Fuzzy Modal Logics of Confluence

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Abstract. In this paper we explore fuzzy semantics for a wide class of normal modal systems enriched with multiple instances of the axiom of confluence and prove a general completeness result for such systems.

Keywords: fuzzy logics, modal logics, confluence

1 Introduction

In [3, 1], the authors study models for a certain kind of fuzzy modal logics and prove weak completeness results for a couple of modal extensions of classic-like fuzzy models of some traditional normal modal systems, viz. K, T, D, B, $S4$ and $S5$. Here we shall follow a similar thread to prove completeness results for a much more inclusive class of fuzzy normal modal systems which contain instances of the *axiom of confluence* $(G^{k,l,m,n})$ $\Diamond^k\Box^m\varphi \supset \Box^l\Diamond^n\varphi$, as the systems $K + G^{k,l,m,n}$ obviously encompass the above traditional systems, and much else. Indeed, one may observe that the characteristic modal axioms (T) $\Box\varphi \supset \varphi$, (D) $\Box\varphi \supset \Diamond\varphi$, (B) $\varphi \supset \Box\Diamond\varphi$, (4) $\Box\varphi \supset \Box\Box\varphi$ and (5) $\Diamond\varphi \supset \Diamond\Box\varphi$ are but particular instances of $(G^{k,l,m,n})$ where $\langle k, l, m, n\rangle$ are $\langle 0, 0, 1, 0\rangle$, $\langle 0, 0, 1, 1\rangle$, $\langle 0, 1, 0, 1\rangle$, $\langle 0, 2, 1, 0\rangle$ and $\langle 1, 1, 0, 1\rangle$, respectively.

The so-called *Geach axiom* $(G^{1,1,1,1})$ is well-known to characterize, in terms of the associated notion of accessibility $\rightsquigarrow$ (and its inverse $\leftsquigarrow$) in the corresponding Kripke frames, the *diamond property*, namely: if $y \leftsquigarrow x \rightsquigarrow z$, then there is some w such that $y \rightsquigarrow w \leftsquigarrow z$. As noted in [5], where $\overset{i}{\rightsquigarrow}$ denotes an i-long sequence of $\rightsquigarrow$ transitions (and similarly for $\leftsquigarrow$ transitions), the natural generalization of the diamond property is the following $\langle k, l, m, n\rangle$-*confluence property*: if $y \overset{k}{\leftsquigarrow} x \overset{l}{\rightsquigarrow} z$, then there is some w such that $y \overset{m}{\rightsquigarrow} w \overset{n}{\leftsquigarrow} z$. From the logical viewpoint, a general completeness proof based directly on the axiom of confluence, thus, is attractive in having the potential to subsume a denumerable number of particular instances of $(G^{k,l,m,n})$.

At any rate, it should be noted that the confluence property has importance on its own. In abstract rewriting systems, for instance, one deals with frames in which accessibility characterizes some appropriate notion of reduction. There, confluence is used together with termination to guarantee convergence of reductions, which on its turn guarantee the existence of normal forms and has applications on the design of decision procedures. Strong normalization, in particular, is a much desirable property of lambda calculi, and is a property guaranteed by theorems of confluence à la Church-Rosser, with applications to programming language theory. The availability of modal logics of confluence, and in fact of fuzzy versions of such logics, allows one to expect to have a local perspective on rewrite systems and on program evaluation, and this time imbued with varying degrees of uncertainty, customized to the user's discretion.

The plan of the paper is as follows: in section 2 we define a set of fuzzy operators, in section 3 we present the concept of classic-like fuzzy semantics, in section 4 we present a particular kind of fuzzy kripke semantics for modal logics, and in section 5 we prove completeness results for the modal system K extended with instances of the axiom of confluence.

2 Fuzzy Operators

We first fix some useful terminology:

Definition 1. *Throughout the paper we shall use* $\mathcal{O}$ *to denote the* ***boolean*** *domain* $\{0,1\}$ *of classical logic, and* $\mathcal{U}$ *to denote the* ***unit*** *interval* $[0,1]$*, typical of fuzzy logics. By* $\leqslant$ *we will always denote the natural* ***total order*** *on* $\mathcal{U}$*. Given an n-ary operator* $\copyright_b$ *on* $\mathcal{O}$ *and an n-ary operator* $\copyright_u$ *on* $\mathcal{U}$*, we shall say that* $\copyright_u$ ***agrees with*** $\copyright_b$ *if* $\copyright_u|_{\mathcal{O}} = \copyright_b$.

