions and then split by way of possible-translations semantics based on 3-valued ingredients.*

KEYWORDS: *possible-translations semantics, paraconsistent logics, truth-functionality.*

DOI:10.3166/JANCL.18.7–28

1. Languages, bivaluations, and sequents

Let $\mathcal{P} = \{p_0, p_1, \ldots, p_m, \ldots\}$ be a denumerable set of sentential letters, and consider the following sets of formulas:

$$\mathcal{S}_0 = \langle \mathcal{P}, \{\}, \{\}, \{\wedge, \vee, \supset\}\rangle \qquad \mathcal{S}_2 = \langle \mathcal{P}, \{\}, \{\sim, \circ\}, \{\wedge, \vee, \supset\}\rangle$$
$$\mathcal{S}_1 = \langle \mathcal{P}, \{\}, \{\sim\}, \{\wedge, \vee, \supset\}\rangle \qquad \mathcal{S}_3 = \langle \mathcal{P}, \{\}, \{\sim, \circ, \bullet\}, \{\wedge, \vee, \supset\}\rangle$$

where $\sim$ ('negation'), $\circ$ ('consistency'), $\bullet$ ('inconsistency') are unary connective symbols, and $\wedge$ ('conjunction'), $\vee$ ('disjunction') and $\supset$ ('implication') are binary connective symbols. As usual, the connective $\equiv$ ('bi-implication') is defined by considering $\varphi \equiv \psi$ as an abbreviation for $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$. Outermost parentheses are omitted whenever there is no risk of confusion.

A mapping $b : \mathcal{S}_i \longrightarrow \{0, 1\}$ is called a bivaluation over $\mathcal{S}_i$. One can easily write some possible axioms governing the set of admissible bivaluations:

(b1.1) $b(\varphi \wedge \psi) = 1 \;\Rightarrow\; b(\varphi) = 1$ and $b(\psi) = 1$
(b1.1$^\mathbf{c}$) $b(\varphi \wedge \psi) = 0 \;\Rightarrow\; b(\varphi) = 0$ or $b(\psi) = 0$
(b1.2) $b(\varphi \vee \psi) = 1 \;\Rightarrow\; b(\varphi) = 1$ or $b(\psi) = 1$
(b1.2$^\mathbf{c}$) $b(\varphi \vee \psi) = 0 \;\Rightarrow\; b(\varphi) = 0$ and $b(\psi) = 0$
(b1.3) $b(\varphi \supset \psi) = 1 \;\Rightarrow\;$ if $b(\varphi) = 1$ then $b(\psi) = 1$
(b1.3$^\mathbf{c}$) $b(\varphi \supset \psi) = 0 \;\Rightarrow\; b(\varphi) = 1$ and $b(\psi) = 0$
(b2) $b({\sim}\varphi) = 0 \;\Rightarrow\; b(\varphi) = 1$
(b3) $b(\circ\varphi) = 1 \;\Rightarrow\; b(\varphi) = 0$ or $b({\sim}\varphi) = 0$
(b3$^\mathbf{c}$) $b(\circ\varphi) = 0 \;\Rightarrow\; b(\varphi) = 1$ and $b({\sim}\varphi) = 1$
(b4) $b({\sim}\circ\varphi) = 1 \;\Rightarrow\; b(\varphi) = 1$ and $b({\sim}\varphi) = 1$
(b5.n) $b(\circ{\sim}^n\circ\varphi) = 1$, given $n \in \mathbb{N}$
(b6) $b({\sim}{\sim}\varphi) = 1 \;\Rightarrow\; b(\varphi) = 1$
(b6$^\mathbf{c}$) $b({\sim}{\sim}\varphi) = 0 \;\Rightarrow\; b(\varphi) = 0$

where ${\sim}^0\varphi \stackrel{\text{def}}{=} \varphi$ and ${\sim}^{n+1}\varphi \stackrel{\text{def}}{=} {\sim}{\sim}^n\varphi$.

The converse of (b4) clearly follows from (b2) and (b3), and the latter two bivaluational axioms are to be respected by most logics we will consider below. Moreover, the reader will surely have noticed the difference between (b4) and (b3$^\mathbf{c}$), the converse of (b3):

FACT 1. — In the presence of (b2), axiom (b3$^\mathbf{c}$) can be derived from (b4). The axiom (b4) can be derived from (b3$^\mathbf{c}$) in the presence of (b3) and (b5.0). □

All the above axioms are in 'dyadic form' (cf. (Caleiro *et al.*, 2005)). As shown in (Béziau, 2001), there is a canonical method for extracting from any such bivaluational axiom a corresponding sequent rule. This results in the following:

(s1.1) $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$
(s1.1$^\mathbf{c}$) $\varphi, \psi \vdash \varphi \wedge \psi$
(s1.2) $\varphi \vee \psi \vdash \varphi, \psi$
(s1.2$^\mathbf{c}$) $\varphi \vdash \varphi \vee \psi$ and $\psi \vdash \varphi \vee \psi$
(s1.3) $\varphi \supset \psi, \varphi \vdash \psi$
(s1.3$^\mathbf{c}$) $\vdash \varphi, \varphi \supset \psi$ and $\psi \vdash \varphi \supset \psi$
(s2) $\vdash \varphi, {\sim}\varphi$
(s3) $\circ\varphi, \varphi, {\sim}\varphi \vdash$
(s3$^\mathbf{c}$) $\vdash \circ\varphi, \varphi$ and $\vdash \circ\varphi, {\sim}\varphi$
(s4) ${\sim}\circ\varphi \vdash \varphi$ and ${\sim}\circ\varphi \vdash {\sim}\varphi$

(s5.n) $\vdash \circ{\sim}^n\circ\varphi$, given $n \in \mathbb{N}$
(s6) ${\sim}{\sim}\varphi \vdash \varphi$
(s6$^\mathbf{c}$) $\varphi \vdash {\sim}{\sim}\varphi$

For the sake of legibility, the side contexts of the above rules were dropped. Any subset of those rules, together with reflexivity, weakening, cut, and the usual structural rules, determines a specific sequent system. We will write $\alpha \dashv\vdash \beta$ as an abbreviation for ($\alpha \vdash \beta$ and $\beta \vdash \alpha$).

The following is a straightforward byproduct of the above:

FACT 2. — Rule (s5.0) is derivable with the help of (s2), (s3) and (s4). Rules (s5.n), for $n > 0$, are all derivable in the presence of (s3), (s4), (s5.0) and (s6). □

2. Some fundamental paraconsistent logics

Let CL^+ denote the positive fragment of classical propositional logic, built over the set of formulas $\mathcal{S}_0$, axiomatized by way of the rules (s1.X) and interpreted through the set of all bivaluations respecting the axioms (b1.X).

The very weak paraconsistent logic PI (cf. (Batens, 1980)) is built over $\mathcal{S}_1$ simply by adding (s2) to the rules of CL^+ or (b2) to its bivaluational axioms. The full classical propositional logic, CL, could be obtained now from PI over $\mathcal{S}_1$ by adding

(b2$^\mathbf{c}$) $b({\sim}\varphi) = 1 \;\Rightarrow\; b(\varphi) = 0$

to the bivaluational axioms of PI, or, equivalently, by adding

(s2$^\mathbf{c}$) $\varphi, {\sim}\varphi \vdash$

to PI's sequent rules. The bivaluational axioms (b2) and (b2$^\mathbf{c}$) together are thus sufficient for interpreting classical negation in isolation from the other connectives, and the sequent rules (s2) and (s2$^\mathbf{c}$) can be seen as the pure characterizing rules of classical negation.

A fundamental logic of formal inconsistency (cf. (Carnielli *et al.*, 2002)) called $\mathbf{mbC}$ is built next over $\mathcal{S}_2$ by adding (s3) to the rules of PI or, equivalently, by adding (b3) to its bivaluational axioms. A 0-ary connective $\bot$ (‘bottom’), characterized semantically by setting $b(\bot) = 0$, can be defined in $\mathbf{mbC}$ if one takes $\bot \stackrel{\text{def}}{=} \circ\psi \wedge (\psi \wedge {\sim}\psi)$, for any fixed formula ψ. As a byproduct:

FACT 3. — A classical negation $\neg$ can be defined in $\mathbf{mbC}$ by setting $\neg\varphi \stackrel{\text{def}}{=} \varphi \supset \bot$. □

The logic $\mathbf{mbC}$, as presented above, had only a primitive consistency connective $\circ$ but no primitive connective for inconsistency. The latter can nonetheless be defined in $\mathbf{mbC}$ if one just sets $\bullet\varphi \stackrel{\text{def}}{=} {\sim}\circ\varphi$. This way one could in fact rebuild $\mathbf{mbC}$ over $\mathcal{S}_3$, if that need be. (But, on that matter, be sure to check Note 7 and the references therein.)

