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On negation: Pure local rules

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Abstract

This is an initial systematic study of the properties of negation from the point of view of abstract deductive systems. A unifying framework of multiple-conclusion consequence relations is adopted so as to allow us to explore symmetry in exposing and matching a great number of positive contextual sub-classical rules involving this logical constant—among others, well-known forms of proof by cases, *consequentia mirabilis* and *reductio ad absurdum*. Finer definitions of paraconsistency and the dual paracompleteness can thus be formulated, allowing for *pseudo-scotus* and *ex contradictione* to be differentiated and for a comprehensive version of the Principle of Non-Triviality to be presented. A final proposal is made to the effect that—pure positive rules involving negation being often fallible—a characterization of what most negations in the literature have in common should rather involve, in fact, a reduced set of negative rules.

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Proposal

> *'Contrariwise', continued Tweedledee, 'if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic.'*
> *—Lewis Carroll, Through the Looking-Glass, and what Alice found there, 1872.*

This is an investigation of negation from the point of view of universal logic, the abstract study of mother-structures (in the sense of Bourbaki) endowed with consequence relations. In that, it has as important predecessors [1,9], and related papers. The general framework adopted here for the study of pure rules for negation—those that do not involve other logical constants but negation— is that of multiple-conclusion consequence relations, as in [31]. Section 0 introduces the general framework and main related definitions and notations. Section 1 presents the most usual axioms regulating the behavior of multiple-conclusion consequence relations, such as overlap, (cautious) cut, (cautious) weakening, compactness and structurality, and shows how several distinct notions of overcompleteness can be defined. The latter notions can be used to catalogue four distinct varieties of triviality, and allow for an extension of da Costa's 'Principle of Tolerance' (or rather 'Principle of Non-Triviality') in the last section. Although the present study is neither proof-theoretical nor semantical in nature, some hints are given on the import of several abstract schematic rules hereby presented from a semantic viewpoint, and reports are often given about the behavior of those rules in the context of some non-classical logics—such as relevance, modal and (sub)intuitionistic or intermediate logics—with which the reader might be familiar. Local, or contextual, rules can be studied in opposition to global rules—positive local schematic rules are meant to hold for any choice of contexts and formulas contained therein, positive global schematic rules are usually weaker rules meant to display relations among local rules. These kinds of rules are contrasted in papers such as [11,19,27]; in [29] the author chooses to present global rules for the connectives as more 'legitimate', here I acknowledge instead that local rules are fairly more common, and concentrate on them. The distinction between local and global rules is reminiscent of the traditional philosophical distinction between inference rules and deduction rules—an elegant modern abstract account of it can be found in ch. 3 of [13].

Section 2 presents a few blocks of local sub-classical rules for negation—among them, some rules that are positive (being universally respected in classical logic) and some rules that are negative (being classically valid for some choices of contexts and formulas but failing for others). The first bunch of rules comes in two dual sets: The first one regulates those properties of negation which are related to 'consistency assumptions' (the inexistence of non-dadaistic models for some formulas together with their negations), the second regulates 'completeness assumptions' (the satisfiability of either a formula or its negation in each non-nihilistic model). Consistency rules include *pseudo-scotus*, which underlies the Principle of Explosion, and *ex contradictione sequitur quodlibet*, and these two rules can be sharply distinguished in the present framework of multiple-conclusion consequence relations; completeness rules include excluded middle, proof by cases and *consequentia mirabilis*; some of those rules will partly span both categories, as for instance the completeness rule of *reductio ad absurdum*, which might interfere with *ex contradictione*. A second bunch of rules deals with other forms of manipulation of negation: Double negation introduction and elimination, contextual contraposition and contextual replacement are among those rules. The various interrelations between those sets of rules are carefully investigated here. The present study teams up and generalizes in part some other foundational studies on negation, such as [4,8,17,21,22,24]. Note that I will not insist here that a negation operator should have *any* of the above mentioned properties. Finally, the last bunch of rules comes again divided into two dual sets, which have the most distinguishing feature of being negative rules, dealing with some minimal properties that a reasonable negation should *not* have in order to reckon minimally interesting interpretations—I would be more reluctant to abandon one of these last negative properties than any of the preceding positive ones.

Paraconsistency, in particular, is equated to the failure of the Principle of Explosion, and this reflects in the failure of the most basic form of *pseudo-scotus*. Dual definitions are offered for paracomplete logics and their subclasses, and some Illustrations are given. Other fine definitions are easily introduced in this framework, as in Section 3, so as to characterize a few interesting subclasses of paraconsistent logics. From the relations established among and inside the three blocks of rules mentioned above, the reader will immediately be able to trace, in particular, some causes and effects of paraconsistency from the point of view of universal logic. For an account of the effects of the above systematization for the praxis of the non-classical designer, Section 3 also illustrates some of the necessary and sufficient conditions for paranormality—either paraconsistency or paracompleteness—in logic.

The first part of Section 4 argues that, while individual classes of logics or classes of negations might well be characterized by positive rules, the *very notions* of logic and of negation, or at least the interesting realizations of those notions, are often best characterized *negatively*, by saying which properties they should not enjoy. Definitions of *minimally decent* classes of logics and classes of negations are then put forward. The section continues by surveying some of the most remarkable attempts to answer the bold question of 'What is negation?' [10,20–22,24], calling attention to some of the merits of each approach and some of their flaws or deficiencies, while at the same time coherently situating them all in the framework set in the present paper for easier comparison.

The last section ends up by listing some of the main novelties and contributions of the present paper (you can go there and read them at any time), and hints at some generalizations and extensions of the basic notions hereby assumed and at directions in which this research should be furthered.

A warning: The intended generality in the exposition of the pure rules for negation, below, might make them hard to read, here and there. It is always easier though to start by looking at the basic cases of each family of rules. The reader should also try not to get psychologically deterred by the formulation of the Facts relating those rules. Some might have the impression that I am trying to draw a map of the empire at a scale 1:1. That is surely *not the intention*. The goal is indeed to be precise about our roads and connections, but, curiously, the full details of the map itself are often not that important here—besides, the map is really easy to draw, once you get an idea of what's going on. Much of what follows is in fact part of many logicians' folklore, now updated into a uniform setting, which reveals relationships already known, and makes it easy to check some new unsuspected relationships... and to introduce some new concepts altogether. The idea, then, avoiding *disorder*, is that you get the spirit, and *don't* lose the feeling (let it out somehow).

0. Background

Logic, n. The art of thinking and reasoning in strict accordance with limitations and incapacities of the human misunderstanding.

—Ambrose Bierce, The Devil's Dictionary, 1881–1906.

After a century of historical reinvention in the field of logic, it rests still rather uncontroversial to admit that there is no general agreement about what a logic or a logical constant *is*. Nonetheless, one might feel quite safe here, yet free, with the forthcoming non-dogmatic definitions.[3] Following a good deal of the recent literature, this short investigation will assume that logics are concerned with the formal study of (patterns of) reasoning, or argumentation, that is, they are concerned with deduction, with 'what follows from what'. Accordingly, let's take a *logic* $\mathcal{L}$ as a structure of the form $\langle \mathsf{S}_{\mathcal{L}}, \Vdash_{\mathcal{L}}\rangle$, where $\mathsf{S}_{\mathcal{L}}$ is a set of (well-formed) *formulas* and $\Vdash_{\mathcal{L}} \subseteq \wp(\mathsf{S}_{\mathcal{L}}) \times \wp(\mathsf{S}_{\mathcal{L}})$ is a (*multiple-conclusion*) *consequence relation*, or *entailment*, defined over sets of formulas (also called *theories*) of $\mathcal{L}$. Using occasionally decorated capital Greek letters as variables for theories, and doing a similar thing with lowercase Greek for formulas, then putting the consequence relation in infix format, I shall often write something as $\Gamma, \alpha, \Gamma' \Vdash_{\mathcal{L}} \Delta', \beta, \Delta$ to say that $\langle \Gamma \cup \{\alpha\} \cup \Gamma', \Delta' \cup \{\beta\} \cup \Delta\rangle$ falls into the relation $\Vdash_{\mathcal{L}}$. Such clauses will be called *inferences*, and their intended reading is that some formula or another among the *alternatives* in the right-hand side of $\Vdash_{\mathcal{L}}$ should follow from the whole set of *premises* in its left-hand side. The theories $\Gamma, \Gamma', \Delta', \Delta$ will be called *contexts* of the inference. A similar move is made by the *canonical model-theoretic account* of a consequence relation: At least one of the alternatives should be true when all the premises are true. Keeping in mind that each such inference should always be relativized to some previously given logic, I shall omit subindices whenever I see no risk of confusion among the plethora of diverse consequence relations and logics which will be allowed to appear below.

The following paragraphs are mostly notational and somewhat boring, so I guess the reader can thread them very quickly and return only when and if they feel the need of it. Note that expressions like '$\neg A$', 'A/B' and '$A//B$' will be used as abbreviations for the metalogical statements 'A is not the case', 'if A then B' and 'A if and only if B', and expressions like '$A \Rightarrow B$ {NN}' and '$A \Leftrightarrow B$ {NN}' will abbreviate the metalinguistic 'A implies B, in the presence of NN', and 'A is equivalent to B, in the presence of NN'. Let $[A_b]_{b \leqslant \mathrm{C}}$ denote some *sequence* of the form '$A_{b_1}, \ldots, A_{b_z}$', whose members are exactly the members of the family $\{A_b\}_{b \leqslant \mathrm{C}}$;[4] whenever the sequence is composed of inference clauses, commas will be read as metalinguistic conjunctions; whenever $\mathrm{C} = 0$, one is simply dealing with an empty sequence. Note that at the metalinguistic level we shall be freely using the mathematical reasoning from classical logic.

[3] By this 'non-dogmatic' I mean that the following definitions and formulations should be taken and investigated as what they are: *proposals*, rather than *prescriptions*. So, I will (try) not (to) be committing myself to any particular set of assumptions, but rather be interested in investigating the effects of each particular choice. A gentle bias towards the concerns of the paraconsistent scenario might though be noted—that is explained by this being the area of my major expertise and experience, and the area whose open questions originated this study.

[4] In general this family will be finite, or at most denumerably finite—ultimately, though, its cardinality will always be supposed here to be limited by the cardinality of the underlying set of formulas S.

In order to add some structure to the set of formulas S, let $\circledcirc_i$ denote some *logical constant* of arity $\mathsf{ar}(i) \in \mathbb{N}$. S will be dubbed *schematic* (*with respect to* $\circledcirc_i$) in case $\circledcirc_i([\alpha_j]_{j\leqslant\mathsf{ar}(i)}) \in \mathsf{S}$ and $\{\beta_j\}_{j\leqslant\mathsf{ar}(i)} \subseteq \mathsf{S}$ imply $\circledcirc_i([\beta_j]_{j\leqslant\mathsf{ar}(i)}) \in \mathsf{S}$. This already embodies some notion of 'logical form'. To make it even stronger, S will be said to have an *algebraic character* in case it is the algebra freely generated over some set LC of logical constants with the help of a convenient set at of *atomic sentences*, thus implying, in particular, that $\{\beta_j\}_{j\leqslant\mathsf{ar}(i)} \subseteq \mathsf{S} \Rightarrow \circledcirc_i([\beta_j]_{j\leqslant\mathsf{ar}(i)}) \in \mathsf{S}$. An *endomorphism* in $\mathcal{L}$ is any mapping $*: \mathsf{S} \to \mathsf{S}$ that preserves the constants of $\mathcal{L}$, that is, such that $(\circledcirc_i([\alpha_j]_{j\leqslant\mathsf{ar}(i)}))^* = \circledcirc_i([\alpha_j^*]_{j\leqslant\mathsf{ar}(i)})$ for any $\circledcirc_i \in \mathsf{LC}$. Given a set S of formulas with algebraic character and a set of generators at, a *uniform substitution*—another commonly required ingredient of the notion of 'logical form'—is the unique endomorphic extension of a mapping $*: \mathsf{at} \to \mathsf{S}$ into the whole set of formulas. Given the aims of this study, I shall assume below that a unary *negation* symbol $\sim$ will always be present as a logical constant in the underlying language of our logics, and S will be assumed to contain at least one formula of the form $\sim\varphi$. This assumption, together with the schematism of S which shall be postulated from here on, will allow us to quantify metalinguistically over formulas. As some further notational help, I will use the following symbols for iterated negations: $\sim^0\alpha := \alpha$ and $\sim^{n+1}\alpha := \sim^n\sim\alpha$—these will be used to inject a bit more of generality into the formulation of the rules in Section 2.

Here, a(n *inference*) *rule* will be simply a relation involving one or more inferences. Given some rule A, I will sometimes be writing $(\forall\mathbf{form})A$ or $(\exists\mathbf{form})A$ in order to quantify in this way over the lowercase Greek elements that appear in A; similarly, I will be writing $(\forall\mathbf{cont})A$ or $(\exists\mathbf{cont})A$ in order to quantify accordingly over its elements in uppercase Greek. A formula φ will be said to *depend only on* its *component* formulas $[\varphi_i]_{i\leqslant\mathrm{I}}$ whenever φ can be written with the sole help of the mentioned component formulas and the logical constants of the language—this shall be denoted by $\varphi\langle[\varphi_i]_{i\leqslant\mathrm{I}}\rangle$. In a similar vein, to denote a theory Φ whose formulas depend only on the formulas $[\varphi_i]_{i\leqslant\mathrm{I}}$, one will write $\Phi\langle[\varphi_i]_{i\leqslant\mathrm{I}}\rangle$. Unless I say something to the contrary, when I state a rule below I shall be referring to the universal closure of this rule, that is, I shall be writing a *schematic rule*, a rule that holds for any choice of contexts and formulas explicitly displayed in it. In the same spirit, when I write by way of $\Gamma, [\alpha_i]_{i\leqslant\mathrm{I}} \nVdash [\beta_j]_{j\leqslant\mathrm{J}}, \Delta$—or, what amounts to the same, $\neg(\Gamma, [\alpha_i]_{i\leqslant\mathrm{I}} \Vdash [\beta_j]_{j\leqslant\mathrm{J}}, \Delta)$—the metalogical denial of a rule, I shall mean that there is *some* choice of contexts Γ and Δ and of formulas $[\alpha_i]_{i\leqslant\mathrm{I}}, [\beta_j]_{j\leqslant\mathrm{J}}$ under which the rule $\Gamma, [\alpha_i]_{i\leqslant\mathrm{I}} \Vdash [\beta_j]_{j\leqslant\mathrm{J}}, \Delta$ does *not* hold. The notation $\Gamma, \alpha \dashv\Vdash \beta, \Delta$ shall abbreviate the metalogical conjunction of $\Gamma, \alpha \Vdash \beta, \Delta$ and $\Gamma, \beta \Vdash \alpha, \Delta$—obviously, this is symmetric, and it results in the same to write $\Gamma, \beta \dashv\Vdash \alpha, \Delta$. To be sure, most statements below will have instances with the format $[A_b]_{b\leqslant\mathrm{C}} \# D$, where each element of $[A_b]_{b\leqslant\mathrm{C}}$ and each D represents an inference clause, and # represents some sort of 'implication': *Positive local* schematic rules such as (C1) and (C2) a few lines below will be constituted of universally quantified schemas, in the form $(\forall\mathsf{form})(\forall\mathsf{cont})([A_b]_{b\leqslant\mathrm{C}} \# D)$; *negative local* schematic rules such as $\neg$(C1) are opposed to positive rules, having thus the form $(\exists\mathsf{form})(\exists\mathsf{cont})\neg([A_b]_{b\leqslant\mathrm{C}} \# D)$; *global positive* schematic rules will have the form $(\forall\mathsf{form})([(\forall\mathsf{cont})A_b]_{b\leqslant\mathrm{C}} \# (\forall\mathsf{cont})D)$; *global negative* schematic rules will have the form $(\exists\mathsf{form})([(\exists\mathsf{cont})\neg A_b]_{b\leqslant\mathrm{C}} \# (\exists\mathsf{cont})\neg D)$. Note that each local, or contextual, rule of the above formats can immediately be given a global version, by suitably distributing some of the metalinguistic contextual quantifiers as expected.

1. Rules for abstract consequence relations

Ex falso nonnumquam sequitur verum, et tamen semper absurdum.