Next we axiomatize the properties of the more or less standard fuzzy operators used to interpret their classical counterparts:

Definition 2. *A* ***fuzzy negation*** *is a unary operation* N *on* $\mathcal{U}$ *such that:* (N0) N *agrees with classical negation,* (N1) N *is antitone, that is, order-reversing.*

Definition 3. *A* ***fuzzy conjunction****, or* ***t-norm****, is a binary operation* T *on* $\mathcal{U}$ *such that:* (T0) T *agrees with classical conjunction,* (T1) T *is commutative,* (T2) T *is associative,* (T3) T *is monotone, that is, order-preserving, on both arguments, and* (T4) T *has* 1 *as neutral element. We call* $x \in \mathcal{U}$ *a* ***zero-divisor*** *of a t-norm* T *if there exists some* $y \in \mathcal{U}$ *such that* $T(x,y) = 0$*; such zero-divisor is called* ***non-trivial*** *if* $\min(x,y) \neq 0$.

Using the properties of t-norms we can easily prove the following result.

Proposition 1. *If* $T(x,y) = 1$*, then* $x = y = 1$.

Proof. Let $x, y \in \mathcal{U}$. As $y \leqslant 1$, by (T3) it follows that $T(x,y) \leqslant T(x,1)$. But by (T4) we have $T(x,1) = x$, so we conclude that (i) $T(x,y) \leqslant x$. For analogous reasons, we know that (ii) $T(x,y) \leqslant y$. From (i) and (ii) it follows that $T(x,y) \leqslant \min(x,y)$. Given $T(x,y) = 1$, then $1 \leqslant \min(x,y)$, so $x = y = 1$.

Definition 4. *A* ***fuzzy disjunction****, or* ***s-norm****, is a binary operation* S *on* $\mathcal{U}$ *such that:* (S0) S *agrees with classical disjunction,* (S1) S *is commutative,* (S2) S *is associative,* (S3) S *is monotone on both arguments, and* (S4) S *has* 0 *as neutral element. We call* $x \in \mathcal{U}$ *a* ***one-divisor*** *of an s-norm* S *if there exists some* $y \in \mathcal{U}$ *such that* $S(x,y) = 1$*; such one-divisor is called* ***non-trivial*** *if* $\max(x,y) \neq 1$.

Definition 5. *A* ***fuzzy implication*** *is a binary operation* I *on* $\mathcal{U}$ *such that:* (I0) I *agrees with classical implication,* (I1) I *is antitone on the first argument, and* (I2) I *is monotone on the second argument.*

Definition 6. *A* ***fuzzy bi-implication*** *is a binary operation* B *on* $\mathcal{U}$ *such that:* (B0) B *agrees with classical bi-implication,* (B1) B *is commutative,* (B2) $B(x,x) = 1$*, and* (B3) *if* $w \leqslant x \leqslant y \leqslant z$ *then* $B(w,z) \leqslant B(x,y)$.

3 Fuzzy Semantics

Let P a denumerable set of propositional variables, and let the set of formulas of classical propositional logic, L_P, be inductively defined by:

$$\varphi ::= p \mid \top \mid \bot \mid (\neg\varphi) \mid (\varphi_1 \wedge \varphi_2) \mid (\varphi_1 \vee \varphi_2) \mid (\varphi_1 \supset \varphi_2) \mid (\varphi_1 \equiv \varphi_2)$$

where p ranges over elements of P.

The following definition employs the standard fuzzy operators in interpreting the above symbols for the classical connectives:

Definition 7. *A **fuzzy evaluation** of the propositional symbols is any total function $e : P \longrightarrow \mathcal{U}$. The structure $\mathbb{S} = \langle N, T, S, I, B \rangle$ will be called a **fuzzy semantics** of the propositional connectives $\langle \neg, \wedge, \vee, \supset, \equiv \rangle$. By way of a fuzzy semantics, an evaluation e may be recursively extended to a function $e^{\mathbb{S}} : L_P \longrightarrow \mathcal{U}$ as follows:*

$$\begin{aligned}
e^{\mathbb{S}}(p) &= e(p) \text{ for each } p \in P\\
e^{\mathbb{S}}(\neg\alpha) &= N(e^{\mathbb{S}}(\alpha))\\
e^{\mathbb{S}}(\alpha \wedge \beta) &= T(e^{\mathbb{S}}(\alpha), e^{\mathbb{S}}(\beta))\\
e^{\mathbb{S}}(\alpha \vee \beta) &= S(e^{\mathbb{S}}(\alpha), e^{\mathbb{S}}(\beta))\\
e^{\mathbb{S}}(\alpha \supset \beta) &= I(e^{\mathbb{S}}(\alpha), e^{\mathbb{S}}(\beta))\\
e^{\mathbb{S}}(\alpha \equiv \beta) &= B(e^{\mathbb{S}}(\alpha), e^{\mathbb{S}}(\beta))
\end{aligned}$$