An important extension of $\mathbf{mbC}$ is the logic $\mathbf{mCi}$, again built over $\mathcal{S}_2$, but now by adding (s4) and (s5.n), $n \in \mathbb{N}$, to the rules of $\mathbf{mbC}$, or (b4) and (b5.n), $n \in \mathbb{N}$, to its bivaluational axioms. The fundamental trait of $\mathbf{mCi}$ is the classical behavior of its consistency connective $\circ$ with respect to the primitive negation $\sim$:

FACT 4. — In $\mathbf{mCi}$:
(i) $b({\sim}{\circ}\alpha) = b(\neg{\circ}\alpha)$,
(ii) $b({\sim}^n{\circ}\alpha) = 1 \;\Leftrightarrow\; b({\sim}^{n+1}{\circ}\alpha) = 0$. □

As a particular consequence, the above mentioned inconsistency connective $\bullet$, in $\mathbf{mCi}$, may in a sense be seen as 'dual' to the consistency connective $\circ$ from the point of view of the paraconsistent negation $\sim$. Indeed:

FACT 5. — In $\mathbf{mCi}$, ${\circ}\alpha \dashv\vdash {\sim}{\bullet}\alpha$. □

Let $\psi[p]$ denote a formula ψ having p as one of its atomic components, and let $\psi[p/\gamma]$ denote the formula obtained from ψ by uniformly substituting all occurrences of p by the formula γ. Given a pair of formulas α and β, we say that they are *logically indistinguishable* if for every formula $\varphi[p]$ we have that $\varphi[p/\alpha] \dashv\vdash \varphi[p/\beta]$. Algebraically, this will mean that α and β will have the 'same reference', and belong thus to the same congruence class. In terms of a bivaluation semantics, this will mean that $b(\varphi[p/\alpha]) = b(\varphi[p/\beta])$, for any formula φ. By the very definition of $\bullet$ we know that the formulas ${\bullet}\alpha$ and ${\sim}{\circ}\alpha$ are logically indistinguishable. However, in spite of the equivalence between the formulas ${\circ}\alpha$ and ${\sim}{\bullet}\alpha$ mentioned in the last fact, such formulas are not logically indistinguishable inside the logics studied in the present paper. We will use our possible-translations tool to check this feature in Example 33, further on.

The logics PIf, $\mathbf{bC}$ and $\mathbf{Ci}$ extend, respectively, the logics PI, $\mathbf{mbC}$ and $\mathbf{mCi}$, by the addition of the bivaluational axiom (b6) or, equivalently, of the sequent rule (s6). The logic PIf appears in ch.4 of (Marcos, 1999) and then at (Carnielli *et al.*, 1999) under the appellation C_{min}. Both $\mathbf{bC}$ and $\mathbf{Ci}$, as well as an enormous number of their extensions, are studied in close detail at (Carnielli *et al.*, 2002). The logic $\mathbf{mCi}$ is suggested at the final section of the latter paper, but axiomatized here for the first time. This logic, together with $\mathbf{mbC}$, constitute the most fundamental logics explored in detail in (Carnielli *et al.*, 2007). Inaccuracies in the axiomatization (as introduced in (Carnielli *et al.*, 2002)) and in the bivaluation semantics (as presented in (Carnielli *et al.*, 2001a; Carnielli *et al.*, 2001b)) of the logic $\mathbf{Ci}$ are also fixed at (Carnielli *et al.*, 2007).

In a similar vein, the logics $PIfe$, $\mathbf{bCe}$ and $\mathbf{Cie}$ can here be introduced as extensions of the previous logics obtained by the further addition of the bivaluational axiom (b6$^{\mathbf{c}}$) or, equivalently, of the sequent rule (s6$^{\mathbf{c}}$). In the light of the preceding facts and comments, it might seem natural that $\mathbf{mCi}$, $\mathbf{Ci}$, and $\mathbf{Cie}$ would from this point on be built instead directly over the extended set of formulas $\mathcal{S}_3$, where $\bullet$ could be introduced by a definition using $\sim$ and $\circ$, as above. For the exact extent in which it does make sense to talk about these logics as if they were pretty much the same if presented using either $\mathcal{S}_2$ or $\mathcal{S}_3$, we advise the reader to check the section 4.3 of (Carnielli *et al.*, 2007).

To summarize the 9 previously mentioned paraconsistent logics:

PI	formulas: $\mathcal{S}_1$ sequent rules: (s1.X) and (s2) axioms on bivaluations: (b1.X) and (b2)
$\mathbf{mbC}$	formulas: $\mathcal{S}_2$ sequent rules: as in PI, plus (s3) axioms on bivaluations: as in PI, plus (b3)
$\mathbf{mCi}$	formulas: $\mathcal{S}_3$ sequent rules: as in $\mathbf{mbC}$, plus (s4) and (s5.n), $n \in \mathbb{N}$ axioms on bivaluations: as in $\mathbf{mbC}$, plus (b4) and (b5.n), $n \in \mathbb{N}$
PIf	formulas: $\mathcal{S}_1$ sequent rules: as in PI, plus (s6) axioms on bivaluations: as in PI, plus (b6) (a.k.a. C_{min})
$\mathbf{bC}$	formulas: $\mathcal{S}_2$ sequent rules: as in PIf, plus (s3) axioms on bivaluations: as in PIf, plus (b3)
$\mathbf{Ci}$	formulas: $\mathcal{S}_3$ sequent rules: as in $\mathbf{bC}$, plus (s4) axioms on bivaluations: as in $\mathbf{bC}$, plus (b4)
$PIfe$	formulas: $\mathcal{S}_1$ sequent rules: as in PIf, plus (s6$^{\mathbf{c}}$) axioms on bivaluations: as in PIf, plus (b6$^{\mathbf{c}}$)
$\mathbf{bCe}$	formulas: $\mathcal{S}_2$ sequent rules: as in $\mathbf{bC}$, plus (s6$^{\mathbf{c}}$) axioms on bivaluations: as in $\mathbf{bC}$, plus (b6$^{\mathbf{c}}$)
$\mathbf{Cie}$	formulas: $\mathcal{S}_3$ sequent rules: as in $\mathbf{Ci}$, plus (s6$^{\mathbf{c}}$) axioms on bivaluations: as in $\mathbf{Ci}$, plus (b6$^{\mathbf{c}}$)

The simplification in the rules and axioms of $\mathbf{Ci}$, as compared to those of $\mathbf{mCi}$, is sanctioned by the results in Fact 2.

For a quick scan, one can find in Figure 1 a schematic illustration displaying the relationships between the 9 paraconsistent logics above. An arrow $\mathcal{L}1 \longrightarrow \mathcal{L}2$ indicates that the logic $\mathcal{L}1$ is (properly) extended by the logic $\mathcal{L}2$.

3. Bivalued entailment, modalities and matrices

Fixed any of the logics presented in the above section, let biv be its set of admissible bivaluations. Given $b \in \mathsf{biv}$, let $\Gamma \vDash_b \Delta$ hold good, for given sets of formulas Γ and Δ, iff $(\exists\gamma \in \Gamma)b(\gamma) = 0$ or $(\exists\delta \in \Delta)b(\delta) = 1$. The canonical *entailment* relation $\vDash_{\mathsf{biv}}$ is defined as usual: $\Gamma \vDash_{\mathsf{biv}} \Delta$ iff $\Gamma \vDash_b \Delta$ for every $b \in \mathsf{biv}$. Moreover,

$$\begin{tikzcd}
\mathbf{mCi} \arrow[r] & \mathbf{Ci} \arrow[r] & \mathbf{Cie} \\
\mathbf{mbC} \arrow[r] \arrow[u] & \mathbf{bC} \arrow[r] \arrow[u] & \mathbf{bCe} \arrow[u] \\
PI \arrow[r] \arrow[u] & PIf \arrow[r] \arrow[u] & PIfe \arrow[u]
\end{tikzcd}$$

Figure 1. *Some fundamental paraconsistent logics*

given a set of sequent rules seq, let $\vdash_{\mathsf{seq}}$ denote the *derivability* relation defined by its canonical notion of (multiple-conclusion) proof-from-premises. Entailment and derivability relations are examples of *consequence* relations. Given any consequence relation $\rhd$ associated to a logic $\mathcal{L}$, we will write $\Gamma \not\rhd \Delta$ to say that the inference $\Gamma \rhd \Delta$ fails according to $\mathcal{L}$, and we will write $\alpha \lhd\!\rhd \beta$ to say that both $\alpha \rhd \beta$ and $\beta \rhd \alpha$ hold good in $\mathcal{L}$.

Can the 9 above paraconsistent logics be given semantics that are more informative than their respective bivaluation semantics? Good question. It should be remarked for instance that those logics cannot be endowed with usual *modal-like semantics*. Indeed, all of them fail the *replacement property*, a property that is typical of normal modal systems:

THEOREM 6. — In any of the logics from Figure 1, $\dashv\vdash$ does not constitute a congruence relation over the set of formulas, that is, there are formulas α and β such that $\alpha \dashv\vdash \beta$, but $\sim\alpha \nvdash \sim\beta$.

PROOF. — Consider the 3-valued matrices of the logic **LFI1**, at Table 1, where F is the only undesignated truth-value.