—*Jakob Bernoulli, XVII century.*

I now proceed to consider some rules which have often been proposed as general properties of 'any' consequence relation. Let's start by:

(C1) Overlap, or Reflexivity: $(\Gamma, \alpha, \Gamma' \Vdash \Delta', \alpha, \Delta)$
(C2) (Full) Cut: $(\Gamma \Vdash \alpha, \Delta$ and $\Gamma', \alpha \Vdash \Delta')$ / $(\Gamma', \Gamma \Vdash \Delta, \Delta')$

To facilitate reference in the following, call *simple* any logic whose consequence relation respects the two above properties (cf. [1]). Given that it is quite usual for a formula to be assumed to follow from itself, most known logics will indeed respect overlap, thus I will not explicitly consider here any weaker versions of this rule (but the reader should be aware of the existence of, for instance, some relevance logics failing the general version of overlap). The full formulation of cut above, however, is quite often more than one needs (or that one can count on) for most practical purposes, as the reader shall see in the following. Many a time, one of the following weaker formulations will suffice:

(C2.1.I) (I-)left cautious cut: $([\Gamma \Vdash \alpha_i, \Delta]_{i \leqslant \mathrm{I}}$ and $\Gamma, [\alpha_i]_{i \leqslant \mathrm{I}} \Vdash \Delta)$ / $(\Gamma \Vdash \Delta)$
(C2.2.J) (J-)right cautious cut: $(\Gamma \Vdash [\alpha_j]_{j \leqslant \mathrm{J}}, \Delta$ and $[\Gamma, \alpha_j \Vdash \Delta]_{j \leqslant \mathrm{J}})$ / $(\Gamma \Vdash \Delta)$

Obviously, (C2.1.1) and (C2.2.1) are identical rules; call them 1-*cautious cut*, and call 1-*simple* those logics respecting (C1) and (C2.k.1). In the Facts I will mention below, I shall often be relying on overlap and 1-cautious cut, and sometimes I will use full cut. There are other interesting 'contextual versions' of cut which dwell in between its cautious versions and the full version, but I shall not study them here.

Other very common rules characterizing general consequence relations are:

(C3) Weakening, or Monotonicity: left weakening plus right weakening
(C3.1) Left weakening: $(\Gamma \Vdash \Delta)$ / $(\Gamma', \Gamma \Vdash \Delta)$
(C3.2) Right weakening: $(\Gamma \Vdash \Delta)$ / $(\Gamma \Vdash \Delta, \Delta')$

Useful information to bear in mind, to fill the gaps in the proofs of the assertions which will be found below, are the easily checkable derivations:

Fact 1.1. *Consider the rules*:

(r1) $(\Gamma, [\alpha_i]_{i \leqslant \mathrm{I}} \Vdash [\beta_j]_{j \leqslant \mathrm{J}}, \Delta)$
(r2) $[\Gamma \Vdash \alpha_i, \Delta]_{i \leqslant \mathrm{I}} / (\Gamma \Vdash [\beta_j]_{j \leqslant \mathrm{J}}, \Delta)$
(r3) $[\Gamma, \beta_j \Vdash \Delta]_{j \leqslant \mathrm{J}} / (\Gamma, [\alpha_i]_{i \leqslant \mathrm{I}} \Vdash \Delta)$

Then:

(i) (r1) $\Rightarrow$ (r2), *for* $\mathrm{I} = 0$ $\{\}$
(r1) $\Rightarrow$ (r2), *for* $\mathrm{J} = 0$ $\{$(C2)$\}$
(r1) $\Rightarrow$ (r2), *in all other cases* $\{$(C2) *and* (C3)$\}$

(ii)	(r1) $\Rightarrow$ (r3), *for* J $=0$	{}
	(r1) $\Rightarrow$ (r3), *for* I $=0$	{(C2)}
	(r1) $\Rightarrow$ (r3), *in all other cases*	{(C2) *and* (C3)}
(iii)	(r2) *or* (r3) $\Rightarrow$ (r1)	{(C1)}

Standard *tarskian* consequence relations (cf. [35]) are characterized by the validity of (C1), (C2) and (C3), but for *non-monotonic* logics this rule (C3) (and also (C2)) fails to obtain in full generality. Thus, the model-theoretic account related to non-monotonic logics should be expected to be an update of the standard one, so as to take contexts into account in evaluating the truth of formulas or the satisfiability of schematic rules. Some interesting milder versions of the weakening rule are the following:

(C3.1.K) (K-)left cautious weakening: $([\Gamma \Vdash \alpha_k]_{k\in \mathrm{K}}$ and $\Gamma \Vdash \Delta)$ / $(\Gamma, [\alpha_k]_{k\in \mathrm{K}} \Vdash \Delta)$
(C3.2.L) (L-)right cautious weakening: $([\alpha_l \Vdash \Delta]_{l\in \mathrm{L}}$ and $\Gamma \Vdash \Delta)$ / $(\Gamma \Vdash [\alpha_k]_{l\in \mathrm{L}}, \Delta)$

Now, many interesting non-monotonic logics—the so-called *plausible* ones (cf. [3]), of which adaptive logics (cf. [6]) under the 'minimal abnormality' strategy constitute a special case—will still respect (C1), (C2.1.I), (C2.2.J), (C3.2) and (C3.1.K). Other exotic consequence relations, such as the one induced by *inferentially many-valued* logics (cf. [25]), will only respect, in general, the properties (C2.1.I), (C2.2.J) and (C3). I will call a logic *cautious tarskian* in case it respects overlap, cautious cut and cautious weakening.

Note that, from this point on, I will often be using italic lowercase / uppercase letters as wildcards for a string of one / finitely-many arbitrary variables. Note also that 'finitely-many' does not exclude the empty string. Separating dots are not parsed. One can then easily check that:

Fact 1.2.

(i)	(C2.k.0) *and* (C3.q.0)	{}
(ii)	(C2) $\Rightarrow$ (C2.x.a)	{}
(iii)	(C3.x) $\Rightarrow$ (C3.x.a)	{ }
(iv)	(Cn.x.a+b) $\Rightarrow$ (Cn.x.a), *for* $n \in \{2, 3\}$	{}
(v)	(C2.x.a) *and* (C2.x.b) $\Rightarrow$ (C2.x.a+b)	{(C3.x)}
(vi)	(C2.x.1) $\Rightarrow$ (C2)	{(C3)}

So, diverting from uninformative rules such as (i), we see that some forms of cut imply others (see (ii) and (iv)), and the same holds for weakening (see (iii) and (iv)). Cautious cut is in fact equivalent to full cut in the presence of weakening (see (v) and (vi)).

Some further important properties of general consequence relations are:

(C4) Compactness: left compactness plus right compactness
(C4.1) Left compactness: for any Γ and Δ such that $(\Gamma \Vdash \Delta)$ there is some finite $\Gamma' \subseteq \Gamma$ such that $(\Gamma' \Vdash \Delta)$

(C4.2) Right compactness: for any Γ and Δ such that $(\Gamma \Vdash \Delta)$ there is some finite $\Delta' \subseteq \Delta$ such that $(\Gamma \Vdash \Delta')$

(C5) Structurality: for any endomorphism $*$, $(\Gamma \Vdash \Delta)$ implies $(\Gamma^* \Vdash \Delta^*)$

Compactness is usually invoked, for instance, to guarantee the finitary character of proofs, and is often equivalent to the axiom of choice in model theory. Typical examples of consequence relations failing compactness are those of higher-order logics. Structurality is the rule that allows for uniform substitutions to preserve entailment. Still some other rules, such as those regulating left- and right-contractions, expansions and permutation will in the present framework come for free, given that I have chosen to express inferences using only *sets*—when the repetition of formulas or their order becomes important, as in the case of linear logics or in categorial grammar, it is convenient to upgrade the previous definitions so as to deal with multi-sets or ordered sets of contexts.

Not all the consequence relations which respect some or even all the above properties are decent and worth of being studied. A particularly striking way of being uninteresting and uninformative occurs when the nature of the formulas of the contexts involved in an inference does not really matter, but only the cardinality of the contexts is determinant of the validity of the inference involving them. Consider thus the following kind of property:

(C0.I.J) I.J-overcompleteness: $(\Gamma, [\alpha]_{i \leqslant \mathrm{I}} \Vdash [\beta]_{j \leqslant \mathrm{J}}, \Delta)$

0.0-overcompleteness says that whatever set of alternatives follows from whatever set of premises. This is clearly not a very attractive situation, as it ceases to draw a difference between inferences. Everything is permitted—one might call this 'Dostoyevski's God-is-dead situation'. But some other instances of overcompleteness may be worth looking at. If you fix a particular sequence of alternatives $[\beta_j]_{j \leqslant \mathrm{J}}$, you might call it an I.J-*alternative* if for some cardinal I and any contexts Γ and Δ one has that $(\Gamma, [\alpha]_{i \leqslant \mathrm{I}} \Vdash [\beta]_{j \leqslant \mathrm{J}}, \Delta)$ holds; call it simply a J-*alternative* if it is an I.J-alternative for any I. Similarly, if you fix a particular sequence of premises $[\alpha_i]_{i \leqslant \mathrm{I}}$, you might call it I.J-*trivializing* if $(\Gamma, [\alpha]_{i \leqslant \mathrm{I}} \Vdash [\beta]_{j \leqslant \mathrm{J}}, \Delta)$ holds for some cardinal J and any contexts Γ and Δ; call it simply I-*trivializing* if it is an I.J-alternative for any J. A particularly interesting case here is that of *finitely trivializing* theories, i.e. those theories which are I-trivializing for some finite I. Of course, if at least overlap holds then the whole set of formulas is both 1-trivializing and a 1-alternative theory. Note, for instance, that the difference between a 1.1-alternative and a 0.1-alternative is only very slight: It is the distinction, if it makes any sense to say that there is any, between a formula being a consequence of *anything* or of *whatever* (in Latin, *quocumque* versus *qualiscumque*). A similar observation can be made about 1.1- and 1.0-trivializing theories.[5] Any formula φ will be called a *top particle*, or simply a *thesis*,[6] whenever it

[5] But the distinction becomes ineffable once you start using single-conclusion instead of multiple-conclusion consequence relations (cf. [26]).

[6] The theses of a given logic are sometimes called its *logical truths*, in the manner of Quine. Some authors would prefer, though, to call logical truths the formulas which are proved under empty contexts (but not necessarily under all other contexts, what makes a difference if your logic is non-monotonic). This terminology is not at

is a 0-alternative, and will be called a *bottom particle*, or an *antithesis*, whenever it is 0-trivializing.

Note that:

Fact 1.3. *By definition*:

(i) *any formula of a* 0.1-*overcomplete logic is a top particle*;
(ii) *any formula of a* 1.0-*overcomplete logic is a bottom particle*;
(iii) *any logic respecting weak cut and having a formula which is both a top and a bottom particle is* 0.0-*overcomplete*;
(iv) *any overcomplete logic is tarskian.*

Moreover:

(v)	$(\mathrm{C0.I.J}) \Leftrightarrow (\mathrm{C0.I+K.J+L})$, *for* $\mathrm{I}, \mathrm{J} > 0$	$\{\}$
(vi)	$(\mathrm{C0.0.0}) \Rightarrow (\mathrm{C0.I.J})$	$\{\}$
(vii)	$(\mathrm{C0.0.1}) \Rightarrow (\mathrm{C0.0.0})$	{*bottom and* (C2.k.j)}
(viii)	$(\mathrm{C0.1.0}) \Rightarrow (\mathrm{C0.0.0})$	{*top and* (C2.k.j)}

From the above we see that all varieties of overcompleteness reduce thus to one among 0.0-, 0.1-, 1.0- and 1.1-overcompleteness. From the point of view of the standard model-theoretic account, 0.1-overcomplete logics can be characterized by a unique model in which everything is true; similarly for 1.0-overcomplete logics and models in which everything is false. The empty set of valuations, with no truth-values, provide an adequate semantics for 0.0-overcomplete logics, and for 1.1-overcomplete logics you might combine two valuation mappings: One which makes all formulas true, and another one which makes them all false. From this point on, I will be calling a logic *dadaistic* in case it is 0.1-overcomplete, *nihilistic* in case it is 1.0-overcomplete, *trivial* in case it is 0.0-overcomplete, and *semitrivial* in case it is I.J-overcomplete for any $\mathrm{I}, \mathrm{J} > 0$.

As we have seen, the four above kinds of overcompleteness collapse into triviality in case weak cut is respected and there are bottoms and tops around. A cheaper way of producing that collapse is by assuming the following properties on consequence relations (extending the proposal in [21]):

(CG) Coherence: left coherence plus right coherence
(CG.1) Left coherence: $(\Gamma \Vdash \beta, \Delta) \Leftrightarrow (\forall\alpha)(\Gamma, \alpha \Vdash \beta, \Delta)$
(CG.2) Right coherence: $(\Gamma, \alpha \Vdash \Delta) \Leftrightarrow (\forall\beta)(\Gamma, \alpha \Vdash \beta, \Delta)$

Although the above properties are clearly admissible in most usual logics, they are also considerably esoteric, and we will not assume them at any point in this paper.

A warning: From this point on, unless otherwise stated, all the above sorts of overcompleteness shall explicitly be avoided.

issue here—I shall rather, in general, just take *invariance* under contexts for granted and assume these definitional matters to be largely *conventional*, in the manner of Carnap.

2. Pure rules for negation

Sameness leaves us in peace, but it is contradiction that makes us productive.
—Johann Wolfgang Von Goethe, Conversations with Eckermann,
March 28, 1827.

Let us now consider some general *pure* sub-classical properties of negation—in the sense that their statement does not involve other logical constants but negation—which often appear in the literature (some of them known since medieval or even ancient times). Be aware that, even though I will be in what follows presenting positive contextual (and, later on, negative contextual) schematic rules for negation and then studying their interrelations in the next Facts by way of local or global schematic tautologies, lack of space will prevent me from analyzing in this paper the (usually weaker) global versions of the same contextual rules hereby presented, in spite of their possible interest.

For each choice of *levels* $\mathrm{m}, \mathrm{n} \in \mathbb{N}$, consider the rules:

(1.1.m) $(\Gamma, \sim^{\mathrm{m}}\alpha, \sim^{\mathrm{m}+1}\alpha \Vdash \Delta)$
pseudo-scotus, or *explosion*

(2.1.n) $(\Gamma \Vdash \sim^{\mathrm{n}+1}\beta, \sim^{\mathrm{n}}\beta, \Delta)$
casus judicans, or *implosion*, or *excluded middle*

(1.1.m.n) $(\Gamma, \sim^{\mathrm{m}}\alpha, \sim^{\mathrm{m}+1}\alpha \Vdash \sim^{\mathrm{n}}\beta, \Delta)$
ex contradictione sequitur quodlibet

(2.1.n.m) $(\Gamma, \sim^{\mathrm{m}}\alpha \Vdash \sim^{\mathrm{n}+1}\beta, \sim^{\mathrm{n}}\beta, \Delta)$
quodlibet sequitur ad casos

Rules of the form (1.1.m) postulate the existence of special kinds of 2-trivializing theories, those containing both a formula and its negation; rules (2.1.n) do the same for some similar 2-alternatives. From the simple schematic character of the rules, it is obvious that (1.1.m.n) follows from (1.1.m), and (2.1.n.m) follows from (2.1.n)—the latter are, in fact, *ex*$\sim$/ *ad nihil* forms of the former. The converses, however, are usually not that immediate, as one can conclude from Fact 1.3(vii) and (viii). One form of the rules in the family (1.X) or another have been in vogue since at least the XIV century, where they could indeed be found in the work of John of Cornwall (the 'Pseudo-Duns Scotus'), commenting on Aristotle's *Prior Analytics*. An emphasis on the validity of all forms of *casus judicans*, as regulating the so-called 'Principle of Excluded Middle' was strongly advocated already by stoics like Chrysippus, in which they would early be opposed, with equal strength, by Epicurus and, more modernly, by Brouwer. The validity of all forms of its dual rule, *pseudo-scotus*, regulates the so-called 'Principle of Explosion'. Accordingly, the rules in family (1.X) will be related to the metatheoretical notion of 'consistency', and those in family (2.X) will be related to '(model-)completeness', or 'determinedness'.

From the point of view of the standard model-theoretic account, (1.1.m) will make sure that no formula (of the form $\sim^{\mathrm{m}}\alpha$) can ever be true together with its negation; (1.1.m.0) will guarantee that any model for $\sim^{\mathrm{m}}\alpha$ and its negation will be dadaistic. A dual remark can be made about (2.1.n), (2.1.n.0), formulas being false together with their negations, and nihilistic models.

The attentive and well-informed reader will have already suspected that general *paraconsistency* has to do with the basic failure of explosion, that is, the failure of rule (1.1.0); dually, general *paracompleteness* has to do with the failure of (2.1.0). Thus, in particular, relevance logics provide examples of paraconsistent logics, and intuitionistic logic is an example of a paracomplete logic. In fact, duality intuitions will guide the statement of most negation rules above and below; sometimes rules from both sides of each dual pair will be well-known from the logico-mathematical praxis, in some other occasions only one of the sides will be really that common, like in the case of (1.1.m.n)—people rarely mention (2.1.n.m) at all. As a matter of fact, it seems that it is only because there is an old tendency to work under the asymmetrical multiple-premise-single-conclusion environments that people even care to look at (1.1.m.n), localizing the issue of (para)consistency over there instead of over (1.1.m). A more detailed discussion of that can be found in [26].