*A formula $\alpha \in L_P$ is an $\mathbb{S}$-**tautology**, denoted by $\models_{\mathbb{S}} \alpha$, if for each fuzzy evaluation e we have $e^{\mathbb{S}}(\alpha) = 1$.*

We shall denote by $\mathbb{T}(L_P)$ the set of all classical tautologies in L_P and by $\mathbb{T}^{\mathbb{S}}(L_P)$ the set of all $\mathbb{S}$-tautologies in L_P. The fact that each fuzzy operator agrees with the corresponding classical operator allows one to immediately prove the following:

Proposition 2. *Let $\mathbb{S} = \langle N, T, S, I, B \rangle$ be a fuzzy semantics. All fuzzy tautologies are classical tautologies, that is, $\mathbb{T}^{\mathbb{S}}(L_P) \subseteq \mathbb{T}(L_P)$.*

The following definition and the subsequent result come from [2], and strive to capture the core of classical semantics from within the context of fuzzy semantics:

Definition 8. *$\mathbb{S}$ is a **classic-like fuzzy semantics** if $\mathbb{T}(L_P) \subseteq \mathbb{T}^{\mathbb{S}}(L_P)$.*

Proposition 3. *A fuzzy semantics $\mathbb{S} = \langle N, T, S, I, B \rangle$ is a classic-like fuzzy semantics iff*

- *N is the fuzzy negation*

$$N_C(x) = \begin{cases} 1, \text{ if } x < 1\\ 0, \text{ if } x = 1 \end{cases}$$

- *all zero-divisors of T are trivial*
- *all one-divisors of S are trivial*
- *$I(x, y) = 1$ if and only if $x < 1$ or $y = 1$*
- *B is the fuzzy bi-implication*

$$B_C(x, y) = \begin{cases} \min(x, y), \text{ if } \max(x, y) = 1\\ 1, \qquad\qquad \textit{otherwise} \end{cases}$$

4 Fuzzy Kripke Semantics

The set of modal formulas, LM_P, is defined by adding $(\Diamond\phi)$ to the inductive clauses defining L_P. The connective $\Box$ may be introduced by definition by setting $\Box\alpha := \neg\Diamond\neg\alpha$.

Definition 9. *Generalizing the notion of a characteristic function to the domain of fuzzy logic, a **fuzzy n-ary relation** B over a universe A is characterized by a membership function $\mu_B : A^n \longrightarrow \mathcal{U}$ which associates to each tuple $\overrightarrow{x} \in A^n$ its degree of membership $\mu_B(\overrightarrow{x})$ in B. In this context, a **fuzzy subset** is characterized by a fuzzy unary relation, or the corresponding unary membership function. A **crisp** n-ary relation is any fuzzy n-ary relation B over a given A such that $\mu_B(A^n) \subseteq \mathcal{O}$, and crisp sets are defined analogously.*

In the following definitions the standard kripke models are fuzzified:

Definition 10. *A **fuzzy frame** $\mathbb{F}$ is a structure $\langle W, \rightsquigarrow \rangle$, where W is a non-empty crisp set (of 'objects', 'worlds', or 'states') and $\rightsquigarrow$ is a fuzzy binary ('reduction', 'accessibility', or 'transition') relation over W. As expected, to characterize m-step accessibility, $\overset{m}{\rightsquigarrow}$, we set:*

- $\mu_{\overset{0}{\rightsquigarrow}}(w_i, w_j) = 1$ *means that* $w_i = w_j$
- $\mu_{\overset{n+1}{\rightsquigarrow}}(w_i, w_j) = 1$ *means that there is some* w_k *such that* $\mu_{\overset{n}{\rightsquigarrow}}(w_i, w_k) = 1$ *and* $\mu_{\rightsquigarrow}(w_k, w_j) = 1$ *(intuitively, this means* w_j *is 'crispily' accessible from* w_i *in* $n+1$ *steps)*

Furthermore, $w_i \overset{m}{\leftsquigarrow} w_j$ *is to denote* $w_j \overset{m}{\rightsquigarrow} w_i$.