Table 1. *Matrices of the logic* **LFI1**

$\wedge$	T	t	F
T	T	t	F
t	t	t	F
F	F	F	F

$\vee$	T	t	F
T	T	T	T
t	T	t	t
F	T	t	F

$\supset$	T	t	F
T	T	t	F
t	T	t	F
F	T	T	T

	$\sim$	$\circ$
T	F	T
t	t	F
F	T	T

It is easy to check that **LFI1** (properly) extends all the above paraconsistent logics — it constitutes in fact a maximally paraconsistent extension of those logics (cf. (Marcos, 1999; Carnielli *et al*., 2000)). Nevertheless, in **LFI1**, while tautologies such as $(p \vee \sim p)$ and $(q \vee \sim q)$ are equivalent, the formulas $\sim(p \vee \sim p)$ and $\sim(q \vee \sim q)$ are not equivalent: To see that, consider any 3-valued valuation that assigns the value t to the atomic sentence p, while q is assigned a different value. ■

NOTE 7 (A SEEMING PARADOX). — The logic of formal inconsistency $\mathbf{mbC}$ (and any of its non-trivial paraconsistent extensions) can be seen both as a conservative extension and as a deductive fragment of classical logic, CL. Indeed, for the first assertion, recall the set of formulas $\mathcal{S}_0$ of positive classical logic (Section 1), and consider now the sets of formulas:

$$\mathcal{S}_4 = \langle \mathcal{P}, \{\}, \{\neg\}, \{\wedge, \vee, \supset\}\rangle \quad \mathcal{S}_5 = \langle \mathcal{P}, \{\}, \{\neg, \sim, \circ\}, \{\wedge, \vee, \supset\}\rangle$$

Interpret the connectives from $\mathcal{S}_4$ as in CL, using the bivaluational axioms (b1.X) and (b2.X) (let in the latter axiom $\neg$ take the place of $\sim$). Interpret the new connectives in $\mathcal{S}_5$ as in $\mathbf{mbC}$, using the bivaluational axioms (b2) and (b3). It is clear that this last move provides just a new way of presenting $\mathbf{mbC}$. Indeed, as we have seen in Fact 3, a classical negation $\neg$ can be defined from the original presentation of $\mathbf{mbC}$. Consider again the matrices of $\mathbf{LFI1}$, from Table 1, a logic that deductively extends $\mathbf{mbC}$. The classical negation $\neg$ in $\mathbf{LFI1}$, defined as above, would be such that $v(\neg\varphi) = T$ if $v(\varphi) = F$, and $v(\neg\varphi) = F$ otherwise. It is easy to see, in that case, that the matrices of $\sim$ and $\circ$, the new connectives of $\mathcal{S}_5$ cannot be defined, in $\mathbf{LFI1}$, from the matrices of the connectives in $\mathcal{S}_4$. If you recall now that CL is a maximal logic, then you have concluded the proof that $\mathbf{mbC}$ can be seen as a (proper) conservative extension of CL. For the second assertion, consider CL to be written in the language of $\mathcal{S}_5$. Recall that classical logic is presupposed consistent, and interpret the connective $\circ$ accordingly, by taking as axiom $b(\circ\varphi) = 1$. Based on the received idea that there is just 'one true classical negation', interpret $\neg$ and $\sim$ both using axioms (b2) and (b2$^{\mathbf{c}}$). In that case $\mathbf{mbC}$ is clearly characterized as a (proper) deductive fragment of CL. Notice that this is, however, a very peculiar fragment of CL —it is a fragment into which all classical reasoning can be internalized by way of a definitional translation. For further details on those translations and their general significance for paraconsistent logics, check (Carnielli *et al.*, 2007). □

NOTE 8 (MORE ON INTERNALIZING STRONGER LOGICS). — Not only can $\mathbf{mbC}$ faithfully internalize classical logic, but it can also internalize the reasoning of other logics of formal inconsistency that are deductively stronger than itself. To see that, consider now the following sets of formulas:

$$\mathcal{S}_6 = \langle \mathcal{P}, \{\bot\}, \{\}, \{\wedge, \vee, \supset\}\rangle$$
$$\mathcal{S}_7 = \langle \mathcal{P}, \{\bot\}, \{\sim\}, \{\wedge, \vee, \supset\}\rangle$$
$$\mathcal{S}_8 = \langle \mathcal{P}, \{\bot\}, \{\sim, \circ\}, \{\wedge, \vee, \supset\}\rangle$$

Interpret the 0-ary connective ('bottom') from $\mathcal{S}_6$ by taking as axiom $b(\bot) = 0$, and interpret the new connectives from $\mathcal{S}_7$ and $\mathcal{S}_8$ as in $\mathbf{mbC}$. Again, this provides just another presentation for $\mathbf{mbC}$, as we have seen in Section 1 that $\bot$ is definable in this logic. On the other hand, a new consistency connective strictly stronger than $\circ$ can be defined using the connectives from $\mathcal{S}_7$. Indeed, as in (Carnielli *et al.*, 2002), consider a connective $\tilde{\circ}$ defined by setting $\tilde{\circ}\varphi \stackrel{\text{def}}{=} (\varphi \supset \bot) \vee (\sim\varphi \supset \bot)$ (or, equivalently, $\tilde{\circ}\varphi \stackrel{\text{def}}{=} \neg\varphi \vee \neg{\sim}\varphi$). This connective is naturally characterizable by axiom (b3) and its converse (b3$^{\mathbf{c}}$), while the original consistency connective of $\mathbf{mbC}$ was characterized by axiom (b3) alone. If you recall Fact 1 you will notice that the last definition determines a logic of formal inconsistency that lies right in between $\mathbf{mbC}$ and $\mathbf{mCi}$. As a matter of fact, this approach provides one way of presenting the logic $\mathbf{CLuN}$, the preferred logic of adaptive logicians (cf. (Batens, 2000)), often used as the lower limit logic of their inconsistency-adaptive systems. Though the first presentations of $\mathbf{CLuN}$ made this logic coincide with PI, it has been more recently presented as a conservative extension of PI obtained by adding a bottom connective to the language of the latter, as in $\mathcal{S}_7$ above. If one writes the whole thing in the language of $\mathcal{S}_8$, using the above defined consistency connective, $\mathbf{CLuN}$ is very naturally recast thus as a logic of formal inconsistency that lies in between $\mathbf{mbC}$ and $\mathbf{mCi}$. (The full details concerning this assertion will be discussed elsewhere. It suffices to say here that the truth of the assertion itself is guaranteed by the completeness results presented in Section 5.) □

PROBLEM 9. — Is there a definitional translation of $\mathbf{mCi}$ into $\mathbf{mbC}$? Can the logic $\mathbf{mbC}$ faithfully internalize in some way the reasoning of $\mathbf{mCi}$? □

NOTE 10 (OTHER LOGICS EXTENDING $\mathbf{mbC}$ BUT NOT $\mathbf{mCi}$). — Besides $\mathbf{CLuN}$, there are many other interesting logics of formal inconsistency that extend $\mathbf{mbC}$ but do not go through $\mathbf{mCi}$. There is even a large class of such logics that satisfies the full replacement property. I have shown in (Marcos, 2005b; Marcos, 2005a), in fact, that any non-degenerate normal modal logic can be easily recast as a logic of formal inconsistency extending $\mathbf{CLuN}$ (and thus extending $\mathbf{mbC}$), but not $\mathbf{mCi}$. □

Before the diversion provided by the above set of notes, we had seen in Theorem 6 that the 9 paraconsistent logics from the last section cannot be endowed with usual modal-like semantics. The reader might now be wondering whether those logics would still stand some chance at least of being *truth-functional*, should they turn out themselves to be characterizable by way of some convenient set of finite-valued matrices (just like their extension $\mathbf{LFI1}$). However, some widely applicable negative results concerning that possibility can be promptly checked as follows. For the first result, let α^1 abbreviate the formula $\sim(\alpha \wedge \sim\alpha)$ and α^{n+1} abbreviate the formula $\sim(\alpha^n \wedge \sim\alpha^n)$ for $n \geq 1$. Consider, additionally, $\alpha^0 \stackrel{\text{def}}{=} \alpha$ for every α in For°. Finally, set $\delta(m) \stackrel{\text{def}}{=} (\bigwedge_{0\leq i<m} \delta^i) \rightarrow \delta^m$ for $m \geq 1$. Then, as shown in (Avron, 2007b), the following holds good:

LEMMA 11. — Any collection $\mathcal{J}$ of n-valued truth-tables for which positive classical logic (CL^+), or some deductive extension thereof, is sound must validate all formulas of the form $\delta(m)$, for $m > n$.

PROOF. — The case $n < 2$ is obvious. The other cases are easy consequences of the Pigeonhole Principle of finite combinatorics and of the cyclic character of the composition of finite functions. Indeed, if $\mathcal{J}$ is n-valued, for some finite n, the truth-table determined by a formula δ^n must be identical to the truth-table of at least one among the formulas $\delta^0, \ldots, \delta^{n-1}$. But in that case, using classical properties of conjunction and implication, it follows that any $\delta(m)$, for $m > n$, is valid according to $\mathcal{J}$. ■

Consider now the following result:

LEMMA 12. — No formula of the form $\delta(m)$ is derivable in the logic **Cie**.