I proceed now to state some other rules which can easily be harvested in the literature:

(1.2.m.↓)	$(\Gamma \Vdash {\sim}^{\mathrm{m}}\alpha, \Delta)$ / $(\Gamma, {\sim}^{\mathrm{m}+1}\alpha \Vdash \Delta)$	(2.2.n.↓)	$(\Gamma, {\sim}^{\mathrm{n}}\beta \Vdash \Delta)$ / $(\Gamma \Vdash {\sim}^{\mathrm{n}+1}\beta, \Delta)$
(1.2.m.↑)	$(\Gamma \Vdash {\sim}^{\mathrm{m}+1}\alpha, \Delta)$ / $(\Gamma, {\sim}^{\mathrm{m}}\alpha \Vdash \Delta)$	(2.2.n.↑)	$(\Gamma, {\sim}^{\mathrm{n}+1}\beta \Vdash \Delta)$ / $(\Gamma \Vdash {\sim}^{\mathrm{n}}\beta, \Delta)$
	dextro-levo symmetry of negation		*levo-dextro symmetry of negation*
(1.3.m.↓)	$(\Gamma, {\sim}^{\mathrm{m}+1}\alpha \Vdash {\sim}^{\mathrm{m}}\alpha, \Delta)$ / $(\Gamma, {\sim}^{\mathrm{m}+1}\alpha \Vdash \Delta)$	(2.3.n.↓)	$(\Gamma, {\sim}^{\mathrm{n}}\beta \Vdash {\sim}^{\mathrm{n}+1}\beta, \Delta)$ / $(\Gamma \Vdash {\sim}^{\mathrm{n}+1}\beta, \Delta)$
(1.3.m.↑)	$(\Gamma, {\sim}^{\mathrm{m}}\alpha \Vdash {\sim}^{\mathrm{m}+1}\alpha, \Delta)$ / $(\Gamma, {\sim}^{\mathrm{m}}\alpha \Vdash \Delta)$	(2.3.n.↑)	$(\Gamma, {\sim}^{\mathrm{n}+1}\beta \Vdash {\sim}^{\mathrm{n}}\beta, \Delta)$ / $(\Gamma \Vdash {\sim}^{\mathrm{n}}\beta, \Delta)$
	causa mirabilis		*consequentia mirabilis*

According to [28], forms of *consequentia mirabilis* were first applied in modern mathematics by Cardano and Clavius, in the XVI century. A century later, Saccheri adopted them as some of his main tools for doing some early work on non-Euclidean geometry. At about the same period, Huygens, and to some extent also Tacquet, argued that one should refrain from merely 'formal' applications of *consequentia mirabilis* to mathematics, adopting instead the more 'intuitive' forms of *reductio ad absurdum* (cf. [7], and below). But then, results from Fact 2.3 will show that such a move is not without consequences: The latter rule is in general much stronger than the former.

Rules of symmetry, from families $(1.2.X)$ and $(2.2.X)$ (cf. [1]), are quite similar to their analogues in the families $(1.3.X)$ and $(2.3.X)$. They are sometimes used, for instance, in presenting the very definition of negation (cf. [17]) for logics intermediate between intuitionistic and classical logic.

Next, consider the rules:

(1.4.m)	$(\Gamma \Vdash {\sim}^{\mathrm{m}}\alpha, \Delta$ and $\Gamma' \Vdash {\sim}^{\mathrm{m}+1}\alpha, \Delta')$ / $(\Gamma', \Gamma \Vdash \Delta, \Delta')$	(2.4.n)	$(\Gamma, {\sim}^{\mathrm{n}}\beta \Vdash \Delta$ and $\Gamma', {\sim}^{\mathrm{n}+1}\beta \Vdash \Delta')$ / $(\Gamma', \Gamma \Vdash \Delta, \Delta')$
	right-redundancy		*left-redundancy*, or *proof by cases*

Forms of proof by cases are some of the most ancient and probably the most common rendering of patterns of reasoning by excluded middle in mathematics and philosophy.

(1.5.m.↓.n) $(\Gamma, \sim^{n}\beta \Vdash \sim^{m}\alpha, \Delta$ and $\Gamma', \sim^{n+1}\beta \Vdash \sim^{m}\alpha, \Delta')$ / $(\Gamma', \Gamma, \sim^{m+1}\alpha \Vdash \Delta, \Delta')$

(1.5.m.↑.n) $(\Gamma, \sim^{n}\beta \Vdash \sim^{m+1}\alpha, \Delta$ and $\Gamma', \sim^{n+1}\beta \Vdash \sim^{m+1}\alpha, \Delta')$ / $(\Gamma', \Gamma, \sim^{m}\alpha \Vdash \Delta, \Delta')$

reductio ex evidentia

(2.5.n.↓.m) $(\Gamma, \sim^{n}\beta \Vdash \sim^{m}\alpha, \Delta$ and $\Gamma', \sim^{n}\beta \Vdash \sim^{m+1}\alpha, \Delta')$ / $(\Gamma', \Gamma \Vdash \sim^{n+1}\beta, \Delta, \Delta')$

(2.5.n.↑.m) $(\Gamma, \sim^{n+1}\beta \Vdash \sim^{m}\alpha, \Delta$ and $\Gamma', \sim^{n+1}\beta \Vdash \sim^{m+1}\alpha, \Delta')$ / $(\Gamma', \Gamma \Vdash \sim^{n}\beta, \Delta, \Delta')$

reductio ad absurdum

One or another form of *reductio ad absurdum* can be found integrating the standard suite of mathematical tools at least since Pythagoras's discovery / invention of irrational numbers—the reduction to absurdity is indeed the gist of methods of indirect proof and of proof by refutation. Zeno of Elea also excelled the use of this rule as applied to argumentation, foreshadowing a sort of dialectical approach to critical thinking which was to become very popular later on. But *reductio* is altogether dispensed by consequence relations such as that of intuitionistic logic (in accordance with results from Fact 2.3), in concert with its general demise of excluded middle.

Continuing, a second set of pure rules for negation which can also be handy and which are often insisted upon are the following—for each choice of levels a, b, c, d, e $\in \mathbb{N}$:

(3.1.a.b.c.d) $(\Gamma, \sim^{a}\gamma \Vdash \sim^{b}\delta, \Delta)$/ $(\Gamma, \sim^{a+2c}\gamma \Vdash \sim^{b+2d}\delta, \Delta)$

(3.2.a.b.c.d) $(\Gamma, \sim^{a+2c}\gamma \Vdash \sim^{b}\delta, \Delta)$ / $(\Gamma, \sim^{a}\gamma \Vdash \sim^{b+2d}\delta, \Delta)$

(3.3.a.b.c.d) $(\Gamma, \sim^{a}\gamma \Vdash \sim^{b+2d}\delta, \Delta)$ / $(\Gamma, \sim^{a+2c}\gamma \Vdash \sim^{b}\delta, \Delta)$

(3.4.a.b.c.d) $(\Gamma, \sim^{a+2c}\gamma \Vdash \sim^{b+2d}\delta, \Delta)$ / $(\Gamma, \sim^{a}\gamma \Vdash \sim^{b}\delta, \Delta)$

double negation manipulation

(4.1.a.e) $(\Gamma, \sim^{a}\gamma \Vdash \sim^{a+2e}\gamma, \Delta)$

double negation introduction

(4.2.a.e) $(\Gamma, \sim^{a+2e}\gamma \Vdash \sim^{a}\gamma, \Delta)$

double negation elimination

(5.1.a.b.c.d) $(\Gamma, \sim^{a}\gamma \Vdash \sim^{b}\delta, \Delta)$ / $(\Gamma, \sim^{b+2d+1}\delta \Vdash \sim^{a+2c+1}\gamma, \Delta)$

(5.2.a.b.c.d) $(\Gamma, \sim^{a+2c+1}\gamma \Vdash \sim^{b}\delta, \Delta)$ / $(\Gamma, \sim^{b+2d+1}\delta \Vdash \sim^{a}\gamma, \Delta)$

(5.3.a.b.c.d) $(\Gamma, \sim^{a}\gamma \Vdash \sim^{b+2d+1}\delta, \Delta)$ / $(\Gamma, \sim^{b}\delta \Vdash \sim^{a+2c+1}\gamma, \Delta)$

(5.4.a.b.c.d) $(\Gamma, \sim^{a+2c+1}\gamma \Vdash \sim^{b+2d+1}\delta, \Delta)$ / $(\Gamma, \sim^{b}\delta \Vdash \sim^{a}\gamma, \Delta)$

contextual contraposition

(6.1.a.b.e) $(\Gamma, \sim^{a}\gamma \dashv\Vdash \sim^{b}\delta, \Delta)$ / $(\Gamma, \sim^{a+e}\gamma \dashv\Vdash \sim^{b+e}\delta, \Delta)$

(6.2.a.b.e) $(\Gamma, \sim^{a+e}\gamma \dashv\Vdash \sim^{b}\delta, \Delta)$ / $(\Gamma, \sim^{a}\gamma \dashv\Vdash \sim^{b+e}\delta, \Delta)$

(6.3.a.b.e) $(\Gamma, \sim^{a}\gamma \dashv\Vdash \sim^{b+e}\delta, \Delta)$ / $(\Gamma, \sim^{a+e}\gamma \dashv\Vdash \sim^{b}\delta, \Delta)$

(6.4.a.b.e) $(\Gamma, \sim^{a+e}\gamma \dashv\Vdash \sim^{b+e}\delta, \Delta)$ / $(\Gamma, \sim^{a}\gamma \dashv\Vdash \sim^{b}\delta, \Delta)$

contextual replacement (for negation)

The above rules regulate some fixed-point and involutive properties of negation. I should here insist that one ought not to confuse any of the above contextual rules with their (weaker) global versions. Note indeed, by way of an example, that basic forms of global contraposition, or even better, basic forms of global replacement will provide exactly what one needs for a negation to be amenable to a Lindenbaum–Tarski algebraization, and to have an adequate standard modal interpretation. But local forms of contextual contraposition and replacement will often fail for non-classical logics such as paraconsistent logics (see Fact 2.5 below), even though some of those logics will in fact be perfectly algebraizable (cf. [32] and the Section 3.12 of [16]).

Let me now invite you to have a look at some of the aftereffects and interrelations among the rules introduced just above, to get a taste of how powerful they can be.[7]

Fact 2.1. *Some relations that hold among the last set of rules for negation are*:

(i)	$(t.u.\mathrm{a}.w.x.Y) \Rightarrow (t.u.\mathrm{a}+\mathrm{b}.w.x.Y)$, *for* $\mathrm{a},\mathrm{b} \in \mathbb{N}$	$\{\}$
(ii)	$(t.u.v.\mathrm{a}.x.Y) \Rightarrow (t.u.v.\mathrm{a}+\mathrm{b}.x.Y)$, *for* $\mathrm{a},\mathrm{b} \in \mathbb{N}$	$\{\}$
(iii)	$(4.u.\mathrm{a}.w) \Rightarrow (4.u.\mathrm{a}+\mathrm{b}.w)$, *for* $\mathrm{a},\mathrm{b} \in \mathbb{N}$	$\{\}$
(iv)	$(3.x.\mathrm{a}.\mathrm{b}.0.0)$	$\{\ \}$
(v)	$(4.x.0.0) \Leftrightarrow (\mathrm{C1})$	$\{\ \}$
(vi)	$(6.x.\mathrm{a}.\mathrm{b}.0)$	$\{\ \}$
(vii)	$(3.x.\mathrm{a}.\mathrm{b}.\mathrm{c}.\mathrm{d}) \Rightarrow (3.x.\mathrm{a}.\mathrm{b}.t \times \mathrm{c}.t \times \mathrm{d})$, *for* $t > 0$	$\{\ \}$
(viii)	$(4.x.\mathrm{a}.\mathrm{e}) \Rightarrow (4.x.\mathrm{a}.t \times \mathrm{e})$, *for* $t > 0$	$\{(\mathrm{C2.k.1})\}$
(ix)	$(6.x.\mathrm{a}.\mathrm{b}.\mathrm{e}) \Rightarrow (6.x.\mathrm{a}.\mathrm{b}.t \times \mathrm{e})$, *for* $t > 0$	$\{\}$
(x)	$(x.1.\mathrm{a}.\mathrm{a}.\mathrm{f}+u.\mathrm{f}+v) \Rightarrow (4.y.\mathrm{a}+2\mathrm{f}+z.\mathrm{e})$, *for* $\langle x,u,v,y,z\rangle \in \{\langle 3,0,\mathrm{e},1,0\rangle, \langle 3,\mathrm{e},0,2,0\rangle, \langle 5,\mathrm{e},0,1,1\rangle, \langle 5,0,\mathrm{e},2,1\rangle\}$	$\{(\mathrm{C1})\}$
(xi)	$(x.2.\mathrm{a}.\mathrm{a}+2\mathrm{c}+y.\mathrm{c}.\mathrm{e}) \Rightarrow (4.z.\mathrm{a}.\mathrm{c}+\mathrm{e}+y)$, *for* $\langle x,y,z\rangle \in \{\langle 3,0,1\rangle, \langle 5,1,2\rangle\}$	$\{(\mathrm{C1})\}$
(xii)	$(x.3.\mathrm{a}+2\mathrm{c}+y.\mathrm{a}.\mathrm{e}.\mathrm{c}) \Rightarrow (4.z.\mathrm{a}.\mathrm{c}+\mathrm{e}+y)$, *for* $\langle x,y,z\rangle \in \{\langle 3,0,2\rangle, \langle 5,1,1\rangle\}$	$\{(\mathrm{C1})\}$
(xiii)	$(x.4.\mathrm{a}+2r.\mathrm{a}+2s.\mathrm{c}+t.\mathrm{c}+u) \Rightarrow (4.y.\mathrm{a}.\mathrm{e})$, *for* $\langle x,r,s,t,u,y\rangle \in \{\langle 3,0,\mathrm{e},\mathrm{e},0,1\rangle, \langle 3,\mathrm{e},0,0,\mathrm{e},2\rangle, \langle 5,\mathrm{e},0,0,\mathrm{e},1\rangle, \langle 5,0,\mathrm{e},\mathrm{e},0,2\rangle\}$	$\{(\mathrm{C1})\}$
(xiv)	$(x.4.0.0.\mathrm{f}+r.\mathrm{f}+s)$ *and* $(4.y.2\mathrm{f}+z.\mathrm{e}) \Rightarrow (\mathrm{C1})$, *for* $\langle x,r,s,y,z\rangle \in \{\langle 3,0,\mathrm{e},1,0\rangle, \langle 3,\mathrm{e},0,2,0\rangle, \langle 5,0,\mathrm{e},1,1\rangle, \langle 5,\mathrm{e},0,2,1\rangle\}$	$\{\}$
(xv)	$(v.x.\mathrm{a}.\mathrm{b}.\mathrm{e}.\mathrm{e})$ *and* $(v.y.\mathrm{b}.\mathrm{a}.\mathrm{e}.\mathrm{e}) \Rightarrow (6.z.\mathrm{a}.\mathrm{b}.2\mathrm{e}+w)$, *for* $\langle v,w\rangle \in \{\langle 3,0\rangle, \langle 5,1\rangle\}$ *and* $\langle x,y,z\rangle \in \{\langle 1,1,1\rangle, \langle 2,3,2\rangle, \langle 3,2,3\rangle, \langle 4,4,4\rangle\}$	$\{\}$
(xvi)	$(3.x.\mathrm{a}+2\mathrm{e}.\mathrm{b}.\mathrm{c}.\mathrm{d})$ *and* $(4.z.\mathrm{a}.\mathrm{c}+\mathrm{e}) \Rightarrow (3.y.\mathrm{a}.\mathrm{b}.\mathrm{e}.\mathrm{d})$, *for* $\langle x,y,z\rangle \in \{\langle 1,2,1\rangle, \langle 2,1,2\rangle, \langle 3,4,1\rangle, \langle 4,3,2\rangle\}$	$\{(\mathrm{C2})\}$, *or* $\{(\mathrm{C2.k.j})$ *and* $(\mathrm{C3.1})\}$, *or* $\{(4.3-z.\mathrm{a}.\mathrm{c}+\mathrm{e})$ *and* $(\mathrm{C2.k.j})$ *and* $(\mathrm{C3.1.p})\}$
(xvii)	$(3.x.\mathrm{a}.\mathrm{b}+2\mathrm{f}.\mathrm{c}.\mathrm{d})$ *and* $(4.z.\mathrm{b}.\mathrm{d}+\mathrm{f}) \Rightarrow (3.y.\mathrm{a}.\mathrm{b}.\mathrm{c}.\mathrm{f})$, *for* $\langle x,y,z\rangle \in \{\langle 1,3,2\rangle, \langle 2,4,2\rangle, \langle 3,1,1\rangle, \langle 4,2,1\rangle\}$	$\{(\mathrm{C2})\}$, *or* $\{(\mathrm{C2.k.j})$ *and* $(\mathrm{C3.2})\}$, *or* $\{(4.3-z.\mathrm{b}.\mathrm{d}+\mathrm{f})$ *and* $(\mathrm{C2.k.j})$ *and* $(\mathrm{C3.2.q})\}$
(xviii)	$(3.x.\mathrm{a}+2\mathrm{e}.\mathrm{b}+2\mathrm{f}.\mathrm{c}.\mathrm{d})$ *and* $(4.w.\mathrm{a}.\mathrm{c}+\mathrm{e})$ *and* $(4.z.\mathrm{b}.\mathrm{d}+\mathrm{f}) \Rightarrow (3.y.\mathrm{a}.\mathrm{b}.\mathrm{e}.\mathrm{f})$, *for* $\langle x,y,w,z\rangle \in \{\langle 1,4,1,2\rangle, \langle 4,1,2,1\rangle\}$	$\{(\mathrm{C2})\}$, *or*

[7] In the next facts, I do not claim of course to present 'all' the interesting results, and not even the 'best' possible results—in the sense of working always with the weakest premises and deriving the strongest conclusions by way of the feeblest set of assumptions, in the most general way. But I have advanced a great deal polishing the results in that direction, and the reader will see they are indeed not that bad.