Definition 11. *Given a fuzzy frame $\mathbb{F}$, an $\mathbb{F}$-**evaluation** is any total function $\rho : P \times W \longrightarrow \mathcal{U}$. A **fuzzy kripke model** is a structure $\mathcal{K} = \langle \mathbb{F}, \mathbb{S}, \rho \rangle$, where $\mathbb{F}$ is a fuzzy frame, $\mathbb{S}$ is a classic-like fuzzy semantics where T is a left-continuous t-norm and ρ is an $\mathbb{F}$-valuation. Given a fuzzy kripke model $\mathcal{K}$, the associated **degree of satisfiability** is a total function $\Vdash : W \times LM_P \longrightarrow \mathcal{U}$ recursively defined as follows (in infix notation, we write $w \Vdash \varphi$ where $w \in W$ and $\varphi \in LM_P$):*

$$\begin{aligned}
w \Vdash \alpha &= \rho(\alpha, w), \text{ if } \alpha \in P \\
w \Vdash \neg\alpha &= N(w \Vdash \alpha) \\
w \Vdash \alpha \wedge \beta &= T(w \Vdash \alpha, w \Vdash \beta) \\
w \Vdash \alpha \vee \beta &= S(w \Vdash \alpha, w \Vdash \beta) \\
w \Vdash \alpha \supset \beta &= I(w \Vdash \alpha, w \Vdash \beta) \\
w \Vdash \alpha \equiv \beta &= B(w \Vdash \alpha, w \Vdash \beta) \\
w \Vdash \Diamond\alpha &= \sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash \alpha) : w' \in W\} \\
w \Vdash \Box\alpha &= N(w \Vdash \Diamond\neg\alpha)
\end{aligned}$$

*A formula $\varphi \in LM_P$ is said to be **true** in a fuzzy kripke model $\mathcal{K}$, denoted by $\models_{\mathcal{K}} \alpha$, if $(w \Vdash \varphi) = 1$ for every $w \in W$. Given a collection $\mathfrak{K}$ of fuzzy kripke models, a formula $\varphi \in LM_P$ is said to be a $\mathfrak{K}$-**tautology** if φ is true in every model in $\mathfrak{K}$.*

Many standard properties of the binary relations have natural 'weakly' fuzzy counterparts, among which we may mention:

Definition 12. *We say the fuzzy accessibility relation $\rightsquigarrow$ is:*

- ***reflexive*** *if* $\mu_{\rightsquigarrow}(x, x) = 1$, *for every* $x \in W$
- ***symmetric*** *if* $\mu_{\rightsquigarrow}(x, y) = \mu_{\rightsquigarrow}(y, x)$, *for every* $x, y \in W$
- ***transitive*** *if* $\mu_{\overset{2}{\rightsquigarrow}}(x, y) = 1$ *implies* $\mu_{\rightsquigarrow}(x, y) = 1$ *for every* $x, y \in W$
- ***euclidean*** *if* $\mu_{\rightsquigarrow}(x, y) = 1$ *and* $\mu_{\rightsquigarrow}(x, z) = 1$ *imply* $\mu_{\rightsquigarrow}(y, z) = 1$ *for every* $x, y, z \in W$

*In general, given natural numbers k, l, m, n, we say that $\rightsquigarrow$ is **(k,l,m,n)-confluent** if for each $x, y, z \in W$ such that $\mu_{\overset{k}{\rightsquigarrow}}(x, y) = \mu_{\overset{l}{\rightsquigarrow}}(x, z) = 1$ there exists $w \in W$ such that $\mu_{\overset{m}{\rightsquigarrow}}(y, w) = \mu_{\overset{n}{\rightsquigarrow}}(z, w) = 1$.*

5 Characterization of Fuzzy Kripke Models for Normal Modal Systems based on Instances of $G^{k,l,m,n}$

The following result, [3], shows that each fuzzy modal semantics may be assumed to be based on a convenient crisp accessibility relation.

Proposition 4. *Let $\mathcal{KM} = \langle W, \rightsquigarrow, \mathbb{S}, \rho \rangle$ be a fuzzy kripke model and define the model $\mathcal{KM}^C = \langle W, \rightsquigarrow^C, \mathbb{S}, \rho \rangle$ to be such that $\mu_{\rightsquigarrow^C}(w, w') = 1$ if $\mu_{\rightsquigarrow}(w, w') = 1$, and $\mu_{\rightsquigarrow^C}(w, w') = 0$ if $\mu_{\rightsquigarrow}(w, w') < 1$. Then, $\alpha \in LM_P$ is true in $\mathcal{KM}$ iff α is true in $\mathcal{KM}^C$.*