PROOF. — Consider, for $n \in \mathbb{N}$, the following sets $\mathcal{J}_n$ of infinitary truth-tables that take the truth-values from the ordinal $\omega + 1 = \omega \cup \{\omega\}$, where ω (the set of natural numbers) is the only undesignated truth-value:

$$x \wedge y = \begin{cases} 0, & \text{if } x = n \text{ and } y = n+1 \\ \mathsf{max}(x, y), & \text{otherwise} \end{cases}$$

$$x \vee y = \mathsf{min}(x, y)$$

$$x \rightarrow y = \begin{cases} \omega, & \text{if } x \in \mathbb{N} \text{ and } y = \omega \\ y, & \text{if } x = \omega \text{ and } y \in \mathbb{N} \\ 0, & \text{if } x = \omega = y \\ \mathsf{max}(x, y), & \text{otherwise} \end{cases}$$

$$\sim x = \begin{cases} \omega, & \text{if } x = 0 \\ 0, & \text{if } x = \omega \\ x + 1, & \text{otherwise} \end{cases} \qquad \circ x = \begin{cases} 0, & \text{if } x \in \{0, \omega\} \\ \omega, & \text{otherwise} \end{cases}$$

It is clear, on the one hand, that all the sequent rules from Section 1 are validated by the above matrices. On the other hand, $\mathcal{J}_{2m+1}$ falsifies the formula $\delta(m+1)$. Indeed, let δ be an atomic sentence p and consider a valuation v such that $v(p) = 1$. It follows then that $v(p^i) = 2i + 1$, for $0 \leq i \leq m$, yet $v(p^{m+1}) = \omega$. But in that case $v(\delta(m+1)) = ((2m+1) \rightarrow \omega) = \omega$. ■

Using the previous lemmas one can now check that:

THEOREM 13 (UNCHARACTERIZABILITY BY FINITE MATRICES, *version* I). — No logic that is written in the language of $\mathcal{S}_3$ (with • introduced by definition, as above) and that is a fragment of **Cie** is finite-valued.

PROOF. — Suppose that the logic **L** is some fragment of **Cie** written over $\mathcal{S}_3$ such that **L** has an adequate finite-valued truth-functional semantics with, say, m truth-values. By Lemma 11 the formula $\delta(m+1)$ is valid with respect to this semantics and so it is a theorem of **L**. But then $\delta(m+1)$ would be a theorem of **Cie**, contradicting Lemma 12. ■

Notice that this theorem covers in particular all the logics from the first two lines of Figure 1. Now, a second negative result can be seen to cover another part of the same figure.

LEMMA 14. — No sequent of the form $\vdash \sim^i\varphi \equiv \sim^j\varphi$ is derivable, for non-negative $i \neq j$, in logics from the first two columns of Figure 1.

PROOF. — Consider a set of infinite-valued matrices that take the natural numbers $\mathbb{N}$ as truth-values, where 0 is the only undesignated truth-value. Define the matrices for the connectives as follows:

$$v(\varphi \wedge \psi) = \begin{cases} 1, & \text{if } v(\varphi) > 0 \text{ and } v(\psi) > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$v(\varphi \vee \psi) = \begin{cases} 1, & \text{if } v(\varphi) > 0 \text{ or } v(\psi) > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$v(\varphi \supset \psi) = \begin{cases} 0, & \text{if } v(\varphi) > 0 \text{ and } v(\psi) = 0 \\ 1, & \text{otherwise} \end{cases}$$

$$v({\sim}\varphi) = \begin{cases} 1, & \text{if } v(\varphi) = 0 \\ v(\varphi) - 1, & \text{otherwise} \end{cases} \qquad v(\circ\varphi) = \begin{cases} 0, & \text{if } v(\varphi) > 1 \\ 1, & \text{otherwise} \end{cases}$$

It is easy to check that all the sequent rules from Section 1 are validated by the above matrices, with the sole exception of (s6$^{\mathbf{c}}$). At the same time, the above matrices can also easily be seen to invalidate all sequents of the form $\vdash {\sim}^i\varphi \equiv {\sim}^j\varphi$, for non-negative $i \neq j$. ■

THEOREM 15 (UNCHARACTERIZABILITY BY FINITE MATRICES, *version* II). — None of the logics from the first two columns of Figure 1 (i.e., the fragments of **Ci**) is finite-valued.

PROOF. — If any of these logics were characterized by matrices with only m truth-values, then, by the Pigeonhole Principle, we would have, fixing an arbitrary $i \in \mathbb{N}$, that some $i < j \leq (i + m^m)$ would be such that $v({\sim}^i p) = v({\sim}^j p)$, for all v. This would in turn validate some sequent of the form $\vdash {\sim}^j\varphi \equiv {\sim}^i\varphi$, for $i < j$. ■

The last theorem and its preceding auxiliary lemma correct and extend in fact a result suggested long ago, in (Arruda, 1975), following the lines of a proposal originally made in (Marcos, 1999).

One logic from Figure 1, however, was not covered by the previous results. Accordingly, the following is here left open:

PROBLEM 16. — Find a proof similar to the above ones to show that $PIfe$ is not characterizable by finite matrices. □

The next section will show how these same logics, while not characterizable neither by way of finite matrices nor by way of standard modal semantics, as we have seen, can as a matter of fact be all perfectly characterized by way of suitable splicing of finite-valued scenarios.

4. Interpretations through possible translations

We will see in this section that all the paraconsistent logics in Figure 1 can still be given adequate interpretations in terms of *combinations* of 3-valued logics, by way of specific possible-translations semantics (PTS). Consider the 3-valued matrices of $\mathcal{M}$, at Table 2), where F is the only undesignated truth-value. Notice in particular that the underlying language of $\mathcal{M}$ has 3 different primitive symbols for negation and 3 different symbols for the consistency connective.

Table 2. *Matrices of* $\mathcal{M}$

$\wedge$	T	t	F
T	t	t	F
t	t	t	F
F	F	F	F

$\vee$	T	t	F
T	t	t	t
t	t	t	t
F	t	t	F

$\supset$	T	t	F
T	t	t	F
t	t	t	F
F	t	t	t

	$\sim_1$	$\sim_2$	$\sim_3$
T	F	F	F
t	F	t	t
F	T	t	T

	$\circ_1$	$\circ_2$	$\circ_3$
T	T	t	F
t	F	F	F
F	T	t	F

Given a 3-valued assignment $a : \mathcal{P} \longrightarrow \{T, t, F\}$, let w be its unique homomorphic extension into the whole language of $\mathcal{M}$, and let $\Gamma \vDash_w \Delta$ hold good, for given sets of formulas Γ and Δ, iff $(\exists\gamma \in \Gamma)w(\gamma) = F$ or $(\exists\delta \in \Delta)w(\delta) \in \{T, t\}$. Then, the canonical (multiple-conclusion) entailment relation $\vDash_{\mathcal{M}}$ determined by the above 3-valued matrices is set by taking $\Gamma \vDash_{\mathcal{M}} \Delta$ iff $\Gamma \vDash_w \Delta$ for every interpretation $w \in \mathcal{M}$.

Consider next the following possible restrictions over the set of admissible translating mappings $* : \mathcal{S}_i \longrightarrow \mathcal{M}$:

(tr0) $p^* = p$, for $p \in \mathcal{P}$
(tr1) $(\varphi \bowtie \psi)^* = (\varphi^* \bowtie \psi^*)$, for $\bowtie \in \{\wedge, \vee, \supset\}$
(tr2.1) $(\sim\varphi)^* \in \{\sim_1\varphi^*, \sim_2\varphi^*\}$
(tr2.2) $(\sim\varphi)^* \in \{\sim_1\varphi^*, \sim_3\varphi^*\}$
(tr2.3) $(\sim^{n+1}\circ\varphi)^* = \sim_1(\sim^n\circ\varphi)^*$
(tr3.1) $(\circ\varphi)^* \in \{\circ_2\varphi^*, \circ_3\varphi^*, \circ_2(\sim\varphi)^*, \circ_3(\sim\varphi)^*\}$
(tr3.2) $(\circ\varphi)^* \in \{\circ_1\varphi^*, \circ_1(\sim\varphi)^*\}$
(tr3.3) if $(\sim\varphi)^* = \sim_1\varphi^*$ then $(\circ\varphi)^* = \circ_1(\sim\varphi)^*$
(tr4) if $(\sim\varphi)^* = \sim_3\varphi^*$ then $(\sim\sim\varphi)^* = \sim_3(\sim\varphi)^*$

One can now select appropriate sets of restrictions in order to split each of the paraconsistent logics from the last section by way of PTS:

Logic	Restrictions over the translating mappings
PI	(tr0), (tr1), (tr2.1)
$\mathbf{mbC}$	(tr0), (tr1), (tr2.1), (tr3.1)
$\mathbf{mCi}$	(tr0), (tr1), (tr2.1), (tr2.3), (tr3.2), (tr3.3)
PIf	(tr0), (tr1), (tr2.2)
$\mathbf{bC}$	(tr0), (tr1), (tr2.2), (tr3.1)
$\mathbf{Ci}$	(tr0), (tr1), (tr2.2), (tr3.2), (tr3.3)
$PIfe$	(tr0), (tr1), (tr2.2), (tr4)
$\mathbf{bCe}$	(tr0), (tr1), (tr2.2), (tr3.1), (tr4)
$\mathbf{Cie}$	(tr0), (tr1), (tr2.2), (tr3.2), (tr3.3), (tr4)

Let Tr denote some set of translating mappings defined according to an appropriate subset of the previously mentioned restrictions. Define a pt-*model* as a pair $\langle w, *\rangle$, where $* \in \mathsf{Tr}$ and $w \in \mathcal{M}$, and let $\Gamma \Vdash^*_w \Delta$ hold good, for given sets of formulas Γ and Δ, iff $\Gamma^* \vDash_w \Delta^*$. A pt-*consequence relation* $\Vdash_{\mathsf{pt}}$ is then set by taking $\Gamma \Vdash_{\mathsf{pt}} \Delta$ iff $\Gamma \Vdash^*_w \Delta$ for every pt-model $\langle w, *\rangle$ admitted by Tr. Equivalently, in the cases presently under consideration, $\Gamma \Vdash_{\mathsf{pt}} \Delta$ also means, more simply, that $\Gamma^* \vDash_{\mathcal{M}} \Delta^*$, for every admissible translation $* \in \mathsf{Tr}$.