$\{(4.3-w.\mathrm{a.c+e})$ *and* $(4.3-z.\mathrm{b.d+f})$ *and* (C2.k.j) *and* (C3.1.p) *and* (C3.2.q)$\}$

(xix) $(4.x.\mathrm{a.c})$ *and* $(4.y.\mathrm{b.d}) \Rightarrow (3.z.\mathrm{a.b.c.d})$, *for* $\langle x,y,z\rangle \in \{\langle 2,1,1\rangle,\langle 1,1,2\rangle,\langle 2,2,3\rangle,\langle 1,2,4\rangle\}$ — $\{$(C2)$\}$, *or* $\{(4.3-w.\mathrm{a.c})$ *and* $(4.3-z.\mathrm{b.d})$ *and* (C2.k.j) *and* (C3.1.p) *and* (C3.2.q)$\}$

(xx) (5.1.a.b.c.d) *and* $(5.1.\mathrm{b+2d+1.a+2c+1.f.e}) \Rightarrow (3.1.\mathrm{a.b.c+e+1.d+f+1})$ — $\{\}$

(xxi) $(5.2.\mathrm{a+2b+1.b.e.d})$ *and* $(5.3.\mathrm{b+2d+1.a.f.c}) \Rightarrow (3.2.\mathrm{a.b.c+e+1.d+f+1})$ — $\{\}$

(xxii) $(5.3.\mathrm{a.b+2d+1.c.f})$ *and* $(5.2.\mathrm{b.a+2c+1.d.e}) \Rightarrow (3.3.\mathrm{a.b.c+e+1.d+f+1})$ — $\{\}$

(xxiii) $(5.4.\mathrm{a+2c+1.b+2d+1.e.f})$ *and* $(5.4.\mathrm{b.a.d.c}) \Rightarrow (3.4.\mathrm{a.b.c+e+1.d+f+1})$ — $\{\}$

(xxiv) $(5.x.\mathrm{a+2e+1.b.c.d})$ *and* $(4.2.\mathrm{a.c+e+1}) \Rightarrow (5.y.\mathrm{a.b.e.d})$, *for* $\langle x,y,z\rangle \in \{\langle 1,2,2\rangle,\langle 2,1,1\rangle,\langle 3,4,2\rangle,\langle 4,3,1\rangle\}$ — $\{$(C2)$\}$, *or* $\{$(C2.k.j) *and* $(\mathrm{C3}.z)\}$, *or* $\{(4.1.\mathrm{a.c+e+1})$ *and* (C2.k.j) *and* $(\mathrm{C3}.z.\mathrm{p})\}$

(xxv) $(5.x.\mathrm{a.b+2f+1.c.d})$ *and* $(4.1.\mathrm{b.d+f+1}) \Rightarrow (5.y.\mathrm{a.b.c.f})$, *for* $\langle x,y,z\rangle \in \{\langle 1,3,1\rangle,\langle 3,1,2\rangle,\langle 2,4,1\rangle,\langle 4,2,2\rangle\}$ — $\{$(C2)$\}$, *or* $\{$(C2.k.j) *and* $(\mathrm{C3}.z)\}$, *or* $\{(4.2.\mathrm{b.d+f+1})$ *and* (C2.k.j) *and* $(\mathrm{C3}.z.\mathrm{p})\}$

(xxvi) $(5.x.\mathrm{a+2e+1.b+2f+1.c.d})$ *and* $(4.1.\mathrm{b.d+f+1})$ *and* $(4.2.\mathrm{a.c+e+1})$ *and* $(4.2.\mathrm{b.d+f+1})$ *and* $(4.1.\mathrm{a.c+e+1}) \Rightarrow (5.y.\mathrm{a.b.e.f})$, *for* $x,y\in\{1,4\}, x\neq y$ — $\{$(C2)$\}$, *or* $\{$(C2.k.j) *and* $(\mathrm{C3}.z.\mathrm{p})\}$

(xxvii) $(6.x.\mathrm{a.b.e}) \Leftrightarrow (6.x.\mathrm{b.a.e})$, *for* $x\in\{1,4\}$ — $\{\}$

(xxviii) $(6.2.\mathrm{a.b.e}) \Leftrightarrow (6.3.\mathrm{b.a.e})$ — $\{\}$

(xxix) $(6.2.\mathrm{a.a+e.e}) \Rightarrow (4.1.\mathrm{a.e})$ *and* $(4.2.\mathrm{a.e})$ — $\{$(C1)$\}$

(xxx) $(6.x.\mathrm{a+e.b.e})$ *and* $(4.1.\mathrm{a.e})$ *and* $(4.2.\mathrm{a.e}) \Rightarrow (6.y.\mathrm{a.b.e})$, *for* $x,y\in\{1,2\}, x\neq y$ — $\{$(C2.k.j) *and* (C3.1.p) *and* (C3.2.q)$\}$

(xxxi) $(6.x.\mathrm{a.b+e.e})$ *and* $(4.1.\mathrm{b.e})$ *and* $(4.2.\mathrm{b.e}) \Rightarrow (6.y.\mathrm{a.b.e})$, *for* $x,y\in\{2,4\}, x\neq y$ — $\{$(C2.k.j) *and* (C3.1.p) *and* (C3.2.q)$\}$

(xxxii) $(6.x.\mathrm{a+e.b+e.e})$ *and* $(4.1.\mathrm{m.e})$ *and* $(4.2.\mathrm{m.e}) \Rightarrow (6.y.\mathrm{a.b.e})$, *for* $x,y\in\{1,4\}, x\neq y$, *and* $\mathrm{m}=\min(\mathrm{a},\mathrm{b})$ — $\{$(C2.k.j) *and* (C3.1.p) *and* (C3.2.q)$\}$

Assuming we are talking about 1-simple logics, that is, taking overlap and 1-cautious cut for granted, let me briefly comment on the above Fact: Note that, by schematism (remember last section), less complex rules—those dealing with fewer negations—usually imply more complex ones (see (i)–(iii)); less generous rules—those introducing or eliminating fewer negations—often imply more generous ones (see (vii)–(ix)), and in the most basic cases they sometimes do not tell you much (see (iv)–(vi)); each form of double negation introduction / elimination is implied by some appropriate form of double negation manipulation or contextual contraposition (see (x)–(xiii)) and a similar thing happens with respect to contextual replacement (see (xv)); moreover, some strong forms of the rules for double negation manipulation or contraposition can only hold together with the introduction / elimination rules for double negation in case the underlying consequence relation respects overlap (see (xiv)); contextual replacement alone can also force double negation introduction / elimination rules to hold (see (xxix) and (xxviii)); all forms of double negation manipulation can in fact be deduced from appropriate forms of double negation introduction / elimination (see (xix)); combinations of appropriate forms of contextual contraposition will also immediately yield some forms of double negation manipulation (see (xx)–(xxiii)); some forms of double negation manipulation will even imply others, given convenient rules for double negation introduction / elimination (see (xvi)–(xviii)), and a similar thing will happen with contextual contraposition (see (xxiv)–(xxvi)); some forms of contextual replacement will also imply others, either in general (see (xxvii) and (xxviii)) or in the presence of appropriate forms of double negation introduction / elimination (see (xxx)–(xxxii)).

It is now easy to conclude from the above that there are some rules which are somehow 'more fundamental' than others. For instance:

Illustration 2.2. Here are a few possible choices of rules from which all the other rules from families $(3.X)$, $(4.X)$, $(5.X)$ and $(6.X)$ follow, inside any cautious tarskian logic:

(1) (5.4.0.0.0.1) and (5.4.0.0.1.0)
(2) (5.1.1.1.0.0) and (5.2.0.1.0.0) and (5.3.1.0.0.0)

To check that, use the last Fact. In case (1), parts (i) and (ii) give you schematism, from which you can conclude (5.4.a.b.0.1) and (5.4.a.b.1.0), for any $\mathrm{a}, \mathrm{b} \in \mathbb{N}$. From that you have in particular that (5.4.2.0.0.1) and (5.4.0.2.1.0), thus (4.1.0.1) and (4.2.0.1) are inferred from (xiii). From (iii), (viii) and (v) you can derive all rules from family $(4.X)$. With the help of those rules and (xxvi) all the rules of the form $(5.1.Y)$ and $(5.4.Y)$ ensue, and using (xxiv) and (xxv) you can derive the rest of the family $(5.X)$. The remaining derivations are left to the reader.

In case (2), (4.2.0.1) follows from (5.2.0.1.0.0) by (xi) and (4.1.0.1) follows from (5.3.1.0.0.0) by (xii). From that, (5.1.1.1.0.0), and schematism, (5.4.a.b.c.d) follows, using (xxvi), and we're back to case (1).

Another interesting set of results concerning the above rules is presented in what follows.

Fact 2.3. *Here are some other relations which can be checked to hold among the above rules for negation*:

(i) $(1.X.\mathrm{a}.Y) \Rightarrow (1.X.\mathrm{a}+\mathrm{b}.Y)$, *for* $\mathrm{a}, \mathrm{b} \neq \downarrow, \uparrow$ $\quad\{\}$

(ii) $(1.x.\mathrm{a}+\mathrm{b}.Y)$ *and* $(4.1.\mathrm{a}.\mathrm{e}) \Rightarrow (1.x.\mathrm{a}.Y)$, *for* $\mathrm{e} > 0$ *and* $\langle x, y, z\rangle \in \{\langle 1,1,1\rangle, \langle 2,1,2\rangle, \langle 4,2,2\rangle, \langle 5,1,2\rangle\}$ $\quad\{(\mathrm{C2})\}$, *or* $\{(\mathrm{C2.k.j})$ *and* $(\mathrm{C3}.y)$ *and* $(\mathrm{C3}.z)\}$, *or* $\{(4.2.\mathrm{a.e})$ *and* $(\mathrm{C2.k.j})$ *and* $(\mathrm{C3}.y.\mathrm{p})$ *and* $(\mathrm{C3}.z.\mathrm{q})\}$

(iii) $(1.x.Y.\mathrm{a}+\mathrm{b})$ *and* $(4.2.\mathrm{a.e}) \Rightarrow (1.x.Y.\mathrm{a})$, *for* $\mathrm{e} > 0$ *and* $\langle x, y, z\rangle \in \{\langle 1,2,2\rangle, \langle 5,1,1\rangle\}$ $\quad\{(\mathrm{C2})\}$, *or* $\{(\mathrm{C2.k.j})$ *and* $(\mathrm{C3}.y)$ *and* $(\mathrm{C3}.z)\}$, *or* $\{(4.1.\mathrm{a.e})$ *and* $(\mathrm{C2.k.j})$ *and* $(\mathrm{C3}.y.\mathrm{p})$ *and* $(\mathrm{C3}.z.\mathrm{q})\}$

(iv) $(1.3.\mathrm{a}+\mathrm{b}.\downarrow)$ *and* $(4.1.\mathrm{a.e}) \Rightarrow (1.3.\mathrm{a}.\downarrow)$, *for* $\mathrm{e} > 0$ $\quad\{(4.2.\mathrm{a}+1.\mathrm{e})$ *and* $(\mathrm{C2})\}$, *or* $\{(4.2.\mathrm{a.e})$ *and* $(\mathrm{C2.k.j})$ *and* $(\mathrm{C3.1.p})$ *and* $(\mathrm{C3.2.q})\}$

(v)	$(1.3.\mathrm{a}+\mathrm{b}.\uparrow)$ *and* $(4.2.\mathrm{a.e}) \Rightarrow (1.3.\mathrm{a}.\uparrow)$,	$\{(4.1.\mathrm{a.e})$ *and* $(\mathrm{C2})\}$, *or*
	for $\mathrm{e}>0$	$\{(4.1.\mathrm{a.e})$ *and* $(\mathrm{C2.k.j})$ *and* $(\mathrm{C3.1.p})$ *and* $(\mathrm{C3.2.q})\}$
(vi)	$(1.1.0.1)$ *and* $(5.4.0.0.0.0) \Rightarrow (\mathrm{C1})$	$\{\}$
(vii)	$(1.1.\mathrm{m}) \Rightarrow (1.1.\mathrm{m}.x)$	$\{\}$
(viii)	$(1.1.\mathrm{m}.0) \Rightarrow (1.1.\mathrm{m})$	$\{$*bottom and* $(\mathrm{C2.k.j})\}$
(ix)	$(1.1.\mathrm{m}) \Rightarrow (1.2.\mathrm{m}.\downarrow)$ *and* $(1.2.\mathrm{m}.\uparrow)$	$\{(\mathrm{C2})\}$ *or* $\{(\mathrm{C2.k.j})+(\mathrm{C3.1})\}$
(x)	$(1.2.\mathrm{m}.\downarrow)$ *or* $(1.2.\mathrm{m}.\uparrow) \Rightarrow (1.1.\mathrm{m})$	$\{(\mathrm{C1})\}$
(xi)	$(1.2.\mathrm{m}.x) \Rightarrow (1.3.\mathrm{m}.x)$	$\{\}$
(xii)	$(1.1.\mathrm{m}) \Rightarrow (1.3.\mathrm{m}.\downarrow)$ *and* $(1.3.\mathrm{m}.\uparrow)$	$\{(\mathrm{C2.k.j})\}$
(xiii)	$(1.3.\mathrm{m}.\downarrow)$ *or* $(1.3.\mathrm{m}.\uparrow) \Rightarrow (1.1.\mathrm{m})$	$\{(\mathrm{C1})\}$
(xiv)	$(1.1.\mathrm{m}) \Rightarrow (1.4.\mathrm{m})$	$\{(\mathrm{C2})\}$
(xv)	$(1.4.\mathrm{m}) \Rightarrow (1.x.\mathrm{m}.Y)$, *for* $x \in \{1,2,3\}$	$\{(\mathrm{C1})\}$
(xvi)	$(1.3.\mathrm{m}.\uparrow)$ *and* $(2.1.\mathrm{m}+1.\mathrm{m}) \Rightarrow (4.1.\mathrm{m}.1)$	$\{\}$
(xvii)	$(1.3.\mathrm{m}+1.\downarrow)$ *and* $(2.1.\mathrm{m.m}+2) \Rightarrow (4.2.\mathrm{m}.1)$	$\{\}$
(xviii)	$(1.5.\mathrm{m}.\uparrow.\mathrm{m}+1) \Rightarrow (4.1.\mathrm{m}.1)$	$\{(\mathrm{C1})\}$
(xix)	$(1.5.\mathrm{m}+1.\downarrow.\mathrm{m}) \Rightarrow (4.2.\mathrm{m}.1)$	$\{(\mathrm{C1})\}$
(xx)	$(1.5.\mathrm{m}+1.x.\mathrm{n})$ *and* $(4.1.\mathrm{m}.1) \Rightarrow (1.5.\mathrm{m}.y.\mathrm{n})$,	$\{(\mathrm{C2})\}$, *or*
	$\langle x,y,z\rangle \in \{\langle\downarrow,\uparrow,1\rangle,\langle\uparrow,\downarrow,2\rangle\}$	$\{(\mathrm{C2.k.j})$ *and* $(\mathrm{C3}.z)\}$, *or*
		$\{(4.2.\mathrm{m}.1)$ *and* $(\mathrm{C2.k.j})\ (\mathrm{C3}.z.\mathrm{p})\}$
(xxi)	$(1.5.\mathrm{m}.x.\mathrm{n})$ *and* $(4.2.\mathrm{m}.1) \Rightarrow (1.5.\mathrm{m}+1.y.\mathrm{n})$,	$\{(\mathrm{C2})\}$, *or*
	$\langle x,y,z\rangle \in \{\langle\downarrow,\uparrow,2\rangle,\langle\uparrow,\downarrow,1\rangle\}$	$\{(\mathrm{C2.k.j})$ *and* $(\mathrm{C3}.z)\}$, *or*
		$\{(4.1.\mathrm{m}.1)$ *and* $(\mathrm{C2.k.j})\ (\mathrm{C3}.z.\mathrm{p})\}$
(xxii)	$(1.5.\mathrm{m}.\downarrow.x)$ *or* $(1.5.\mathrm{m}.\uparrow.x) \Rightarrow (1.1.\mathrm{m})$	$\{(\mathrm{C1})\}$
(xxiii)	$(1.5.\mathrm{m}.\uparrow.\mathrm{n}) \Rightarrow (2.1.\mathrm{n.m})$	$\{(\mathrm{C1})\}$
(xxiv)	$(1.5.\mathrm{m}.\downarrow.\mathrm{n}) \Rightarrow (2.1.\mathrm{n.m}+1)$	$\{(\mathrm{C1})\}$
(xxv)	$(1.2.\mathrm{m}.x)$ *and* $(2.4.\mathrm{n}) \Rightarrow (1.5.\mathrm{m}.x.\mathrm{n})$	$\{\}$
(xxvi)	$(1.3.0.\uparrow)$ *and* $(2.1.0.0) \Rightarrow (\mathrm{C1})$	$\{\}$
(xxvii)	$(1.3.\mathrm{m}.\uparrow)$ *and* $(2.1.\mathrm{n.m}) \Rightarrow (1.5.\mathrm{m}.\uparrow.\mathrm{n})$	$\{(\mathrm{C2})\}$
(xxviii)	$(1.3.\mathrm{m}.\downarrow)$ *and* $(2.1.\mathrm{n.m}+1) \Rightarrow (1.5.\mathrm{m}.\downarrow.\mathrm{n})$	$\{(\mathrm{C2})\}$