5.1 The *K*-Modal System

Definition 13. *The K-modal system is the triple $\langle LM_P, \Delta \cup \{(\mathrm{K})\}, \{(\mathrm{MP}), (\mathrm{Nec})\}\rangle$, where Δ is an axiomatization of Classical Propositional Logic and* (K) *is the axiom*

$$\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$$

(MP) *and* (Nec) *are respectively the rules of Modus Ponens and Necessitation, namely:*

$$(\mathrm{MP}) : \frac{\alpha, \alpha \supset \beta}{\beta}$$

and

$$(\mathrm{Nec}) : \frac{\vdash \alpha}{\vdash \Box\alpha}$$

Definition 14. *A fuzzy kripke model $\mathcal{K}$ is **K-like** if each theorem in the K-modal system is true in $\mathcal{K}$ and, conversely, each formula that is true in $\mathcal{K}$ is a theorem in the K-modal system.*

Proposition 5. *Let $\alpha \in LM_P$, α is a theorem in the K-modal system iff $\models_{\mathcal{K}} \alpha$ for each fuzzy kripke model $\mathcal{K} = \langle W, \rightsquigarrow, \mathbb{S}, \rho\rangle$.*

Proof. ($\Rightarrow$) We already know that the theorems of classical logic are all valid in any classic-like fuzzy semantics. It remains to be proven that the axiom *(K)* is valid and that the inferences rules preserve validity. Suppose that there exists a $w \in W$ such that $(w \Vdash \Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)) < 1$. So by Prop. 3 it follows that:

$$(w \Vdash \Box(\alpha \supset \beta)) = 1 \tag{1}$$

And

$$(w \Vdash \Box\alpha \supset \Box\beta) < 1 \tag{2}$$

By (2) and Prop. 3

$$(w \Vdash \Box\alpha) = 1 \tag{3}$$

And

$$(w \Vdash \Box\beta) < 1 \tag{4}$$

By (4) and Def. 11

$$N_C(\sup\{T(\mu_{\rightsquigarrow}(w, w'), N_C(w' \Vdash \beta))/w' \in W\}) < 1 \tag{5}$$

By (5) and Prop. 3

$$\sup\{T(\mu_{\rightsquigarrow}(w, w'), N_C(w' \Vdash \beta))/w' \in W\} = 1 \tag{6}$$

By (6) there exists a $w^* \in W$ such that

$$T(\mu_{\rightsquigarrow}(w, w^*), N_C(w^* \Vdash \beta)) = 1 \tag{7}$$

By (7) and the Prop. 1

$$\mu_{\rightsquigarrow}(w, w^*) = 1 \tag{8}$$

and

$$N_C(w^* \Vdash \beta) = 1 \tag{9}$$

From (9), by Prop. 3 we know that

$$(w^* \Vdash \beta) < 1 \tag{10}$$

By (1)

$$\sup\{T(\mu_{\rightsquigarrow}(w, w'), N_C(w' \Vdash \alpha \supset \beta))/w' \in W\} < 1 \tag{11}$$

By (11) and (8) in particular when $w' = w^*$ we have $N_C(w^* \Vdash \alpha \supset \beta)) < 1$, by Prop. 3, that is,

$$(w^* \Vdash \alpha \supset \beta) = 1 \tag{12}$$

Using (3),(8) and Prop. 3 analogously we conclude that

$$(w^* \Vdash \alpha) = 1 \tag{13}$$

By (12), (13) and the interpretation of classic-like fuzzy implication it follows that

$$(w^* \Vdash \beta) = 1 \tag{14}$$

But (14) contradicts (10).

Assume now that $\models_{\mathcal{K}} \beta$. Suppose by contradiction that $\models_{\mathcal{K}} \Box\beta$ is not the case. So there exists a $w \in W$ such that $(w \Vdash \Box\beta) < 1$, that is, $N_C(\sup\{T(\mu_{\rightsquigarrow}(w, w'), N_C(w' \Vdash \beta))/w' \in W\}) < 1$. It follows by Prop. 3 that $\sup\{T(\mu_{\rightsquigarrow}(w, w'), N_C(w' \Vdash \beta))/w' \in W\} = 1$. For some $w^* \in W$ it is the case that $T(\mu_{\rightsquigarrow}(w, w^*), N_C(w^* \Vdash \beta)) = 1$. From the latter we conclude that $(w' \Vdash \beta) < 1$, and this contradicts the assumption.

Assume next that $\models_{\mathcal{K}} \varphi$ and $\models_{\mathcal{K}} \varphi \supset \psi$. Suppose again by contradiction that $\models_{\mathcal{K}} \psi$ fails. So there exists a v such that $(v \Vdash \psi) < 1$. However, $(v \Vdash_{\mathcal{K}} \varphi) = 1$. By Prop. 3 it follows that $(v \Vdash \varphi \supset \psi) < 1$, so $\varphi \supset \psi$ is not a $\mathcal{K}$-tautology.