NOTE 17 (THE DEVELOPMENT OF PTS). — A logic $\mathcal{L}$ is said to have a *possible-translations semantics* when it can be given an adequate interpretation in terms of pt-models as above, for some appropriate set of translating mappings. Each translation can then be seen as a sort of interpretation scenario for $\mathcal{L}$. This intuition is good enough for the purposes of the present paper, but the possible-translations tool is in fact more general than that. For a generous and clear formal definition of these semantic structures, check (Marcos, 2004). For other more specific and carefully explained examples, check (Marcos, 1999; Carnielli *et al.*, 1999; Carnielli, 2000). The interested reader will notice that the PTS offered for $\mathbf{Ci}$ above is distinct from the one presented in (Carnielli *et al.*, 2001a). Possible-translations semantics were first introduced in (Carnielli, 1990), restricted to the splitting of a logic into finite-valued truth-functional scenarios. The embryo was then frozen for a period, and in between 1997 and 1998 it was publicized under the denomination 'non-deterministic semantics', in (Carnielli *et al.*, 1997), and in several talks by Carnielli and a few by myself. Noticing that the non-deterministic element was but a particular accessory of the more general picture, from 1999 on the semantics returned to its earlier denomination, bearing the qualifier 'possible-translations'. More recently, in chapter 9 of (Carnielli *et al.*, 2008), possible-translations semantics have been presented as one of the main tools for the analysis of complex logics through 'splitting' them into simpler components. □

NOTE 18 (PTS AND NON-DETERMINISTIC SEMANTICS). — PTS are related to (but are more general than) the *non-deterministic semantics* (NDS) proposed by Avron & Lev (cf. (Avron *et al.*, 2005b)) in ways that are still to be more carefully clarified. On what concerns the logics studied in the present paper, it should be noticed that (Avron *et al.*, 2005a) proposes a 2-valued NDS for PI, (Avron, 2007a) proposes a 3-valued NDS for PIf which is an alternative to the PTS presented for this logic above (and that comes from (Marcos, 1999; Carnielli *et al.*, 1999)), and (Avron *et al.*, 2005b) illustrates the cases of both PI and PIf. Moreover, (Avron, 2005) offers 3-valued NDS also for the logics **mbC**, **bC**, **bCe**. Roughly speaking, in the light of a classification put forward in (Avron *et al.*, 2005a), one could say that *dynamic* NDS are based on clauses having the same format of (tr0)–(tr2.2), and *static* NDS additionally impose constraints having the format of (tr2.3) or (tr4) for each of the involved connectives. There is a mechanical way, thus, to move from a given NDS to an equivalent PTS. Further discussion of that issue shall be postponed to a future study. □

We now have a number of quite diverse consequence relations associated to each of the above logics. Of course we want to keep this fauna under control — in the best of all possible worlds we want to be able to prove that all those consequence relations deliver just the same the result, for each given logic, that is, we want to prove that:

$$\vdash_{\mathsf{seq}} = \vDash_{\mathsf{biv}} = \Vdash_{\mathsf{pt}}$$

That is the subject of the next, and final, section.

5. Adequacy of each of the newly proposed PTS

As mentioned in Section 1, the technology that solves the first part of our problem is well-known, and its outcome will here be taken for granted: $\vdash_{\mathsf{seq}} = \vDash_{\mathsf{biv}}$.

Now, to check soundness of each of the paraconsistent logics in section 2 with respect to its specific PTS in section 4, one has two alternatives from the start. The first is to prove it directly from the axiomatizations in section 1 and the appropriate sets of translating mappings:

THEOREM 19 (SOUNDNESS). — $\vdash_{\mathsf{seq}} \subseteq \Vdash_{\mathsf{pt}}$.

PROOF. — Just translate each sequent axiom in all possible ways allowed by Tr and check that these translations are validated by $\mathcal{M}$. ■

The second alternative is to prove that each pt-model is bisimulated by some appropriate bivaluation:

THEOREM 20 (CONVENIENCE). —

$$(\forall w \in \mathcal{M})(\forall * \in \mathsf{Tr})(\exists b \in \mathsf{biv})\ \vDash_b \alpha \ \Leftrightarrow\ \Vdash^*_w \alpha.$$

PROOF. — Define a total bivaluation by setting the condition ($\Diamond$): $b(\alpha) = 0$ iff $w(\alpha^*) = F$, for any formula α (and $b(\alpha) = 1$ otherwise). Then check that the axioms in biv are all respected, in each case.

The strategy is pretty much mechanical, thus we will not delve into the details of the 'convenience' results for our present logics. Just consider, by way of an illustration, the case of PI and its bivaluational axiom (b2). Given $b(\sim\varphi) = 0$, and considering ($\Diamond$), we must be talking about a situation in which $w((\sim\varphi)^*) = F$. From (tr2.1), a restriction on the set of admissible translating mappings that characterize PI, we know, however, that $(\sim\varphi)^* \in \{\sim_1\varphi^*, \sim_2\varphi^*\}$. Thus, from the matrices of $\mathcal{M}$ we will conclude that $w(\varphi^*) \in \{T, t\}$. In that case, considering again ($\Diamond$) we must be talking about a situation in which $b(\alpha) = 1$, exactly in accordance with (b2). The other bivaluational axioms are all, in each case, verified using a similar strategy. ■

COROLLARY 21 (SOUNDNESS AGAIN). — $\vDash_{\mathsf{biv}} \subseteq \Vdash_{\mathsf{pt}}$.

Now for completeness. Given that the evaluation of the consistency connective, $\circ$, in the way we have defined it, takes into account the evaluation of the negation connective, $\sim$, it will be helpful, when doing some of the next proofs by induction on the complexity of the formulas, to make use of the following non-canonical measure of complexity, $\mathbf{mc}$:

(mc0) $\mathbf{mc}(p) = 0$, for $p \in \mathcal{P}$
(mc1) $\mathbf{mc}(\varphi \bowtie \psi) = \mathsf{max}(\mathbf{mc}(\varphi), \mathbf{mc}(\psi)) + 1$, for $\bowtie \in \{\wedge, \vee, \supset\}$
(mc2) $\mathbf{mc}(\sim\varphi) = \mathbf{mc}(\varphi) + 1$
(mc3) $\mathbf{mc}(\circ\varphi) = \mathbf{mc}(\sim\varphi) + 1$

With such apparatus in hands, we can start looking for a proof that each particular bivaluation is bisimulated by some appropriate pt-model:

THEOREM 22 (REPRESENTABILITY). —

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})\ \Vdash^*_w \alpha \ \Leftrightarrow\ \vDash_b \alpha.$$

From what it would easily follow that:

COROLLARY 23 (COMPLETENESS). — $\vDash_{\mathsf{biv}} \supseteq \Vdash_{\mathsf{pt}}$.

With respect to the above mentioned representability result, still to be proven, the safest strategy at this point seems to be that of checking it for each of our paraconsistent logics on its own turn, refining the statements and proofs to better suit each case. So, here we go:

THEOREM 24 (PI-REPRESENTABILITY). —

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})$$
$$w(\alpha^*) = t \ \Leftrightarrow\ b(\alpha) = 1, \textit{ and}$$
$$w(\alpha^*) = F \ \Leftrightarrow\ b(\alpha) = 0.$$

PROOF. — To take care of w, set, for $p \in \mathcal{P}$:

(rw) $a(p) = F$ if $b(p) = 0$, and
$a(p) = t$ otherwise

and extend a into w homomorphically, according to the strictures of $\mathcal{M}$.

On what concerns $*$, make the following choices on the translating mappings:

(rt0) $p^* = p$, for $p \in \mathcal{P}$
(rt1) $(\varphi \bowtie \psi)^* = (\varphi^* \bowtie \psi^*)$, for $\bowtie \in \{\wedge, \vee, \supset\}$
(rt2) $(\sim\varphi)^* = \sim_1\varphi^*$, if $b(\sim\varphi) = 0$
$(\sim\varphi)^* = \sim_2\varphi^*$, otherwise

Notice that these choices are indeed allowed by the restrictions (tr0), (tr1) and (tr2.1) that characterize the admissible translating mappings of PI.