This much for the $(1.X)$*-column. Dual results obtain for the* $(2.X)$*-column, if one only uniformly substitutes, in the formulation of the above items, each*: $(1.X)$ *for* $(2.X)$, *and vice-versa*; $(4.1.X)$ *for* $(4.2.X)$, *and vice-versa*; *tops for bottoms*; $(\mathrm{C3}.1.X)$ *for* $(\mathrm{C3}.2.X)$, *and vice-versa.*

To make things more concrete, if we assume we are talking about simple consequence relations then the non-obvious parts of the previous Fact boil down to something like this: Again, by schematism, less complex rules imply more complex ones (see (i)), but then, in the presence of appropriate forms of double negation introduction elimination, complex rules can on their turn be simplified (see (ii)–(v)); there are always equivalent forms of *pseudo-scotus*, dextro-levo symmetry of negation, *causa mirabilis* and right-redundancy (see (ix)–(xv)); *ex contradictione* is in reality weaker than *pseudo-scotus*[8] (see (vii)

[8] Recall for instance the semitrivial logic from the last section. That specific 1.1-overcomplete logic respects *ex contradictione* but not *pseudo-scotus*. A more general realization of that phenomenon as applied to non-overcomplete logics was explored in [26].

and (viii)); $\uparrow$-forms and $\downarrow$-forms of *reductio ex evidentia* can in fact imply each other if appropriate forms of double negation introduction or elimination are available (see (xx) and (xxi)); moreover, some double negation rules are implied by *reductio* (see (xviii) and (xix)); *reductio ex evidentia* also gives you *pseudo-scotus* (see (xxii)) and some forms of *quodlibet sequitur ad casos* (see (xxiii) and (xxiv)); in fact, you can only count on both 'full consistency' and 'semicompleteness' then you can get *reductio ex evidentia* back (see (xxv), (xxvii) and (xxviii)); no surprise, appropriate forms of *causa mirabilis* and *ad casos* can tell you something about double negation (see (xvi) and (xvii)). Note also, for more general classes of logics, that a consequence relation cannot fail overlap once it respects, for instance, either some basic forms of *causa mirabilis* and *ad casos*, or some forms of *ex contradictione* and contextual contraposition (see (xxvi) and (vi)). This much for the 'consistency' column $(1.X)$; dual readings are readily available for the column of 'determinedness', $(2.X)$. Consequently, in case you have a (simple) paraconsistent or paracomplete logic you are bound to lose some forms of symmetry of negation, some of its miraculous and redundancy rules, and some forms of *reductio*.

In a single-conclusion framework, rules such as *pseudo-scotus*, symmetry, proof by cases and *reductio ex evidentia* are not expressible in the way they were here presented—so it will happen, for instance, that *pseudo-scotus* and *ex contradictione* will be indistinguishable. Observe that, if your (multiple-conclusion) consequence relation respects overlap, then the validity of *reductio ad absurdum* implies the validity of *ex contradictione*; differently from the single-conclusion case, though, *pseudo-scotus* can now still fail in such a situation. The attentive reader will have noticed that not everything is completely symmetrical, however, even in the multiple-conclusion framework. For instance:

Illustration 2.4. Inside simple logics:

(1) $(x.5.\mathrm{m}.\uparrow.\mathrm{n}) \Rightarrow (x.5.\mathrm{m}.\downarrow.\mathrm{n})$, for $\mathrm{n} \leqslant \mathrm{m}+1$
(2) $(x.5.\mathrm{m}.\downarrow.\mathrm{n}) \Rightarrow (x.5.\mathrm{m}+1.\uparrow.\mathrm{n})$, for $\mathrm{n} \leqslant \mathrm{m}$

To check those assertions, use parts (i) and (xviii)–(xxi) from the last Fact. Moreover, you can now easily check that all rules from families $(1.X)$ and $(2.X)$ become valid once both $(1.5.0.\uparrow.0)$ and $(2.5.0.\uparrow.0)$ are verified by a simple logic. Another option to generate a basis for all the other rules is to include a top particle together with $(1.5.0.\uparrow.0)$, or else to include a bottom particle together with $(2.5.0.\uparrow.0)$.

Notice, at any rate, that one can easily think of a simple logic for which $(x.5.\mathrm{m}.\downarrow.\mathrm{n})$ holds good, for all levels $\mathrm{m}, \mathrm{n} \in \mathbb{N}$, and where tops and bottoms are present, while $(x.5.0.\uparrow.0)$ is still not inferable—such is the case, for instance, of intuitionistic logic.

In [8], Béziau pointed out an interesting way of correcting the above asymmetry, which runs like this. Recall from Section 0 that we have added and have been using symbols for iterated negations, defined in terms of a single negation, $\sim$, by setting $\sim^0\alpha := \alpha$ and $\sim^{\mathrm{n}+1}\alpha := \sim^{\mathrm{n}}\sim\alpha$, for $\mathrm{n} \in \mathbb{N}$. Now, take instead all such symbols $\sim^{\mathrm{n}}$ as primitive symbols, and consider the 'symmetric domain' given by the integers, requiring only the schematic axiom $\sim^{\mathrm{a}+\mathrm{b}}\alpha = \sim^{\mathrm{a}}\sim^{\mathrm{b}}\alpha$, for every $\mathrm{a}, \mathrm{b} \in \mathbb{Z}$, to be respected. Keeping the above rules exactly as they were presented, it is clear that all the Facts that we proved (or else some slightly modified versions of them) keep provable with this new definition. But now the above pathology cannot obtain, and if $(x.5.\mathrm{m}.{\downarrow}.\mathrm{n})$ holds good, for a given simple logic and any given levels $\mathrm{m}, \mathrm{n} \in \mathbb{Z}$, then $(x.5.\mathrm{m}.{\uparrow}.\mathrm{n})$ will also hold good, as a consequence, for all $\mathrm{m}, \mathrm{n} \in \mathbb{Z}$. So far, so well. The author of [8], however, after using this symmetrization on the content of the above Illustration to suggest that, in a symmetric domain, the differences between classical and intuitionistic negation will vanish, also proceeded to use particular cases of the derivations in Fact 2.3 in order to point some forms of the above rules from which all the other rules would be derived. More specifically, in the single-conclusion environment that he works in, he points out that the validity of *reductio ad absurdum* in a simple logic will be enough to allow for the derivation of all the other rules for negation. But, as we have seen above, in case we use a multiple-conclusion environment and there is no bottom present in the language of the logic, one might quite well have all forms of *reductio ad absurdum* holding good while *pseudo-scotus* still fails; in case there is no top in the logic, all forms of *reductio ex evidentia* might be available and still *casus judicans* might fail. (It does not really help to point out that canonical sequent-style presentations of intuitionistic logic are single-conclusion. Multiple-conclusioned presentations for that same logic have been known since long—check [34], for instance.) So, to be sure, contrarily to what Béziau asserts, here we see that *reductio ad absurdum* alone does *not* sanction the derivation of all the other rules for negation. One always has to be alert not to let a particular choice of framework fool oneself into deceivingly general conclusions.

Fact 2.5. *Some further interesting relations among the two above sets of rules for negation are* (*let* $\mathsf{opt} = \{\langle\downarrow,\downarrow,1\rangle, \langle\uparrow,\downarrow,2\rangle, \langle\downarrow,\uparrow,3\rangle, \langle\uparrow,\uparrow,4\rangle\}$):

(i)	$(1.2.\mathrm{a}+r.x)$ *and* $(1.2.\mathrm{b}+s.y)$ *and* $(2.2.\mathrm{a}+t.x)$ *and* $(2.2.\mathrm{b}+u.x) \Rightarrow$ $(3.z.\mathrm{a.b.c.d})$, *for* $\langle x,y,z\rangle \in \mathsf{opt}$ *and* $\langle z,r,s,t,u\rangle \in \{\langle 1,1,0,0,1\rangle, \langle 2,0,0,1,1\rangle, \langle 3,1,1,0,0\rangle, \langle 4,0,1,1,0\rangle\}$	$\{\}$
(ii)	$(w.2.\mathrm{a}.x)$ *and* $(3-w.2.\mathrm{a}+1.x) \Rightarrow (4.y.\mathrm{a.e})$, *for* $w \in \{1,2\}$	$\{(\mathrm{C1})\}$
(iii)	$(1.2.\mathrm{b}.y)$ *and* $(2.2.\mathrm{a}.x) \Rightarrow (5.z.\mathrm{a.b}.0.0)$, *for* $\langle x,y,z\rangle \in \mathsf{opt}$	$\{\}$
(iv)	$(1.2.\mathrm{a}+1.x)$ *and* $(1.2.\mathrm{b}.y)$ *and* $(2.2.\mathrm{a}.x)$ *and* $(2.2.\mathrm{b}+1.y) \Rightarrow$ $(5.z.\mathrm{a.b.c.d})$, *for* $\langle x,y,z\rangle \in \mathsf{opt}$	$\{\}$
(v)	$(1.2.\mathrm{a}.x)$ *and* $(1.2.\mathrm{b}.y)$ *and* $(2.2.\mathrm{a}.x)$ *and* $(2.2.\mathrm{b}.y) \Rightarrow$ $(6.z.\mathrm{a.b}.w)$, *for* $w>0$ *and* $\langle x,y,z\rangle \in \mathsf{opt}$	$\{\}$
(vi)	$(1.2.\mathrm{a}+1.\uparrow)$ *and* $(4.1.\mathrm{a}.1) \Rightarrow (1.1.\mathrm{a})$	$\{\}$
(vii)	$(1.2.\mathrm{a}.\downarrow)$ *and* $(4.2.\mathrm{a}.1) \Rightarrow (1.1.\mathrm{a}+1)$	$\{\}$
(viii)	$(1.2.\mathrm{a}+2\mathrm{e}.x)$ *and* $(4.1.\mathrm{a.e}) \Rightarrow (1.1.\mathrm{a})$	$\{(\mathrm{C2.k.j})\}$
(ix)	$(1.2.\mathrm{a}.x)$ *and* $(4.2.\mathrm{a.e}) \Rightarrow (1.1.\mathrm{a}+2\mathrm{e})$	$\{(\mathrm{C2.k.j})\}$
(x)	$(5.1.\mathrm{a.b.c.d})$ *and* $(4.2.\mathrm{b.d}) \Rightarrow (1.1.\mathrm{b}+2\mathrm{d.a}+2\mathrm{c}+1)$	$\{\}$
(xi)	$(5.2.\mathrm{a.b.c.d})$ *and* $(4.2.\mathrm{b.d}) \Rightarrow (1.1.\mathrm{b}+2\mathrm{d.a})$	$\{\}$
(xii)	$(5.3.\mathrm{a.b.c.d})$ *and* $(4.1.\mathrm{b}+1.\mathrm{d}) \Rightarrow (1.1.\mathrm{b.a}+2\mathrm{c}+1)$	$\{\}$
(xiii)	$(5.4.\mathrm{a.b.c.d})$ *and* $(4.1.\mathrm{b}+1.\mathrm{d}) \Rightarrow (1.1.\mathrm{b.a})$	$\{\}$
(xiv)	$(5.1.\mathrm{a.b.c.d})$ *and* $(4.1.\mathrm{b}+1.\mathrm{d}) \Rightarrow (1.1.\mathrm{b.a}+2\mathrm{c}+1)$	$\{(\mathrm{C1})$ *and* $(\mathrm{C2.k.j})\}$
(xv)	$(5.2.\mathrm{a.b.c.d})$ *and* $(4.1.\mathrm{b}+1.\mathrm{d}) \Rightarrow (1.1.\mathrm{b.a})$	$\{(\mathrm{C1})$ *and* $(\mathrm{C2.k.j})\}$
(xvi)	$(5.3.\mathrm{a.b.c.d})$ *and* $(4.2.\mathrm{b.d}) \Rightarrow (1.1.\mathrm{b}+2\mathrm{d.a}+2\mathrm{c}+1)$	$\{(\mathrm{C1})$ *and* $(\mathrm{C2.k.j})\}$
(xvii)	$(5.4.\mathrm{a.b.c.d})$ *and* $(4.2.\mathrm{b.d}) \Rightarrow (1.1.\mathrm{b}+2\mathrm{d.a})$	$\{(\mathrm{C1})$ *and* $(\mathrm{C2.k.j})\}$

(xviii)	$(\mathrm{C}0.0.1) \Rightarrow (x.y.Z)$, *for* $x \in \{2, 3, 4, 5, 6\}$ *and* $x.y \neq 2.4$	{}
(xix)	$(\mathrm{C}0.0.1) \Rightarrow (x.y.Z)$, *for* $x.y = 2.4$	{(C2)}
(xx)	$(\mathrm{C}0.0.0) \Rightarrow (Z)$	{}

Dual results hold if one uniformly substitutes, in the above items, each: $(1.X)$ *for* $(2.X)$, *and vice-versa;* $(4.1.X)$ *for* $(4.2.X)$, *and vice-versa;* (3.z.b.a.X) *for* (3.z.a.b.X); (5.z.b.a.d.c) *for* (5.z.a.b.c.d); (3.2.X) *for* (3.3.X), *and vice-versa;* (5.2.X) *for* (5.3.X), *and vice-versa;* (C0.1.0) *for* (C0.0.1).

So, at least as far as simple logics are concerned, one sees that appropriate forms of symmetry rules from the consistency and the completeness families together are enough to imply each rule from the second bunch of rules, that is, those rules involving double negation, contraposition or contextual replacement (see (i)–(v), and recall also Fact 2.1(xix)); furthermore, in the presence of appropriate forms of double negation introduction / elimination, one sees how symmetry rules imply *pseudo-scotus* and *casus judicans*, and how contextual contraposition rules imply *ex contradictione* and *ad casos* (see (vi)–(ix) and (x)–(xvii)). Finally, note that overcompleteness might give you the positive properties for free (see (xviii)–(xx)). As a particularly interesting base for deriving all the other rules, one might consider:

Illustration 2.6. Inside any logic respecting overlap (rule (C1)), all the rules from families $(1.X)$–$(6.X)$ follow from the validity of basic rules such as $(1.4.0)$ together with $(2.4.0)$.

To check that all rules from families $(1.X)$ and $(2.X)$ follow from $(1.4.0)$ and $(2.4.0)$, recall parts (xv), (vii), (xxv) and (i) of Fact 2.3. For the remaining rules, use parts (ii)–(v) of Fact 2.5, together with parts (iv)–(vi) and (xx)–(xxiii) of Fact 2.1.