($\Leftarrow$) The K system is known to be complete with respect the class of all kripke models. So, by Prop. 4, if $\models_{\mathcal{K}} \alpha$ then $\vdash_K \alpha$.

5.2 Modal Normal System $KG^{k,l,m,n}$

Definition 15. *Given a fuzzy kripke model $\mathcal{KG}$ is KG-**like** if each theorem in the KG modal system is true in $\mathcal{KG}$ and, conversely, each formula that is true in $\mathcal{KG}$ is a theorem in the KG-modal system.*

5.3 Completeness of $KG^{k,l,m,n}$

Lemma 1. *Let $\mathcal{M} = \langle W, R, \mathbb{S}, \rho\rangle$ be a fuzzy kripke model. If $(w \Vdash \Diamond^z \phi) = 1$, then there exists a w_z such that both $\mu_{\overset{z}{\rightsquigarrow}}(w, w_z) = 1$ and $(w_z \Vdash \phi) = 1$.*

Proof. The proof proceeds by induction on z.

[Basis] $z = 1$

If $(w \Vdash \Diamond\beta) = 1$, then, by Def. 7, $\sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash \beta)/w' \in W\} = 1$. So, there is a $w_1 \in W$ such that $T(\mu_{\rightsquigarrow}(w, w_1), w_1 \Vdash \beta) = 1$. By Prop. 1 we have $\mu_{\rightsquigarrow}(w, w_1) = 1$ and $(w_1 \Vdash \beta) = 1$.

[Step] Suppose by Induction Hypothesis that for $z = k$ the property is valid.

Note that:

If $(w \Vdash \Diamond^{k+1}\beta) = 1$, then, by Def. 11,

$$\sup\{T(\mu_{\rightsquigarrow}(w, w'), w' \Vdash \Diamond^k\beta)/w' \in W\} = 1 \tag{15}$$

From (15) there exists a w_1 such that:

$$T(\mu_{\rightsquigarrow}(w, w_1), w_1 \Vdash \Diamond^k\beta) = 1 \tag{16}$$

By (16) and Prop. 1 we have:

$$\mu_{\rightsquigarrow}(w, w_1) = 1 \tag{17}$$

and

$$(w_1 \Vdash \Diamond^k\beta) = 1 \tag{18}$$

By (18) and Induction Hypothesis it follows there exists a $w_{k'}$ such that $\mu_{\overset{k}{\rightsquigarrow}}(w_1, w_{k'}) = 1$ and $(w_{k'} \Vdash \beta) = 1$. Using (17) and setting $w_{k+1} = w_{k'}$ we conclude $\mu_{\overset{k+1}{\rightsquigarrow}}(w, w_{k+1}) = 1$ and $(w_{k+1} \Vdash \beta) = 1$.

Lemma 2. *Let* $\mathcal{M} = \langle W, R, \mathbb{S}, \rho \rangle$ *be a fuzzy kripke model. If* $\mu_{\overset{m}{\rightsquigarrow}}(w, v) = 1$ *and* $(w \Vdash \Box^m \phi) = 1$, *then* $(v \Vdash \phi) = 1$.

Proof. The proof is carried out by induction on m.
[*Basis*] $m = 1$
Assume that:

$$\mu_{\rightsquigarrow}(w, v) = 1 \tag{19}$$

and

$$(w \Vdash \Box\beta) = 1 \tag{20}$$

Expanding (20) *we have*

$$(w \Vdash \neg\Diamond\neg\beta) = 1 \tag{21}$$

By Prop. 3 it follows that

$$N_c(w \Vdash \Diamond\neg\beta) = 1 \tag{22}$$

By (22)

$$\sup\{T(\mu_{\rightsquigarrow}(w, w'), N_c(w' \Vdash \beta)/w' \in W\} < 1 \tag{23}$$

From (23), *for every* $w' \in W$,

$$T(\mu_{\rightsquigarrow}(w, w'), N_c(w' \Vdash \beta)) < 1 \tag{24}$$

In particular, for $w' = v$,

$$T(\mu_{\rightsquigarrow}(w, v), N_c(v \Vdash \beta)) < 1 \tag{25}$$

By (19), (25) and Prop. 1

$$N_c(v \Vdash \beta) < 1 \tag{26}$$

We conclude from (26) and Prop. 3 that $(v \Vdash \beta) = 1$.
[*Step*] $m = k + 1$
The Induction Hypothesis states that for $m = k$, *if* $\mu_{\overset{k}{\rightsquigarrow}}(w, v) = 1$ *and* $(w \Vdash \Box^k\beta) = 1$ *then* $(v \Vdash \beta) = 1$.
Note that, given

$$\mu_{\overset{k+1}{\rightsquigarrow}}(w, v) = 1 \tag{27}$$

and

$$(w \Vdash \Box^{k+1}\beta) = 1 \tag{28}$$

it follows from (28) and Prop. 3 that

$$\sup\{T(\mu_{\rightsquigarrow}(w, w'), N_C(w' \Vdash \Box^k\beta))/w' \in W\} < 1 \tag{29}$$