The main statement above can now easily be proven by induction on the complexity measure $\mathbf{mc}$.

The atomic case follows immediately from (rw) and (rt0). As induction hypothesis, (IH), assume that both (A) $w(\alpha^*) = t \;\Leftrightarrow\; b(\alpha) = 1$ *and* (B) $w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0$ are indeed the case for any formula α with $\mathbf{mc}(\alpha) \leq k$, for some given k, and consider in turn the case of formulas immediately more complex than α, obtained by adding a further constructor from $\mathcal{S}_1$. For the case of formulas containing an extra binary connective, the result easily follows from (rt1), using the (IH). Consider now in detail the case of a formula of the form $\sim\alpha$:

– Part (A)

- ($\Rightarrow$) Suppose $w((\sim\alpha)^*) = t$. Then, by the matrices of $\mathcal{M}$ and (rt2), we must be talking about $(\sim\alpha)^* = \sim_2\alpha^*$ and $b(\sim\alpha) = 1$.

- ($\Leftarrow$) Suppose $b(\sim\alpha) = 1$. By (rt2), we have $(\sim\alpha)^* = \sim_2\alpha^*$. Now, suppose, on the one hand, that $b(\alpha) = 1$. By part (A) of the (IH), $w(\alpha^*) = t$, and from the matrices of $\mathcal{M}$ it follows that $w((\sim\alpha)^*) = t$. Suppose, on the other hand, that $b(\alpha) = 0$. Then, by part (B) of the (IH), $w(\alpha^*) = F$. Again, this means that $w((\sim\alpha)^*) = t$.

– Part (B)

- ($\Rightarrow$) Suppose $b(\sim\alpha) = 1$. Then, by (rt2), $(\sim\alpha)^* = \sim_2\alpha^*$. Suppose, on the one hand, that $b(\alpha) = 1$. By part (A) of the (IH), $w(\alpha^*) = t$. So, $w((\sim\alpha)^*) = t \neq F$. Suppose, on the other hand, that $b(\alpha) = 0$. By part (B) of the (IH), this means that $w(\alpha^*) = F$. But in that case we must have $w((\sim\alpha)^*) \neq F$.

- ($\Leftarrow$) Suppose $b(\sim\alpha) = 0$. Then, by (rt2), $(\sim\alpha)^* = \sim_1\alpha^*$. By the bivaluational axiom (b2) it also follows that $b(\alpha) = 1$. Thus, by part (A) of the (IH), $w(\alpha^*) = t$. So, $w((\sim\alpha)^*) = F$.

That completes the inductive step. ■

THEOREM 25 ($\mathbf{mbC}$-REPRESENTABILITY). —

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})$$
$$w(\alpha^*) = T \;\Rightarrow\; b(\sim\alpha) = 0, \textit{ and}$$
$$w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.$$

PROOF. — To take care of w, set, for $p \in \mathcal{P}$:

(rw) $a(p) = F$ if $b(p) = 0$,
$a(p) = T$ if $b(\sim p) = 0$, and
$a(p) = t$ otherwise

and extend a into w homomorphically, according to the strictures of $\mathcal{M}$.

On what concerns $*$, set:

(rt0) $p^* = p$, for $p \in \mathcal{P}$
(rt1) $(\varphi \bowtie \psi)^* = (\varphi^* \bowtie \psi^*)$, for $\bowtie \in \{\wedge, \vee, \supset\}$
(rt2) $({\sim}\varphi)^* = {\sim}_1\varphi^*$, if $b({\sim}\varphi) = 0$ or $b(\varphi) = 0 = b({\sim}{\sim}\varphi)$
$({\sim}\varphi)^* = {\sim}_2\varphi^*$, otherwise
(rt3) $(\circ\varphi)^* = \circ_3\varphi^*$, if $b(\circ\varphi) = 0$
$(\circ\varphi)^* = \circ_2({\sim}\varphi)^*$, if $b(\circ\varphi) = 1$ and $b({\sim}\varphi) = 0$
$(\circ\varphi)^* = \circ_2\varphi^*$, otherwise

Once more, the above choices do not go against the restrictions (tr0), (tr1), (tr2.1) and (tr3.1) that characterize the admissible translating mappings of $\mathbf{mbC}$.

The result is again proven by induction on $\mathbf{mc}$. The atomic case is checked as before. The induction hypothesis, (IH), now assumes that (A) $w(\alpha^*) = T \Rightarrow b({\sim}\alpha) = 0$ *and* (B) $w(\alpha^*) = F \Leftrightarrow b(\alpha) = 0$, for any formula α with $\mathbf{mc}(\alpha) \leq k$, for some given k. The case of binary connectives is straightforward. We will check in detail the cases of formulas of the forms ${\sim}\alpha$ or $\circ\alpha$.

Case of ${\sim}\alpha$:

– Part (A). Suppose $w(({\sim}\alpha)^*) = T$. By (rt2) and $\mathcal{M}$, we must have $({\sim}\alpha)^* = {\sim}_1\alpha^*$, and so $w(\alpha^*) = F$. Then, by part (B) of the (IH), $b(\alpha) = 0$. In view of the bivaluational axiom (b2), using (rt2) again, and given that $({\sim}\alpha)^* = {\sim}_1\alpha^*$, it now follows that $b({\sim}{\sim}\alpha) = 0$.
– Part (B)
- ($\Rightarrow$) Suppose $b({\sim}\alpha) = 1$. On the one hand, suppose further that $b(\alpha) = 0$. By part (B) of the (IH), $w(\alpha^*) = F$, so, in any case allowed by (rt2), we have $w(({\sim}\alpha)^*) \neq F$. On the other hand, suppose now that $b(\alpha) = 1$. In that case, by (rt2), we must have $({\sim}\alpha)^* = {\sim}_2\alpha^*$. By part (B) of the (IH), $b(\alpha) = 1$ implies that $w(\alpha^*) \neq F$, and by part (A) of the (IH), $b({\sim}\alpha) = 1$ implies that $w(\alpha^*) \neq T$. Thus, we must have $w(\alpha^*) = t$, from what it follows, given that $({\sim}\alpha)^* = {\sim}_2\alpha^*$, that $w(({\sim}\alpha)^*) = t$, and so in fact $w(({\sim}\alpha)^*) \neq F$.
- ($\Leftarrow$) Suppose $b({\sim}\alpha) = 0$. By (rt2), $({\sim}\alpha)^* = {\sim}_1\alpha^*$. By the bivaluational axiom (b2), $b(\alpha) = 1$, and by part (B) of the (IH), it follows that $w(\alpha^*) \neq F$. This is enough information to conclude that $w(({\sim}\alpha)^*) = F$.

Case of $\circ\alpha$:

– Part (A). Immediate, for $w(\circ\alpha)^* = T$ is impossible, in view of (rt3) and the matrices of $\mathcal{M}$.
– Part (B)
- ($\Rightarrow$) Suppose $b(\circ\alpha) = 1$. By the bivaluational axiom (b3), this means that either $b(\alpha) = 0$ or $b({\sim}\alpha) = 0$ (but not both, in view of axiom (b2)). If, on the one hand, $b(\alpha) = 0$, it follows by part (B) of the (IH) that $w(\alpha^*) = F$, and, by (rt3), $(\circ\alpha)^* = \circ_2\alpha^*$. But then we have $w((\circ\alpha)^*) = t \neq F$. If, on the other hand, $b({\sim}\alpha) = 0$, one might recall the condition (mc3) on the definition of the complexity measure $\mathbf{mc}$, according to which $\mathbf{mc}(\circ\alpha) > \mathbf{mc}({\sim}\alpha)$, and conclude by part (B) of the (IH) that $w(({\sim}\alpha)^*) = F$. Further, by (rt3), we must have $(\circ\alpha)^* = \circ_2({\sim}\alpha)^*$, thus $w((\circ\alpha)^*) = w(\circ_2({\sim}\alpha)^*) = t \neq F$.

- ($\Leftarrow$) Suppose $b(\circ\alpha) = 0$. By (rt3), $(\circ\alpha)^* = \circ_3\alpha^*$. But from the matrices of $\mathcal{M}$ we know that $w(\circ_3\alpha^*) = F$.

It is interesting to notice, in particular, how the non-standard clause (mc3) of the previously defined non-canonical measure of complexity proves to be useful at the Part B($\Rightarrow$). ■

THEOREM 26 (**mCi**-REPRESENTABILITY). —

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})$$
$$w(\alpha^*) = T \;\Rightarrow\; b(\sim\alpha) = 0, \textit{ and}$$
$$w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.$$

PROOF. — Do as in parts (rt0)–(rt2) of Theorem 25, but now set:

(rt3) $(\circ\varphi)^* = \circ_1(\sim\varphi)^*$, if $b(\sim\varphi) = 0$
$(\circ\varphi)^* = \circ_1\varphi^*$, otherwise
(rt4) $(\sim^{n+1}\circ\varphi)^* = \sim_1(\sim^n\circ\varphi)^*$

Notice that such choices are indeed allowed by the restrictions (tr2.3), (tr3.2) and (tr3.3) that govern the set of admissible translations that characterize **mCi**. Again, the result is proven by complete induction on **mc**, and the (IH) is identical to the previous one: assume that (A) $w(\alpha^*) = T \;\Rightarrow\; b(\sim\alpha) = 0$ *and* (B) $w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0$, for any formula α with $\mathbf{mc}(\alpha) \leq k$, for some given k. We check again in detail only the cases of the formulas of the form $\sim\alpha$ or $\circ\alpha$.