We might now reasonably ask ourselves: Have we not been too permissive? Is there anything *in common*, after all, among 'all negations'? I have prudently not said a word about that matter this far. More interesting for me was to note the consequences of each set of rules assumed to hold at each given moment. For instance, taking Fact 2.3 into consideration, if you are talking about a simple paraconsistent logic, then you should first allow for inconsistent models, thus you cannot expect any of the rules of the form $(1.x.0.Y)$ to be valid—except perhaps for *ex contradictione*, and this only in case there is no bottom particle present in your logic. Now, if *ex contradictione* is also not valid, as it is usually the case, then *reductio ad absurdum* must also fail. Moreover, taking Fact 2.1 into consideration, if your logic also lacks some form of double negation introduction / elimination, then not all forms of contextual contraposition will be interderivable, and not all forms of contextual replacement will be interderivable; in fact, some forms of contextual contraposition and of contextual replacement will be simply *prevented* from holding. Finally, taking Fact 2.5 into consideration, any double negation manipulation, contextual contraposition or contextual replacement rule that might be lacking will cause a failure of symmetry, and your simple logic might end up being either paraconsistent or paracomplete, in the presence of appropriate forms of double negation introduction / elimination; the failure of *pseudo-scotus* at given levels is incompatible with both symmetry and double negation rules at related levels; the failure of *ex contradictione* will condemn either some form of contextual contraposition or of double negation, and so forth. Dual results hold for paracomplete logics and undetermined models.

All that said and done, it might come as no surprise the acknowledgment that some of the few things which are common to all negations in the literature are not 'positive properties', but 'negative' ones. In fact, it is not that they *have* something in common, but that they *lack* some things in unison. Consider the following set of negative rules, for each level $\mathrm{a} \in \mathbb{N}$:

(7.1.a)	$(\Gamma, \sim^{\mathrm{a}+1}\varphi \nVdash \Delta)$ nonbot	(8.1.a)	$(\Gamma \nVdash \sim^{\mathrm{a}+1}\varphi, \Delta)$ nontop
(7.2.a)	$(\Gamma, \sim^{\mathrm{a}+1}\varphi \nVdash \sim^{\mathrm{a}}\varphi, \Delta)$ *verificatio*	(8.2.a)	$(\Gamma, \sim^{\mathrm{a}}\varphi \nVdash \sim^{\mathrm{a}+1}\varphi, \Delta)$ *falsificatio*

Of course, I continue to consider above only sub-classical properties of negation: The negative rules stated above are rules which *can* hold in classical logic for some particular choice of contexts and of (negated) formulas, but that should *not*, I contend, hold in general for an object we intend to call 'negation'.[9]

From a semantic point of view, (7.1.a) makes sure that our negation is not an operator which produces only bottom particles, and (8.1.a) poses a similar restriction on operators which produce only top particles—these could be held as some sort of very basic requirements for a decent version of this logical constant. Now, a decent negation operator should also embody some reasonable notion of 'opposition': Accordingly, (7.2.a) requires that the negation of some formula can be true while that formula itself is false, and (8.2.a) requires, dually, that some true sentence should have a false negation—thus, no extreme case will be allowed in which all models are dadaistic (that is, thoroughly inconsistent) or nihilistic (that is, thoroughly undetermined). In particular, any of those last two rules preclude identity as an interpretation of negation. This negative axiomatic outlook seems rare, but, I submit, is not really that controversial—in fact, I am unaware of *any* connective which has been seriously proposed intending to represent some sort of 'negation' and that does not respect all the above negative rules. Some interesting results involving the last set of rules follow:

Fact 2.7. *Some further interesting relations among the three above sets of rules for negation are*:

(i) $\neg(7.1.\mathrm{a}) \Rightarrow (1.1.\mathrm{a})$ $\{\}$

(ii) $\neg(7.1.\mathrm{a}) \Rightarrow (4.x.\mathrm{a}+1.e)$ $\{\}$

(iii) $(7.x.\mathrm{a}+\mathrm{b}) \Rightarrow (7.x.\mathrm{a})$ $\{\}$

(iv) $(7.1.\mathrm{a})$ *and* $(4.1.\mathrm{a}+1.\mathrm{e}) \Rightarrow (7.1.\mathrm{a}+\mathrm{b})$, *for* $\mathrm{e} > 0$ $\{(\mathrm{C2})\}$, *or* $\{(\mathrm{C2.k.j})$ *and* $(\mathrm{C3.1})\}$, *or* $\{(4.2.\mathrm{a}+1.\mathrm{e})$ *and* $(\mathrm{C2.k.j})$ *and* $(\mathrm{C3.1.p})\}$

(v) $(7.2.\mathrm{a})$ *and* $(3.4.\mathrm{a}+1.\mathrm{a.e.e}) \Rightarrow (7.2.\mathrm{a}+\mathrm{b})$, *for* $\mathrm{e} > 0$ $\{\}$

[9] Note that I did *not* at any point require—and I will not require—that logics should have any theses / theorems / tautologies / top particles, as much as I also did not require at any point that logics should have any antitheses. Important logics such as Kleene's 3-valued logic have no theses at all. In particular, I surely did not require that logics should have *negated* theses, that is, theses of the form $\sim\alpha$. An example of paraconsistent logic extending positive classical logic by the addition of (2.1.0) and (4.2.0.1) and which can be proven to have no negated theses nor bottom particles is the logic studied under the name C_{min} in [15].

(vi)	(7.1.a) *and* (1.3.a.$\downarrow$) $\Rightarrow$ (7.2.a)	{}
(vii)	(7.2.a) $\Rightarrow$ (7.1.a)	{}
(viii)	(7.1.a $+$ 1) *and* (1.2.a $+$ 1.$\downarrow$) $\Rightarrow$ (8.1.a)	{}
(ix)	(7.1.a) *and* (1.2.a $+$ 1.$\uparrow$) $\Rightarrow$ (8.1.a $+$ 1)	{}
(x)	(7.2.a $+$ 1) $\Rightarrow$ (8.1.a)	{}
(xi)	(7.1.a) *and* (1.3.a $+$ 1.$\uparrow$) $\Rightarrow$ (8.2.a $+$ 1)	{}
(xii)	(7.2.a) *and* (5.4.a.a $+$ 1.e.e) $\Rightarrow$ (8.2.a $+$ 2e $+$ 1)	{}
(xiii)	(C0.0.1) $\Rightarrow \neg$(8.x.y)	{}
(xiv)	(2.1.0) *and* $\neg$(7.x.0) $\Rightarrow$ (C0.0.1)	{(C2.k.j)}

Dual results hold if one uniformly substitutes, in the above items, each: $(1.X)$ *for* $(2.X)$*, and vice-versa;* $(7.X)$ *for* $(8.X)$*, and vice-versa;* $(4.1.X)$ *for* $(4.2.X)$*, and vice-versa;* $(x.4.\text{a.a}+1.\text{e.e})$ *for* $(x.4.\text{a}+1.\text{a.e.e})$*, and vice-versa;* (C3.2.q) *for* (C3.1.p); (C0.1.0) *for* (C0.0.1).

So we see that: If a logic disrespects nonbot then it cannot fail *pseudo-scotus* nor double negation elimination (see (i) and (ii)); this time more complex negative rules imply simpler ones by schematism (see (iii)), the converses being true in some special cases, in an appropriate logical environment, given some appropriate form of double negation introduction / elimination or some form of double negation manipulation (see (iv) and (v)); nonbot implies *verificatio* in the presence of *causa mirabilis*, while the converse is always true in virtue of schematism (see (vi), (vii) and (iii)); nonbot implies an appropriate form of nontop in the presence of an appropriate form of dextro-levo-symmetry (see (viii) and (ix)); *verificatio* always implies nontop in virtue of schematism (see (x)); *falsificatio* is implied by nonbot by way of an appropriate form of *causa mirabilis*, and is implied by *verificatio* in the presence of a conveniently strong form of contraposition (see (xi) and (xii)). This much if we put the family $(7.X)$ at the side of the premises; dual readings can be effected if we now put the family $(8.X)$ there. Notice also that, on the one hand, basic forms of overcompleteness imply the failure of the rules from the last two families (see (xiii)) and, on the other hand, a failure of any of the most basic forms of the last given rules occasions overcompleteness in the appropriate positive environment (see (xiv))—or, to put it differently, non-overcompleteness together with determinedness might imply *verificatio*, together with consistency it might imply *falsificatio*.

One can conclude from this last Fact that no paraconsistent logic can disrespect nonbot (and a similar restriction applies to logics without double negation introduction / elimination); on the other hand, if you fix a logic which respects weak cut, any explosive negation in it had better respect nontop, or else it can occasion overcompleteness. Moreover, if a logic respects *verificatio* then it automatically respects nonbot as well, and similarly for *falsificatio* and nontop; besides, in the presence of appropriate forms of levo-dextro-symmetry of negation, nonbot implies nontop. If a logic respects some of the above negative rules, then we are safeguarded against the most basic forms of overcompleteness. Non-overcomplete logics respecting weak cut and some of the above positive rules will also often respect some of the above negative rules, but a logic can respect all the given negative rules and yet respect none of the given positive rules (ok, I concede: This would be quite weak of a 'negation'—but check the next sections). Dual results can easily be checked for paracomplete logics.

Another pure negative rule which might occur to the reader at this point is the following:

(9.a) $\neg(\Gamma, \sim^{\mathrm{a}}\varphi \dashv\Vdash \sim^{\mathrm{a}+1}\varphi, \Delta)$
paradoxical inequivalence

Many set-theoretical paradoxes end up by sanctioning a paradoxical inference which fails some form of (9.a), rather than directly proving a pair of contradictory formulas. But the failure of (9.a) means the failure of both of the corresponding rules (7.2.a) and (8.2.a), and from that it follows, using the last Fact, that those failures leave us standing a very short step from some form of overcompleteness.

3. Causes and consequences for paranormal logics

It is contrary to common sense to entertain apprehensions or terrors upon account of any opinion whatsoever, or to imagine that we run any risk hereafter, by the freest use of our reason. Such a sentiment implies both an absurdity and an inconsistency.

—David Hume, Dialogues Concerning Natural Religion, 1779.

As I see it, a natural continuation of the last section should include an analysis of the consequences of the 'paraconsistent attitude', that is, a brief list of properties enjoyed or avoided by logics for which the positive rule (1.1.0) fails, in the light of all previous Facts. Calculating this is a purely mechanical task, so this section will only provide some Illustrations of such calculations, instead of trying the reader's patience with further lengthy enumeration of facile results.

To make things even more interesting, I will in fact start by quickly showing how the present environment can help in the specification of some interesting specializations of the notion of paraconsistency (see [16]). Recall that in paraconsistent logics the rule $(\Gamma, \alpha, \sim\alpha \Vdash \Delta)$ does not hold in general, that is, it is not valid for some choice of contexts Γ and Δ and some formula α. Of course, the rule *does* hold, for instance, in case either α or $\sim\alpha$ are bottom particles. Now, suppose there is some formula $\varphi\langle[\varphi_i]_{i\in\mathrm{I}}\rangle$ of a special format such that neither φ nor $\sim\varphi$ are bottom particles for all choices of components $[\varphi_i]_{i\in\mathrm{I}}$, but such that the rule $(\Gamma, \varphi, \sim\varphi \Vdash \Delta)$ always holds. In that case the logic will be said to be *controllably explosive* (*in contact with* φ). Explosive logics are those which are controllably explosive in contact with any formula φ to which the definition applies, and controllably explosive logics are always non-1.0-overcomplete, by definition. Paraconsistent logics cannot be explosive, but they *can* be controllably explosive, and they often are. Consider for instance the case of a logic in which (1.1.m) fails only for some $\mathrm{m} < \mathrm{a}$, where $\mathrm{m}, \mathrm{a} \in \mathbb{N}$, and suppose that (7.1.a) holds good—this logic will obviously be paraconsistent yet controllably explosive in contact with $\sim^{\mathrm{a}}\alpha$. An example of logic with that property is given by the 3-valued maximal paraconsistent logic P^1, studied in [30]. Dual definitions can easily be offered for paracompleteness and *controllable implosion*. Next, remember that the failure of the rule $(\Gamma, \alpha, \sim\alpha \Vdash \beta, \Delta)$ is equivalent to the failure of the rule $(\Gamma, \alpha, \sim\alpha \Vdash \Delta)$ in the presence of a bottom particle and (C2.k.j). Of course, $(\Gamma, \alpha, \sim\alpha \Vdash \beta, \Delta)$ *does* hold, for instance, in case β is a top particle. Suppose then that $\varphi\langle[\varphi_i]_{i\in\mathrm{I}}\rangle$ is a formula of a special format such that φ is not a top particle for all choices of components $[\varphi_i]_{i\in\mathrm{I}}$, but such that the rule $(\Gamma, \alpha, {\sim}\alpha \Vdash \varphi, \Delta)$ always holds. Logics with that property are called *partially explosive* (*with respect to* φ). Given a theory $\Phi\langle[\varphi_i]_{i\in\mathrm{I}}\rangle$ which happens not to make a J-alternative for every choice of its components, but such that $(\Gamma, \alpha, {\sim}\alpha \Vdash \Phi, \Delta)$ always holds, one may now naturally extend the previous definition so as to call the underlying logic *partially explosive with respect to* Φ. Explosive logics are partially explosive with respect to any formula φ or theory Φ to which the definition applies, and partially explosive logics are always non-0.1-overcomplete, by definition. Paraconsistent logics can be partially explosive with respect to some formulas, but not with respect to all sets of alternatives. Consider the case of a logic having a bottom and such that (1.1.0.n) fails only for some $\mathrm{n} < \mathrm{a}+1$, where $\mathrm{n}, \mathrm{a} \in \mathbb{N}$, and suppose that (8.1.a) holds good—this logic will obviously be paraconsistent yet partially explosive with respect to ${\sim}^{\mathrm{a}+1}\beta$. Kolmogorov–Johánsson's minimal intuitionistic logic gives an example of a partially explosive paraconsistent logic, since (1.1.0.0) fails in it while (1.1.0.n) holds good for every $\mathrm{n} > 0$. Finally, a logic is called *boldly paraconsistent* in case it is not partially explosive; obviously, boldly paraconsistent logics are, in particular, paraconsistent. Dual definitions can be offered for paracompleteness and both its partial and its bold varieties of implosion. Note that most paraconsistent logics are in practice designed, expected or even required to be boldly paraconsistent (see [33]). Relevance logics, in particular, are always boldly paraconsistent, in virtue of their *variable-sharing property*: Any inference $(\Gamma \Vdash \Delta)$ can only hold good in case Γ and Δ depend on some common atomic sentences. It is not true though that every boldly paraconsistent logic must have the variable-sharing property.

Say that a logic is foo *paranormal* in case it is either foo paraconsistent or foo paracomplete, where foo is one of the above varieties of paraconsistency / paracompleteness. Can we spell out some of the sufficient and some of the necessary conditions for foo paranormality? Surely. Note, for instance, that: From parts (xii) and (xxvi) of Fact 2.3, any logic respecting weak cut and the Principle of Excluded Middle but failing overlap will forcibly be paraconsistent; from parts (x)–(xiii) and (xx)–(xxiii) of Fact 2.1 and parts (x)–(xvii) of Fact 2.5 it follows that contextual contraposition and double negation rules are incompatible with each other, inside any 1-simple boldly paraconsistent logic; from part (i) and the qualification of part (ix) of Fact 2.3 we see that there is no reason to suppose, given a non-monotonic logic, that the failure of dextro-levo-symmetry should be held as a characterizing mark of paraconsistency. And, of course, similar things can always be said and done about the other paranormal class of logics, the paracomplete ones. In the way we have formulated, in the last section, the positive local rules for negation, from families $(1.X)$–$(6.X)$, it turns out that no rule alone has all the others as consequences, given some convenient set of properties of the underlying consequence relation, and, in the same spirit, there is no single rule whose failure causes the failure of all the other rules at once. But, in general, neither the validity nor the failure of a given rule, or set of rules, will be without consequences for some of the other rules. In particular, one could conclude from what has been seen in the above Illustrations and Facts that all positive rules are inferable, for instance, from *pseudo-scotus*, (1.1.0), and *casus judicans*, (2.1.0), via overlap and cut.

Here are a few other selected causes and consequences of the paraconsistent stance:

Illustration 2.8. Let's look first for some possible *causes* for paraconsistency, that is, some (combinations of) conditions leading to the failure of (1.1.0). The following logics are paraconsistent:

(1) Simple logics respecting all rules from families (2.1.X) to (2.4.X) but failing any other rule from families (1.X) to (6.X).
(2) Logics respecting weak cut and some rule from family (7.X), while failing a rule at the same level from family (8.X) (e.g., respecting (7.1.a) and failing (8.2.a)).
(3) Non-nihilistic logics respecting weak cut and failing basic forms of the rules from family (8.X) (viz. (8.1.0) or (8.2.0)).