On the other hand, for every $w' \in W$ *we have*

$$T(\mu_{\rightsquigarrow}(w, w'), N_C(w' \Vdash \Box^k\beta)) < 1 \tag{30}$$

By (27) there is a v_0 *such that* $\mu_{\rightsquigarrow}(w, v_0) = 1$. *Note that for such* v_0 *it is thus the case that* $T(\mu_{\rightsquigarrow}(w, v_0), N_C(v_0 \Vdash \Box^k\beta)) < 1$. *It follows that*

$$(v_0 \Vdash \Box^k\beta) = 1 \tag{31}$$

We conclude, by (31), that $\mu_{\overset{k}{\rightsquigarrow}}(v_0, v) = 1$ *and from the Induction Hypothesis it follows that* $(v \Vdash \beta) = 1$.

The following result concerns equivalences between formulas with nested modalities.

Lemma 3. *If* $\mathcal{M} = \langle W, R, \mathbb{S}, \rho \rangle$ *is a fuzzy kripke model, and* w *is a element of* W, *then* $(w \Vdash \neg\Diamond^m\phi) = 1$ *iff* $(w \Vdash \Box^m\neg\phi) = 1$.

Proof. It is not hard to check this by induction on m. The basis and inductive step follow by Def. 11, using Prop. 3.

Lemma 4. *Let $\mathcal{M} = \langle W, R, \mathbb{S}, \rho \rangle$ be a fuzzy kripke model. If $(w \Vdash \neg\Diamond^n \phi) = 1$ and $\mu_{\overset{n}{\leadsto}}(w, v) = 1$, then $(v \Vdash \phi) < 1$.*

Proof. This is a straightforward consequence of the previous results. Indeed, note first that by Lemma 3 we have $(v \Vdash \neg\Diamond^m \phi) = 1$ iff $(v \Vdash \Box^m \neg\phi) = 1$. So we know that $(w \Vdash \Box^m \neg\phi) = 1$ and $\mu_{\overset{n}{\leadsto}}(w, v) = 1$, and by applying Lemma 2 it follows that $(v \Vdash \neg\phi) = 1$. By Prop. 3 we conclude that $(v \Vdash \phi) < 1$.

Lemma 5. *Let $\mathcal{M} = \langle W, R, \mathbb{S}, \rho \rangle$ be a fuzzy kripke model. If $(w \Vdash \Box^n \phi) < 1$, then there exists some w_n such that $\mu_{\overset{n}{\leadsto}}(w, w_n) = 1$ and $(w_n \Vdash \neg\phi) = 1$.*

Proof. This is checked by induction on n. The basis is straightforward from Def. 11. If $(w \Vdash \Box^{k+1} \phi) < 1$ we have for some w_1 that $\mu_{\leadsto}(w, w_1) = 1$ and $(w_1 \Vdash \Box^k \phi) < 1$. So, using the Induction Hypothesis it follows that $\mu_{\overset{k+1}{\leadsto}}(w, w_{k+1}) = 1$ and $(w_{k+1} \Vdash \neg\phi) = 1$

The next lemma show that the axiom $G^{k,l,m,n}$ is sound with respect fuzzy kripke models where $\leadsto$ is (k, l, m, n)-confluent:

Lemma 6. *If α is a formula of form $G^{k,l,m,n}$ and $\mathcal{G}$ is a fuzzy kripke model where $\leadsto$ is (k, l, m, n)-confluent, then $\models_{\mathcal{G}} \alpha$.*