Case of $\sim\alpha$, with α of the form $\sim^n\circ\beta$:
Notice that, from (rt4), $(\sim^{n+1}\circ\beta)^* = \sim_1(\sim^n\circ\beta)^*$. Thus, from the matrices of $\mathcal{M}$, either $w(\sim_1(\sim^n\circ\beta)^*) = T$ or $w(\sim_1(\sim^n\circ\beta)^*) = F$. Now, on the one hand, $w(\sim_1(\sim^n\circ\beta)^*) = T$ iff $w((\sim^n\circ\beta)^*) = F$, and on the other hand $w(\sim_1(\sim^n\circ\beta)^*) = F$ iff $w((\sim^n\circ\beta)^*) \neq F$. But part (B) of the (IH) informs us that $w((\sim^n\circ\beta)^*) = F$ iff $b(\sim^n\circ\beta) = 0$. However, fact 4(ii), relying on (b4) and (b5.n), guarantees that $b(\sim^m\circ\beta) = 0$ iff $b(\sim^{m+1}\circ\beta) = 1$, for any $m \in \mathbb{N}$. Thus, we may conclude that $w((\sim^{n+1}\circ\beta)^*) = T$ iff $b(\sim(\sim^{n+1}\circ\beta)) = 0$, and $w((\sim^{n+1}\circ\beta)^*) = F$ iff $b(\sim^{n+1}\circ\beta) = 0$.

The case of $\sim\alpha$, with α *not* of the form $\sim^n\circ\beta$ looks exactly the same as in the previous theorem, and we will not repeat it here.

Case of $\circ\alpha$:

– Part (A). Suppose $b(\sim\circ\alpha) = 1$. Then, by the bivaluational axiom (b4), he can assume that $b(\alpha) = 1 = b(\sim\alpha)$. By axiom (b3) it also follows that $b(\circ\alpha) = 0$. Note that the new (rt3) now says that $(\circ\alpha)^* = \circ_1\alpha^*$. By part (A) of the (IH), $b(\sim\alpha) = 1$ implies $w(\alpha^*) \neq T$, and by the part (B) of the (IH), $b(\alpha) = 1$ implies $w(\alpha^*) \neq F$. So, we are forced to conclude that $w(\alpha^*) = t$, thus $w((\circ\alpha)^*) = F \neq T$.

– Part (B)

- ($\Rightarrow$) Suppose $b(\circ\alpha) = 1$. By (b3) we conclude that $b(\alpha) = 0$ or $b(\sim\alpha) = 0$. If, on the one hand, $b(\alpha) = 0$, it follows by part (B) of the (IH) that $w(\alpha^*) = F$. By (b2), we also know that $b(\sim\alpha) = 1$, thus, by (rt3), we have $(\circ\alpha)^* = \circ_1\alpha^*$. In that case, $w((\circ\alpha)^*) = T \neq F$. If, on the other hand, $b(\sim\alpha) = 0$, (rt3) now says that

$(\circ\alpha)^* = \circ_1(\sim\alpha)^*$. Given that $\mathbf{mc}(\circ\alpha) > \mathbf{mc}(\sim\alpha)$, by the condition (mc3) on the definition of the complexity measure $\mathbf{mc}$, from $b(\sim\alpha) = 0$ the part (B) of the (IH) guarantees that $w((\sim\alpha)^*) = F$. So, again, $w((\circ\alpha)^*) = w(\circ_1(\sim\alpha)^*) = T \neq F$.

- ($\Leftarrow$) Suppose $b(\circ\alpha) = 0$. By (b2), $b(\sim\circ\alpha) = 1$, and by (b4) $b(\alpha) = 1 = b(\sim\alpha)$. As in Part (A), we can again conclude that $w(\alpha^*) = t$. By (rt3), $b(\sim\alpha) = 1$ implies that $(\circ\alpha)^* = \circ_1\alpha^*$. So, $w((\circ\alpha)^*) = F$.

That concludes the case analysis that belong to the inductive step. ■

The remaining 'representability' results are variations and combinations of the 3 above ones, and we are sure the reader can now check by herself the details of the proofs.

THEOREM 27 (PIf-REPRESENTABILITY). —

$$\begin{gathered}(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})\\ w(\alpha^*) = T \;\Rightarrow\; b(\sim\alpha) = 0, \textit{ and}\\ w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.\end{gathered}$$

PROOF. — Do as in Theorem 24, except that in now setting:

(rt2) $(\sim\varphi)^* = \sim_3\varphi^*$, if $b(\varphi) = 1 = b(\sim\varphi)$
$(\sim\varphi)^* = \sim_1\varphi^*$, otherwise

Check the result by induction on $\mathbf{mc}$. A slightly different proof of this fact —check clause (rw)— can be found in the ch.4 of (Marcos, 1999) and in (Carnielli *et al.*, 1999) —bear in mind though that this logic PIf shows up there under the name C_{min}.) ■

THEOREM 28 ($\mathbf{bC}$-REPRESENTABILITY). —

$$\begin{gathered}(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})\\ w(\alpha^*) = T \;\Rightarrow\; b(\sim\alpha) = 0, \textit{ and}\\ w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.\end{gathered}$$

PROOF. — Do as in Theorem 25, except that in now setting (rt2) as in Theorem 27. Check the result by induction on $\mathbf{mc}$. ■

THEOREM 29 ($\mathbf{Ci}$-REPRESENTABILITY). —

$$\begin{gathered}(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})\\ w(\alpha^*) = T \;\Rightarrow\; b(\sim\alpha) = 0, \textit{ and}\\ w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.\end{gathered}$$

PROOF. — Do as in Theorem 28, except that in now setting:

(rt3) $(\circ\varphi)^* = \circ_1(\sim\varphi)^*$, if $b(\circ\varphi) = 1$
$(\circ\varphi)^* = \circ_1\varphi^*$, otherwise

Check the result by induction on $\mathbf{mc}$. ■

On what concerns the last theorem, one might notice that the PTS offered for $\mathbf{Ci}$ in the paper (Carnielli *et al.*, 2001a) uses different interpretations for the consistency connective and is based on a stricter set of restrictions over the set Tr. The present semantics seems, in a sense, to be more in accordance with the classical behavior of $\circ$ with respect to $\sim$.

THEOREM 30 ($PIfe$-REPRESENTABILITY). —

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})$$
$$w(\alpha^*) = T \;\Rightarrow\; b({\sim}\alpha) = 0, \textit{ and}$$
$$w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.$$

PROOF. — Do as in Theorem 27, except that in now setting the extra requirement:

(rt4) if $({\sim}\varphi)^* = {\sim}_3\varphi^*$, then $({\sim}{\sim}\varphi)^* = {\sim}_3({\sim}\varphi)^*$

Check the result by induction on $\mathbf{mc}$. Be sure to consider in separate the extra case of complex formulas preceded by at least two negation symbols. ■

THEOREM 31 ($\mathbf{bCe}$-REPRESENTABILITY). —

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})$$
$$w(\alpha^*) = T \;\Rightarrow\; b({\sim}\alpha) = 0, \textit{ and}$$
$$w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.$$

PROOF. — Do as in Theorem 28, except that in now setting (rt4) as in Theorem 30. Check the result by induction on $\mathbf{mc}$. ■

THEOREM 32 ($\mathbf{Cie}$-REPRESENTABILITY). —

$$(\forall b \in \mathsf{biv})(\exists w \in \mathcal{M})(\exists * \in \mathsf{Tr})$$
$$w(\alpha^*) = T \;\Rightarrow\; b({\sim}\alpha) = 0, \textit{ and}$$
$$w(\alpha^*) = F \;\Leftrightarrow\; b(\alpha) = 0.$$

PROOF. — Do as in Theorem 26, except that in now setting (rt4) as in Theorem 30. Check the result by induction on $\mathbf{mc}$. ■