Here are some selected *consequences* of paraconsistency, that is, some conditions inferable from the failure of (1.1.0):

(4) If a logic respects overlap, then the basic forms of most rules from family (1.X), namely (1.2.0.x), (1.3.0.x), (1.4.0) and (1.5.0.$x.y$), will fail. Moreover, some basic forms of contextual contraposition, namely (5.2.0.0.z.0) and (5.4.0.0.z.0), will also automatically fail.
(5) The most basic form of nonbot (viz. (7.1.0)) will always be respected.
(6) The underlying logic will not be nihilistic.

If a logic respects the rules from family (8.X) and is *not* controllably explosive then:

(7) The logic is paraconsistent.
(8) All forms of nonbot are also respected.

Finally, here are a few consequences of *bold* paraconsistency:

(9) *Ex contradictione* will fail alongside with *pseudo-scotus* (and there is no need for a bottom to get that result).
(10) Several other basic forms of contextual contraposition, namely (5.1.0.0.z.0) and (5.3.0.0.z.0), will also fail inside logics respecting overlap. If the logic also respects weak cut, that is, if the logic is simple, then it will in general fail every rule of the form (5.$x.y$.0.z.0).

As usual, the whole thing is easily dualizable for the paracomplete case.

4. Oh yes, why not?...(But then again, what is negation, after all?)

> *There are only two means by which men can deal with one another: guns or logic. Force or persuasion. Those who know that they cannot win by means of logic, have always resorted to guns.*
> —*Ayn Rand, Faith and Force: Destroyers of the Modern World, 1960.*

The results in the above sections have painfully illustrated the intricate links that tie the several positive contextual sub-classical rules for negation together. You might have noticed that, inside the appropriate logical environment, all positive rules were derivable, for instance, from (1.1.0) and (2.1.0), the most basic forms of *pseudo-scotus* and *casus judicans*. Alternatively, in a similar logical environment, some rules for contextual contraposition were also shown to be sufficient for deriving all the positive rules. Besides, if non-overcompleteness was also guaranteed, then you could also derive the negative rules from the above mentioned positive rules. The requisites for checking each link have also been made clear. You might have noticed, in particular, that full monotonicity had little use in the previous Facts. Anyway, one of the basic lessons one should draw from the whole thing is that the failure of each positive rule carries forward to the failure of some, but not necessarily all, of the other positive rules.

But there is **more**. I now discuss another, perhaps even more basic lesson, that one should learn from the above. It is easy to run into 'triviality', in an intuitive sense, if one does not explicitly try to regulate and avoid it. So, 0.0-overcomplete logics respect all the positive rules for negation, but at the same time respect none of the negative rules. Moreover, if an arbitrary logic does not respect (7.1.0) then it will automatically respect explosion, if only for silly reasons, and silliness will also guide you from the failure of (8.1.0) to the failure of implosion. Together with basic *casus judicans*, the failure of either (7.1.0) or (7.2.0) will lead you to a dadaistic logic, and together with basic *pseudo-scotus* the failure of either (8.1.0) or (8.2.0) will lead you to a nihilistic logic. What seems to be the safest thing to do about that? To be sure that you have *some* negative rules about logics and about negation around! This way you can at least avoid both the nonsensical situation of overcompleteness and the uncomfortable situation in which you have a sample of a logical constant—negation— which turns out to lack any real substance.[10]

This connects to the difficult trouble of defining what a logic or a logical constant *is* (or, in this case, what it *is not*). Well, one might complain that this discussion does not lead us anywhere, and that it is very likely that researchers will never reach anything like a general and final agreement about those notions (though they are very likely to keep on trying, perhaps by use of force or by appeal to some argument stemming from some unformalizable consideration about aesthetics or about the ultimate goal of science). Hey, but why should there be an agreement? This is not what we should be striving for! It seems to me that we should rather, as scientists and (meta-)logicians, be quite content in investigating, comparing and argumenting for and against each possible 'interesting' definition. Then, as the Western Canon says, "by their fruits ye shall know them". Irrespective of religious backgrounds, one might always aspire to find a bit more of impartiality and tolerance around...

Suppose you want to define a class of objects falling under the denomination **D**. If **D** has some common sense meaning(s) in ordinary language, that might give you a good start.

[10] This approach is in fact an application of a certain metaphysical stance focused in some sort of *accidentalism:* The really 'essential' properties in certain characterizations might in some cases turn out to be the accidental ones—you enumerate the properties which your class of objects *should not* possess from among the ones which are actualizable, and then you have at least some necessary conditions for that class of objects to be 'meaningfully defined'. It is a bit like deciding what you will be when you grow up by listing all the things you do *not* want to be. There is of course no space for better defending this strategy here, from a more abstract point of view, so this had better be left for another occasion.

You begin by abstracting from that meaning toward some specific direction, but it might happen that you do not want to give neither a purely normative nor a purely descriptive characterization of the **D**-objects. What should you do then? You might say, "Listen, I am only interested in **D**-objects in case they have the positive property bunda". The problem about positive properties is that there will often be some smart guy to come and say, "Now look how interesting is the class of **D**-objects which *do not* have bunda!" What is left of **D** in such a case? Some people say that you cannot negotiate all your positive properties (and our present commitment to negative properties is at least *consistent* with the idea that positive properties are important). For instance, you might define the class of non-monotonic logics as the class of logics given by consequence relations which do *not* have such-and-such property; but then, why should you still think that such consequence relations should still be said to define a *logic*? Fixed a given logic, it might be quite all right that you define a *paraconsistent* negation as the negation which lacks such-and-such property; but then, how can you really be sure in that case that you have a paraconsistent *negation* (cf. [12])? The problem about positive properties is that they can easily mutate from a happy finding into a heavy burden. And, depending on the way you write them down and insist on them, your preferred set of positive properties might easily make you oblivious of other interesting classes of objects which are very much related to your original intuitions about **D**, but remain excluded by your rigid dogmatic definition of it. On the other hand, having positive properties can be very convenient, for you to get a good glimpse of what rests ahead. It is just so easy to work with them.

So, suppose next that we all agree that 'decent' **D**-objects should *not* have the property favela. We might still have an argument as to whether **D**-objects should have bunda or not, as bunda and favela might be but slightly related properties, and turn out to be quite independent from each other. Now, the advantage of such a negative property is that it *does* give you a necessary condition for the objects to fall into an 'decent' compartment of the class **D**. To be sure, there might be trivial examples of **D** around, but now you are at least confident about having avoided some of them. Anyway, it seems hard to you and me to negotiate property favela. What is 'decent' though might not be 'decent enough'! So now we might go on to discuss whether 'decent' **D**-objects should not suffer from the property pipoca, in addition to (or instead of) their not having the property favela. Well, I do have my doubts as to whether we will be able to reach a complete and undisputable set of sufficient conditions for characterizing **D**—we might soon have a debate on the status of the next negative property that we consider: Is the denial of property pipoca 'really innegotiable'? Does it make sense to strive endlessly towards a really 'comprehensive' definition? Anyway, no matter the answer we will give to that, now we have at least agreed in avoiding favela, right?

Which positive properties are the *indisputable* ones, if any? I will not take a stand on that. I do not aim to convince you here of adopting any of the above positive properties about logics or logical constants. Just look at their consequences and make up your own mind about them, in the face of the particular application you might be targeting. Now, I do hope we will agree in avoiding inanity. In that case, take my hand and follow me to a cut-and-dried territory where we will look for '**mi**nimally **d**ecent' versions of our objects of discourse. Note that I will not maintain that what is not minimally decent does not fall under the scope of those definitions, but only that I will not *care* about what is not minimally decent, and I can only hope to convince you that you should also not care about that. Anyway, feel free to disagree and propose and study some other smaller or incomparable set of minimally decent properties, at any point!

I hope you did not get tired with the previous long abstract argumentative digression. Here is the meat. Given some set of formulas, I now proceed to define a **mid**-*consequence relation* as a binary relation over theories (subsets of the initial set of formulas) which is not I.J-overcomplete, for any finite I and J. We get rid thus of trivial, semitrivial, dadaistic and nihilistic logics, besides all other logics suffering from other kinds of finite overcompleteness. One might call this the *Principle of Non-Triviality*, (PNT): "Thou shalt not trivialize!" Newton da Costa has proposed some sort of such principle many decades ago (check [16, 18]): "From the syntactical-semantical standpoint, every mathematical theory is admissible, unless it is trivial" (notice that he does not say what 'theory' or 'triviality' mean).[11] Interestingly, much more recently, people like Avron, with a completely different background and intentions, have been incorporating some instances of such a principle: In [2,4] this author requires consequence relations to be (simple and) non-0.0-overcomplete. People in the paraconsistent logic community working with single-conclusion consequence relations have accordingly interpreted (PNT) as requiring only that a logic should not be 0.1-overcomplete. They have thus explicitly tried to avoid both trivial and dadaistic logics, while they theoretically allowed for semitrivial and nihilistic logics to linger (a further discussion of this can be found in [26]). The above definition of a **mid**-consequence relation, however, clearly extends all the preceding definitions in a natural way—of course, in view of Fact 1.3, if the logic has both a bottom and a top particles and respects weak cut, then the present requirement is identical to Avron's.[12] By the way, in view of the same Fact, it is only reasonable to define a **mid**-top as a top particle that is not also a bottom, and a **mid**-bottom as a bottom particle that is not also a top.

Now, for us here a **mid**-*negation* will be any unary operator satisfying the negative properties from families $(7.X)$ and $(8.X)$. Note that this requirement alone safeguards us against 0.0-, 0.1- and 1.0-overcompleteness. In view of the Facts from the last section, on the one hand, even if a logic respects the above positive properties, nothing guarantees that it will respect the negative ones as well, and that it will escape overcompleteness. On the other hand, if some of the positive properties fail for a given logic, then this logic will often respect some negative properties as well, but not necessarily all of them. So, the safest thing to do seems to be just to strive for a **mid**-negation from the start.

By the bye, if our negative sub-classical properties alone are so weak, as one might complain, how is it that one can arrive from them to a full characterization of classical negation? One possibility is to guarantee, from a semantic perspective, that (7.2.0) and (8.2.0) come

[11] Da Costa dubbed this methodological principle the 'Principle of Tolerance in Mathematics', by analogy to Carnap's homonymous principle in syntax (check p. 52 of [14]).

[12] One should notice, though, that the present requirement on non-triviality, which sets all 0.0-, 0.1-, 1.0-, 1.1-overcomplete consequence relations into a class of their own, is exactly the same requirement to be found, later on, in Avron and Lev's [5]. The only methodological difference is that in the last paper the structures corresponding to such relations are somehow "excluded from our [theirs] definition of a *logic*"; in the present paper, instead, they are just said to constitute not 'minimally decent' such relations, but are allowed to stay as 'trivial' (that is, 'degenerate') examples of logics. Do notice also that the entailment relation usually associated to relevance logics, with its characterizing variable-sharing property, automatically respects the present formulation of (PNT).

together with truth-functionality and two-valuedness. The margins of this paper are however too narrow to contain the truly marvelous demonstration of that proposition.

Ways of nay-saying. Before putting an end to this, let me now make a brief comparison among the present necessary properties of a (**mid**-)negation, and other characterizations which have been recently proposed in the literature (all the following proposals appeared in single-conclusion form, so here I will work with their straightforward reformulations into the multiple-conclusion environment).

In [21], Gabbay proposes a few increasingly complex 'definitions of negation', based on a couple of necessary and sufficient sets of properties. The idea behind his most sophisticated definition was the following. Suppose you are working with structural tarskian logics. Let $\Theta = \{[\theta_k]_{k \leqslant \mathrm{K}}\}$ be a non-empty set of 'undesirable results' of 'unwanted sentences' of a logic $\mathcal{L}1 = \langle \mathsf{S}_{\mathcal{L}1}, \Vdash_{\mathcal{L}1}\rangle$, subject to the restriction that Θ should not be a K-trivializing set. Let $\mathcal{L}2 = \langle \mathsf{S}_{\mathcal{L}2}, \Vdash_{\mathcal{L}2}\rangle$ be called a *conservative extension* of $\mathcal{L}1$ if $\Gamma \Vdash_{\mathcal{L}2} \Delta \Leftrightarrow \Gamma \Vdash_{\mathcal{L}1} \Delta$, whenever $\Gamma \cup \Delta \subseteq \mathsf{S}_{\mathcal{L}1}$. Consider next a binary connective $\circledcirc$ such that:

(G1) $(\alpha \circledcirc \beta \Vdash \alpha)$ and $(\alpha \circledcirc \beta \Vdash \beta)$
(G2) $(\alpha, \beta \Vdash \alpha \circledcirc \beta)$
(G3) $(\gamma \circledcirc \top \dashv\Vdash \gamma)$ and $(\top \circledcirc \gamma \dashv\Vdash \gamma)$, for any top particle $\top$
(G4) $(\alpha \Vdash \beta)$ / $(\alpha \circledcirc \gamma \Vdash \beta \circledcirc \gamma)$ and $(\alpha \Vdash \beta)$ / $(\gamma \circledcirc \alpha \Vdash \gamma \circledcirc \beta)$

Notice that a connective having properties (G1)–(G4) will behave just like a classical conjunction. Now, a connective $\sim$ of $\mathcal{L}1$ is said to be a *negation* if, for some conservative extension $\mathcal{L}2$ of $\mathcal{L}1$ having a connective $\circledcirc$ with properties (G1), (G3) and (G4):

(GB) $(\gamma \Vdash_{\mathcal{L}1} {\sim}\alpha) \Leftrightarrow (\gamma \circledcirc \alpha \Vdash_{\mathcal{L}2} \theta)$, for some $\theta \in \Theta$

For an intuition about that sort of negation, you might understand (GB) as conveying the idea that γ and α are 'in conflict' in the presence of the undesirable sentence θ.

How can one capture the set of unwanted sentences, when it exists? Easy: Just consider the set of all negated 1-alternatives, that is, $\Theta = \{\theta : (\forall\gamma, \Gamma, \Delta)(\Gamma, \gamma \Vdash_{\mathcal{L}1} {\sim}\theta, \Delta)\}$. In case $\mathcal{L}1$ has some top particle, then Θ turns to be more simply the set of all formulas whose negations are theses of this logic. You might recall though from Footnote 9 that this already goes much beyond our present general requirements on logics. Let me note in passing a few particular features of the above definition. Suppose that this connective $\circledcirc$ of $\mathcal{L}2$ also respects property (G2), that is, suppose that it behaves like a classical conjunction. Then, by overlap we have that ${\sim}\alpha \Vdash_{\mathcal{L}1} {\sim}\alpha$ and so, by (GB), ${\sim}\alpha \circledcirc \alpha \Vdash_{\mathcal{L}2} \theta$, for some $\theta \in \Theta$. From (G2) and cut, together with the fact that $\mathcal{L}2$ is a conservative extension of $\mathcal{L}1$, one can conclude that $\alpha, {\sim}\alpha \Vdash_{\mathcal{L}1} \theta$. Similarly, from (G1), (G2) and (GB) again, one also concludes that $\alpha \Vdash_{\mathcal{L}1} {\sim}{\sim}\alpha$. Moreover, in this case the underlying logic will be at least partially explosive: $\alpha, {\sim}\alpha \Vdash_{\mathcal{L}1} {\sim}\beta$, for every $\alpha, \beta \in \mathsf{S}_{\mathcal{L}1}$. Obviously, Θ should not contain a top particle, under pain of causing $\mathcal{L}1$ to fail (8.2.0), thus producing a negation that is not **mid**. For similar reasons, $\mathcal{L}1$ should not be 0.1-overcomplete, and we know that it is not 1.0-overcomplete from the very postulated existence of a non-trivializing set Θ. In case $\mathcal{L}1$ counts on some top particle $\top$, and the connective $\circledcirc$ of $\mathcal{L}2$ not only respects properties (G1)–(G4) but it is already expressible in $\mathcal{L}1$, then $\Vdash_{\mathcal{L}1} {\sim}(\alpha \circledcirc {\sim}\alpha)$ (use (G3) to check that). The interested reader will find in [23] an extension of the above definition of negation so as to cover also a class of non-monotonic logics.

In [22], the authors propose a 'simplified version' of (GB). Starting from full classical propositional logic, for each formula α they explicitly introduce the connective $\sim_\alpha$ for '*graded negation*', together with another set of connectives for 'graded tolerance', in order to axiomatize what they claim to be a conservative extension of classical logic. Next, they require graded negation to respect the following property:

(GH) $(\Gamma \Vdash {\sim_\alpha}\beta) \Leftrightarrow (\Gamma \Vdash \alpha)$ and $(\alpha \wedge \beta \Vdash)$

The idea, again, is that the inference of α from Γ is 'in contention with' β. As the authors claim that "it is becoming more widely acknowledged that we need to develop more sophisticated means for handling inconsistent information", one might be led to think that graded negations are non-explosive. This is surely not the case. Indeed, given overlap and any unary connective $\star$, it is easy to check that both $({\sim_\varphi}\star\varphi \Vdash \varphi)$ and $(\varphi \wedge \star\varphi \Vdash)$ should hold good in their logic. The last inference seems quite puzzling, given that it holds for *any* definable unary connective $\star$ (thus also for identity, and for any negation originally intended to be non-explosive), and $\wedge$ is classical conjunction. Thus, we finally conclude, in particular, that $(\varphi \Vdash)$. This renders the present 'extension of classical logic' both non-conservative and nihilistic, thus non-paraconsistent—and so the paper seems not really to delivers what it promises. (To go back to single-conclusion consequence relations and write $(\alpha \wedge \beta \Vdash \gamma)$ instead of $(\alpha \wedge \beta \Vdash)$ at the right-hand side of (GH) does not help at all: The resulting logic will not be **mid**, being at least semitrivial.) The proposal is glaringly unsound.

Another fascinating investigation of negation was made by Lenzen, in [24]. One can find in that paper a list of 'necessary conditions for negation-operators', namely (check Proposal 42):

(L1) $(\Gamma, \alpha \nVdash {\sim}\alpha, \Delta)$
(L2) $(\Gamma, \alpha \Vdash \beta, \Delta) \Rightarrow (\Gamma', {\sim}\beta \Vdash {\sim}\alpha, \Delta')$
(L3) $(\Gamma \Vdash \alpha, \Delta) \Rightarrow (\Gamma' \Vdash {\sim}{\sim}\alpha, \Delta')$
(L4) If the logic has a top, then $\exists\alpha(\Gamma \Vdash {\sim}\alpha, \Delta)$

Now, (L1) is simply our own property (8.2.0). Even though the paper by Lenzen aims to give a special account of paraconsistent negations, it seems ungainly not to find in the above list of necessary properties for negation the dual of property (L1) in family $(7.X)$. I cannot say much here about (L2) and (L3)—they are global properties, and I have postponed the discussion of such properties to a future paper. But again, it is a bit strange not to find other versions of (L2)—global contraposition—in the above list, and also not to find the dual version of (L3) there. At any rate, for the purposes of algebraization and modalization, (L2) is surely more than one needs (as keenly pointed out in [32]), given that the following version of global replacement is already enough:

(L2*) $(\Gamma, \alpha \dashv\Vdash \beta, \Delta) \Rightarrow (\Gamma', {\sim}\alpha \dashv\Vdash {\sim}\beta, \Delta')$

There are, though, an awful amount of interesting logics, algebraizable or not, with known modal interpretations or not, which are supposed to have a 'negation' that respects neither (L2) nor (L2*) (check [16] for many remarkable paraconsistent samples of such logics). As a final remark, in a multiple-conclusion consequence environment, it would of course seem only natural to add and study also the dual of (L4):

(L4$^{\mathrm{d}}$) If the logic has a bottom, then $\exists\alpha(\Gamma, {\sim}\alpha \Vdash \Delta)$

Let's leave it as a suggestion for further development.

Here is a last case study. In [10], Béziau aims to propose "a definition of negation not depending on explicit logical laws but on a conceptual idea". To that purpose, the author tries to formulate a semantical constraint which would be such that the following condition (BZ) is respected: Given a set of 'true' (designated) truth-values and a disjoint set of 'false' (undesignated) truth-values, it would always be possible to find models M1 and M2 such that φ and $\circledcirc\varphi$ would not be both true in M1 nor both false in M2, for some symbol '$\circledcirc$' aimed to model 'negation', as opposed to 'affirmation' (check Fig. 1). Clearly, our rules (7.2.0) and (8.2.0), from the end of Section 2, are just what one needs for the job, under a structural tarskian interpretation of semantics, but that's not the path trodden by the author. What he does in that paper, in fact, amounts to the following. Call any true value T and any false value F, and define the natural order among them, that is, set $F \preccurlyeq F$, $F \preccurlyeq T$, and $T \preccurlyeq T$. Next, call a unary operator $\circledcirc$ *positive* in case it is monotonic over $\preccurlyeq$, that is, in case $\S_1(\varphi) \preccurlyeq \S_2(\varphi)$ implies $\S_1(\circledcirc\varphi) \preccurlyeq \S_2(\circledcirc\varphi)$, for any choice of valuations $\S_1$ and $\S_2$. Finally, call $\circledcirc$ *negative* in case it is not positive. Béziau proposes that negative connectives have all the right to be called *negations*. Indeed, the identical operator ($\circledcirc_2^1$ in Fig. 1), for one, is surely not negative. But then, unfortunately, the last definition is not strong enough to get rid of the other forms of affirmation. Mind you, consider the operator $\circledcirc_2^2$ in Fig. 1, and consider valuations $\S_1$ and $\S_2$ and a formula φ such that $\S_1(\varphi) = F = \S_2(\varphi)$, but $\S_1(\circledcirc_2^2\varphi) = T$ while $\S_2(\circledcirc_2^2\varphi) = F$. Those valuations would characterize $\circledcirc_2^2$ as a negative

	$\circledcirc_2^3$
T	T
T	F
F	F

	$\circledcirc_2^2$
T	T
F	T
F	F

	$\circledcirc_2^1$
T	T
F	F

kinds of affirmation

kinds of negation

	$\circledcirc_1^1$
T	F
F	T

	$\circledcirc_1^2$
T	F
F	F
F	T

	$\circledcirc_1^3$
T	F
T	T
F	T

	$\circledcirc_1^4$
T	F
T	T
F	F
F	T

Fig. 1. Affirmation × negation.

operator, contrary to our expectations, and a similar example can be written with $\odot_2^3$, this time taking $\S_1(\varphi) = T = \S_2(\varphi)$. In neither case can we say that condition (BZ) holds good. The proposal thus is not sound.

A full stop comes. I will make no further inquiries here into what negation *is* (or what it *is not*). I just wanted to convince you that the connective that is studied in this paper has some right to be *called* ‘negation’. My feeling, though, is that a really good theory of ‘what negation is’ can only come as a byproduct of a more general and modern and comprehensive version of a *theory of opposition*, as we learned from good ol’ Aristotle. My interest here, however, was much more modest: This was rather a study about what negation *could be*, and what it *should not be*.

5. Directions

> *‘Would you tell me, please, which way I ought to go from here?’*
> *‘That depends a good deal on where you want to get to,’ said the Cat.*
> *‘I don’t much care where...’ said Alice.*
> *‘Then it doesn’t matter which way you go,’ said the Cat.*
> *‘...so long as I get somewhere,’ Alice added as an explanation.*
> *‘Oh, you’re sure to do that,’ said the Cat, ’if you only walk long enough.’*
> *—Lewis Carroll, Alice’s Adventures in Wonderland, 1865.*

The present paper aimed at making several different contributions, suggestions, and some forceful yet not always claimed to be original remarks, among which:

(1) An elaborate illustration is given on the general use of *multiple-conclusion* consequence relations in the abstract study of deductive systems and logical connectives. Most studies in abstract (universal) logic, such as those by Béziau, have concentrated on single-conclusion consequence relations, and so have missed a lot of what you can get straightforwardly by considerations of *symmetry*. Other studies of multiple-conclusion consequence relations have usually not been made in a purely *abstract* setting, but more frequently in a *proof-theoretical* setting (as in the case of some excellent papers by Avron) or in a *semantical* setting (as in the case of the excellent book by Shoesmith and Smiley). The present paper should be read, then, as a call for integration.
(2) Many local sub-classical rules for consequence relations and for the negation connective are systematically studied here in multiple-conclusion format, and *negative* rules are given so much emphasis—or even more emphasis—as *positive* ones. In fact, *failing* those negative rules can be much more dangerous than failing the positive rules, as you can check at the end of Section 2. Negative rules are argued to be, in a sense, more ‘essential’ than positive ones. An extensive justification for that argument is presented in the first part of Section 4.
(3) Important general approaches to those same rules in the literature (Avron, Béziau, Curry, Gabbay, Hunter, Lenzen, Wansing, etc.) are surveyed, all along the paper. Cor-

rections are made on some proposals and results by Béziau, and a proposal by Gabbay and Hunter is shown to apply only to overcomplete logics (though that flagrant limitation seems to have gone unnoticed up to this moment).

(4) A small yet comprehensive taxonomy of the most well-known classes of consequence relations is presented in Section 1.
(5) The prerequisites for proving each Fact interrelating rules for consequence relations and rules for negations are in each case clearly highlighted. This is quite useful for you to know at once whether you shall make use, say, of monotonicity (weakening) or of rules for double negation to prove each given relation.
(6) General rules that make consequence relations 'trivial' are presented, generalizing many other distinguished approaches from the literature.
(7) The multiple-conclusion environment allows us to present 'consistency' rules as *dual* to 'completeness' rules, in a clear and compelling way. As a consequence, rules that are duals to *ex contradictione*, *consequentia mirabilis*, proof-by-cases, and *reductio ad absurdum* are here introduced, apparently for the very first time.
(8) The same environment, again, allows one in fact to draw a sharp distinction between *pseudo-scotus* ad *ex contradictione sequitur quodlibet*. This is certainly new, as new as the accompanying proposal to draw the very *definition* of paraconsistency as the failure of the former rules instead of the latter, in direct duality to the (most) characterizing feature of (intuitionistic-like) paracomplete systems: the failure of excluded middle.
(9) The definitions of paraconsistency and paracompleteness are precisely stated, and clearly shown not to bear any compulsory effect, for instance, on the invalidation of rules for double negation (and vice-versa). Some definitions of important subclasses of paraconsistent and paracomplete logics (partial, controllable, and bold) are also presented and exemplified, under a new generality and always having symmetry in mind.
(10) Studies of *consequentia mirabilis* (e.g., Pagli and Bellissima) have at times proposed to identify *mirabilis* with *reductio*. This is a historical and a technical abuse, clarified in the present paper.
(11) An illustrative list of sufficient and necessary conditions for (bold) (non-controllable) paraconsistency is presented, in Section 3.
(12) Other proposals of characterizations of negation are offered and analyzed in Section 4. Proposals by other authors are summarized and criticized. Incidentally, having already been mentioned by other authors, n-ary negations can also in this paper be seriously be taken into consideration (see below, in the present section), as they pretty smoothly fit the general framework.

The present study of negation was made quite general, this far, under the natural liberties and restrictions of the chosen framework and our decision to concentrate on pure local sub-classical rules for negation. The picky reader might observe, though, that even some seemingly innocuous assumptions that we made may turn out disputable, or at least limited from their very inception. Thus, I have assumed from the start, for example, that, in this paper, "a unary *negation* symbol $\sim$ will always be present as a logical constant in the underlying language of our logics". Now, why should negation be *unary*? One might think instead that it is much more natural to think of 'negation as conflict', as in the second part of Section 4. With that idea in mind, consider the following rules for a *binary* negation connective:

(A1.1) $(\Gamma \Vdash \sim(\alpha_1, \alpha_2), \Delta)$ / $(\Gamma, \alpha_1, \alpha_2 \Vdash \Delta)$
(A1.2) $(\Gamma \Vdash \alpha_1, \Delta)$ and $(\Gamma \Vdash \alpha_2, \Delta)$ / $(\Gamma, \sim(\alpha_1, \alpha_2) \Vdash \Delta)$
(A2.1) $(\Gamma, \alpha_1, \alpha_2 \Vdash \Delta)$ / $(\Gamma \Vdash \sim(\alpha_1, \alpha_2), \Delta)$
(A2.2) $(\Gamma, \sim(\alpha_1, \alpha_2) \Vdash \Delta)$ / $(\Gamma \Vdash \alpha_1, \Delta)$ and $(\Gamma \Vdash \alpha_2, \Delta)$

Clearly, a unary negation for a formula α can be defined from the above binary connective by considering $\sim(\alpha, \alpha)$. The rules of the preceding connective are analogous to the rules of NAND, also known as Sheffer stroke, or *alternative denial*. One could also look at the rules of its dual, *joint denial*, also known as NOR:

(J1.1) $(\Gamma \Vdash \sim(\alpha_1, \alpha_2), \Delta)$ / $(\Gamma, \alpha_1 \Vdash \Delta)$ and $(\Gamma, \alpha_2 \Vdash \Delta)$
(J1.2) $(\Gamma \Vdash \alpha_1, \alpha_2, \Delta)$ / $(\Gamma, \sim(\alpha_1, \alpha_2) \Vdash \Delta)$
(J2.1) $(\Gamma, \alpha_1 \Vdash \Delta)$ and $(\Gamma, \alpha_2 \Vdash \Delta)$ / $(\Gamma \Vdash \sim(\alpha_1, \alpha_2), \Delta)$
(J2.2) $(\Gamma, \sim(\alpha_1, \alpha_2) \Vdash \Delta)$ / $(\Gamma \Vdash \alpha_1, \alpha_2, \Delta)$

The above connectives obviously generalize our symmetry rules $(1.2.X)$ and $(2.2.X)$. Exercises for the reader: Check what should be done for generalizing the other positive and negative rules in accordance with the above binary connectives, and check what happens when other n-ary 'negations' are defined, including—don't be lazy—infinitary versions. (By the way, as you have the pencil in hand: I have checked the results in the above sections to exhaustion, but I would not be so surprised if some errors had slipped into the easy but general calculations. Have fun on the search for mistakes! I just hope the whole thing has worked well as an illustration of the idea behind the systematization.)

Finally, I must acknowledge that all of this was but an initial step into the realm of negation. I had better just add a last note of intentions. The reader should not assume that I am defending the pure negative rules from the families $(7.X)$ and $(8.X)$, which I used in the last section in the definition of 'minimally decent negations', to be THE rules common to all negations. By no means. Not only do I want to leave, on the one hand, also those very rules open to debate, but on the other hand I also think that those rules are not even enough if you are serious about the notion of a decent negation. In fact, in most normal modal logics, operators such as the necessity operator *are* also expected to respect rules from families $(7.X)$ and $(8.X)$. But we surely do not want negation to be interpreted as necessity, or necessity to be read as a kind of negation! So, a 'minimally decent negation' is more likely to be the one that, besides being a **mid**-negation, also respects some *non-local negative* rules such as the following ones:

(G1.1.a) $\neg[(\Vdash \sim^{\mathrm{a}}\varphi) \Rightarrow (\Vdash \sim^{\mathrm{a}+1}\varphi)]$
(G1.2.a) $\neg[(\Gamma \Vdash \sim^{\mathrm{a}}\varphi, \Delta) \Rightarrow (\sim^{\mathrm{a}+1}\Gamma \Vdash \sim^{\mathrm{a}+1}\varphi, \sim^{\mathrm{a}+1}\Delta)]$
(G2.1.a) $\neg[(\sim^{\mathrm{a}}\varphi \Vdash) \Rightarrow (\sim^{\mathrm{a}+1}\varphi \Vdash)]$
(G2.2.a) $\neg[(\Gamma, \sim^{\mathrm{a}}\varphi \Vdash \Delta) \Rightarrow (\sim^{\mathrm{a}+1}\Gamma, \sim^{\mathrm{a}+1}\varphi \Vdash \sim^{\mathrm{a}+1}\Delta)]$

where $\sim^{\mathrm{a}}\Sigma$ denotes, as you might expect, $\{\sim^{\mathrm{a}}\sigma\colon \sigma \in \Sigma\}$.

A follow-up to the present investigation should include statements of rules mixing negation and other more usual logical constants, such as conjunction, disjunction, implication and bi-implication, always from the point of view of universal logic, and maybe a survey of the effects of paraconsistency also in this terrain—it is well known for instance that some laws of implication might have dreadful consequences for paraconsistency, that rules such as disjunctive syllogism will often fail, that De Morgan laws will not always be convenient, that even *modus ponens* might in some situations be problematic, that adjunctive conjunctions might be dangerous, and so on. The present results will surely be decisive in the future investigation of the mixed rules. It would also be interesting and important, at some moment, to have a good look at global versions of most preceding contextual rules. This discussion also relates to the trouble of algebraization, which should be clarified in detail, and the whole thing will be easily dualizable from paraconsistent to paracomplete logics.

The next step should include the study of some recent contributions to the field: the consistency connective, and its dual completeness (or determinedness) connective, which can help internalizing the homonymic metatheoretical notions at the object language level, recovering through them the inference rules which might be lacking in columns $(1.X)$ and $(2.X)$. Such connectives also allow us to translate and talk about many (sub-)classical properties inside 'gentle' logics which do not enjoy them.

All that and we are still talking, in a sense, about sub-classical properties of negation. By way of closure, a few notes should also be added—without any intention of gauging the full ramifications of the subject in the literature—about some rules for negation which are 'really non-classical': This is the case of MacColl & McCall's connexive negation (depending on how you look at it), Post's cyclic negation, Humberstone's demi-negation, and so on and so forth. This much for the future.

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