EXAMPLE 33. — We could now use the above defined PTS to check that, in $\mathbf{Cie}$ (thus, also in $\mathbf{Ci}$, $\mathbf{bC}$, $\mathbf{mCi}$, $\mathbf{CLuN}$ or $\mathbf{mbC}$), the formulas $\circ\alpha$ and ${\sim}\bullet\alpha$ are logically *distinguishable* even if equivalent, as announced in Section 2. Indeed, by the definition of $\bullet$, the formula ${\sim}\bullet\alpha$ is logically indistinguishable from the formula ${\sim}{\sim}\circ\alpha$. Yet, given a formula φ of the form ${\sim}p$ and a formula ψ of the form $\varphi[p/(p \wedge p)]$, it is easy to see that, in spite of the equivalence between $\varphi[p/\circ p]$ and $\varphi[p/{\sim}{\sim}\circ p]$ in logics as weak as $\mathbf{mCi}$, formulas such as $\psi[p/\circ p]$ and $\psi[p/{\sim}{\sim}\circ p]$ are not equivalent even in logics as strong as $\mathbf{Cie}$. To check that, select some $\mathbf{Cie}$-admissible translating mapping such that $(\circ p)^* = \circ_1{\sim}_1 p$, $({\sim}(\circ p \wedge \circ p))^* = {\sim}_1(\circ p \wedge \circ p)^*$ and $({\sim}({\sim}{\sim}\circ p \wedge {\sim}{\sim}\circ p))^* = {\sim}_3({\sim}{\sim}\circ p \wedge {\sim}{\sim}\circ p)^*$, and then select a 3-valued model $w \in \mathcal{M}$ for which $w(p) = t$. This provides, of course, yet another example of how our present family of paraconsistent logics may easily fail the replacement property, as illustrated in Theorem 6. □

NOTE 34 (DUALIZING THE ABOVE CONSTRUCTIONS). — One might now start everything all over again, back from Section 1, and easily dualize all results for paracomplete counterparts of all the above paraconsistent logics. To such an effect, one only needs to explore the symmetry of the present multiple-conclusion environment, exchange each bivaluational axiom (bi) and each sequent rule (si) for their converses (b$i^{\mathbf{c}}$) and (s$i^{\mathbf{c}}$), and exchange the consistency connective for a completeness, or determinedness, connective (as in (Marcos, 2005b)), and so on and so forth. The case of the dual of PIf was already studied in ch.4 of (Marcos, 1999) and in (Carnielli *et al.*, 1999), under the appellation D_{min}. □

Acknowledgements

This investigation was supported by the Fundação para a Ciência e a Tecnologia (Portugal) with FEDER (European Union), via the grant SFRH / BD / 8825 / 2002 and the Security and Quantum Information Group (SQIG). The author wishes to thank Juliana Bueno, Marcelo Finger and especially Marcelo Coniglio for their comments on earlier drafts of this paper. The final version of the paper also benefited a lot from the valuable feedback of Beata Konikowska, Walter Carnielli and especially Arnon Avron. To the author's great frustration, though, all possible remaining mistakes are to be credited only to him.

6. References

Arruda A. I., "Remarques sur les systèmes C_n", *Comptes Rendus de l'Académie de Sciences de Paris (A-B)*, vol. 280, pp. 1253–1256, 1975.

Avron A., "Non-deterministic matrices and modular semantics of rules", *in* J.-Y. Béziau (ed.), *Logica Universalis*, Birkhäuser Verlag, Basel, Switzerland, pp. 149–167, 2005.

Avron A., "Non-deterministic semantics for families of paraconsistent logics", *in* J.-Y. Béziau, W. Carnielli, D. Gabbay (eds), *Handbook of Paraconsistency*, vol. 9 of *Studies in Logic*, College Publications, pp. 285–320, 2007a.

Avron A., "Non-deterministic semantics for logics with a consistency operator", *International Journal of Approximate Reasoning*, vol. 45, pp. 271–287, 2007b.

Avron A., Konikowska B., "Multi-valued calculi for logics based on non-determinism", *Logic Journal of IGPL*, vol. 13, pp. 365–387, 2005a.

Avron A., Lev I., "Non-deterministic multiple-valued structures", *Journal of Logic and Computation*, vol. 15, pp. 241–261, 2005b.

Batens D., "Paraconsistent extensional propositional logics", *Logique et Analyse (N.S.)*, vol. 90/91, pp. 195–234, 1980.

Batens D., "A Survey of Inconsistency-Adaptive Logics", *in* D. Batens, C. Mortensen, G. Priest, J. P. Van Bendegem (eds), *Frontiers of Paraconsistent Logic*, Research Studies Press, Baldock, pp. 49–73, 2000.

Béziau J.-Y., "Sequents and bivaluations", *Logique et Analyse (N.S.)*, vol. 44, num. 176, pp. 373–394, 2001.

Caleiro C., Carnielli W., Coniglio M. E., Marcos J., "Two's company: "The humbug of many logical values"", *in* J.-Y. Béziau (ed.), *Logica Universalis*, Birkhäuser Verlag, Basel, Switzerland, pp. 169–189, 2005. Preprint available at: `http://wslc.math.ist.utl.pt/ftp/pub/CaleiroC/05-CCCM-dyadic.pdf`.

Carnielli W., "Many-valued logics and plausible reasoning", *Proceedings of the* XX *International Congress on Many-Valued Logics,* held at the University of Charlotte / NC, US, 1990, IEEE Computer Society, pp. 328–335, 1990.

Carnielli W. A., "Possible-translations semantics for paraconsistent logics", *in* D. Batens, C. Mortensen, G. Priest, J. P. Van Bendegem (eds), *Frontiers of Paraconsistent Logic*, Research Studies Press, Baldock, pp. 149–163, 2000.

Carnielli W. A., Coniglio M. E., Marcos J., "Logics of Formal Inconsistency", *in* D. Gabbay, F. Guenthner (eds), *Handbook of Philosophical Logic*, 2nd edn, vol. 14, Kluwer, pp. 1–93, 2007. Preprint available at:
http://wslc.math.ist.utl.pt/ftp/pub/MarcosJ/03-CCM-lfi.pdf.

Carnielli W. A., D'Ottaviano I. M. L., "Translations between logical systems: A manifesto", *Logique et Analyse (N.S.)*, vol. 40, num. 157, pp. 67–81, 1997.

Carnielli W. A., Marcos J., "Limits for paraconsistent calculi", *Notre Dame Journal of Formal Logic*, vol. 40, num. 3, pp. 375–390, 1999.
http://projecteuclid.org/euclid.ndjfl/1022615617.

Carnielli W. A., Marcos J., "Ex contradictione non sequitur quodlibet", *in* R. L. Epstein (ed.), *Proceedings of the* II *Annual Conference on Reasoning and Logic,held in Bucharest, RO, July 2000*, vol. 1, Advanced Reasoning Forum, pp. 89–109, 2001a. Preprint available at:
http://wslc.math.ist.utl.pt/ftp/pub/MarcosJ/01-CM-ECNSQL.pdf.

Carnielli W. A., Marcos J., "Tableaux for logics of formal inconsistency", *in* H. R. Arabnia (ed.), *Proceedings of the International Conference on Artificial Intelligence* (IC-AI'2001), vol. II, CSREA Press, Athens / GA, pp. 848–852, 2001b.
http://wslc.math.ist.utl.pt/ftp/pub/MarcosJ/01-CM-tableaux.pdf.

Carnielli W. A., Marcos J., "A taxonomy of **C**-systems", *in* W. A. Carnielli, M. E. Coniglio, I. M. L. D'Ottaviano (eds), *Paraconsistency: The Logical Way to the Inconsistent*, vol. 228 of *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, pp. 1–94, 2002. Preprint available at:
http://wslc.math.ist.utl.pt/ftp/pub/MarcosJ/02-CM-taxonomy.pdf.

Carnielli W. A., Marcos J., de Amo S., "Formal inconsistency and evolutionary databases", *Logic and Logical Philosophy*, vol. 8, num. 2, pp. 115–152, 2000. Preprint available at:
http://wslc.math.ist.utl.pt/ftp/pub/MarcosJ/00-CdAM-evolutionary.pdf.

Carnielli W., Coniglio M. E., Gabbay D., Gouveia P., Sernadas C., *Analysis and Synthesis of Logics: How to cut and paste reasoning systems*, vol. 35 of *Applied Logic*, Springer, 2008.

Marcos J., *"Possible-Translations Semantics (in Portuguese)"*, Master's thesis, State University of Campinas, BR, 1999.
http://libdigi.unicamp.br/document/?code=vtls000224326.

Marcos J., "Possible-translations semantics", *in* W. A. Carnielli, F. M. Dionísio, P. Mateus (eds), *Proceedings of the Workshop on Combination of Logics: Theory and applications* (CombLog'04), held in Lisbon, PT, July 28–30, 2004, Dep. de Matemática, Instituto Superior Técnico, pp. 119–128, 2004. Extended version available at:
http://wslc.math.ist.utl.pt/ftp/pub/MarcosJ/04-M-pts.pdf.

Marcos J., "Modality and paraconsistency", *in* M. Bilkova, L. Behounek (eds), *The Logica Yearbook 2004*, Filosofia, Prague, pp. 213–222, 2005a. Proceedings of the XVIII International

Symposium promoted by the Institute of Philosophy of the Academy of Sciences of the Czech Republic. Preprint available at:
http://wslc.math.ist.utl.pt/ftp/pub/MarcosJ/04-M-ModPar.pdf.

Marcos J., "Nearly every normal modal logic is paranormal", *Logique et Analyse (N.S.)*, vol. 48, pp. 279–300, 2005b. Preprint available at:
http://wslc.math.ist.utl.pt/ftp/pub/MarcosJ/04-M-Paranormal.pdf.