Proof. Let α be $\Diamond^k \Box^m \beta \supset \Box^l \Diamond^n \beta$. Suppose that $(w \Vdash_{\mathcal{G}} \Diamond^k \Box^m \beta \supset \Box^l \Diamond^n \beta) < 1$ for some $w \in W$. Then by Def. 7

$$(w \Vdash_{\mathcal{G}} \Diamond^k \Box^m \beta) = 1 \tag{32}$$

and

$$(w \Vdash_{\mathcal{G}} \Box^l \Diamond^n \beta) < 1 \tag{33}$$

By (32) and Lemma 1 there exists a w_k such that

$$\mu_{\overset{k}{\leadsto}}(w, w_k) = 1, \tag{34}$$

and

$$(w_k \Vdash_{\mathcal{G}} \Box^m \beta) = 1 \tag{35}$$

By (33) and Lemma 5 there exists a w_l such that

$$\mu_{\overset{l}{\leadsto}}(w, w_l) = 1 \tag{36}$$

and

$$(w_l \Vdash_{\mathcal{G}} \neg\Diamond^n \beta) = 1 \tag{37}$$

By (34), (36) and the appropriate instance of the confluence property of $\leadsto$ there exists a $x \in W$ such that

$$\mu_{\overset{m}{\leadsto}}(w_k, x) = 1 \tag{38}$$

and

$$\mu_{\overset{n}{\leadsto}}(w_l, x) = 1 \tag{39}$$

By (35), (38) and Lemma 2 we conclude that

$$(x \Vdash_{\mathcal{G}} \beta) = 1 \tag{40}$$

By (37), (39) and Lemma 4, on the other hand, we conclude that

$$(x \Vdash_{\mathcal{G}} \beta) < 1 \tag{41}$$

Note that (41) contradicts (40).

Theorem 1. *For any $\alpha \in LM_p$, we have that α is a theorem of $KG^{k,l,m,n}$ iff $\models_{\mathcal{KG}} \alpha$ for each fuzzy kripke model $\mathcal{KG} = \langle W, \rightsquigarrow, \mathbb{S}, \rho\rangle$ such that $\rightsquigarrow$ is (k,l,m,n)-confluent.*

Proof. ($\Rightarrow$) Let α be a theorem of the $KG^{k,l,m,n}$ and $\mathcal{KG}^{k,l,m,n}$ be a fuzzy kripke model where $\rightsquigarrow$ is (k,l,m,n)-confluent. We will prove that $\models_{\mathcal{KG}} \alpha$. In view of Prop. 5, however, it is sufficient to check the case where α is an instance of the $G^{k,l,m,n}$-axiom, i.e., to check that $(w \Vdash_{\mathcal{KG}} \Diamond^k\Box^m\beta \supset \Box^l\Diamond^n\beta) = 1$ for each $w \in W$ and $\beta \in LM_P$, but from the Lemma 6 it is immediate.
($\Leftarrow$) In [4] the completeness of system $KG^{k,l,m,n}$ with respect the class of models that satisfies the confluence accessibility relation is established. By Prop. 4 it follows that the system $\mathcal{KG}$ is complete with respect the $KG^{k,l,m,n}$ system. So, if $\models_{\mathcal{KG}} \beta$ then $\vdash_{KG} \beta$.

The completeness results proven in Prop. 1 can be shown to hold not only for singular instances of $G^{k,l,m,n}$, but also for several such instances combined. Indeed:

Proposition 6. *Let $G^{k_1,l_1,m_1,n_1}, \ldots, G^{k_p,l_p,m_p,n_p}$ be instances of the schema $G^{k,l,m,n}$. Let $K + G^{k_1,l_1,m_1,n_1} + \ldots + G^{k_p,l_p,m_p,n_p}$ be the system which results from extending K with $G^{k_1,l_1,m_1,n_1}, \ldots, G^{k_p,l_p,m_p,n_p}$. A formula α is a theorem of $K + G^{k_1,l_1,m_1,n_1} + \ldots + G^{k_p,l_p,m_p,n_p}$ iff $\Vdash_{\mathcal{KG}^+} \alpha$ for each fuzzy kripke model $\mathcal{KG}^+ = \langle W, \rightsquigarrow, \mathbb{S}, \rho\rangle$ such that $\rightsquigarrow$ is (k_1,l_1,m_1,n_1)-confluent, ..., (k_p,l_p,m_p,n_p)-confluent.*

Proof. ($\Rightarrow$) By Theorem 1 this result is valid for $K+G^{k_1,l_1,m_1,n_1}$. If we add G^{k_2,l_2,m_2,n_2} and use Lemma 6 we can conclude that $K + G^{k_1,l_1,m_2,n_2} + G^{k_2,l_2,m_2,n_2}$ is sound in all fuzzy kripke models such that $\rightsquigarrow$ is (k_1,l_1,m_1,n_1)-confluent and (k_2,l_2,m_2,n_2)-confluent. Using the same reasoning we can extend the result for each system $K + G^{k_1,l_1,m_1,n_1} + \ldots + G^{k_p,l_p,m_p,n_p}$. ($\Leftarrow$) From Prop. 4 this proof is analogous to the proof of completeness for extensions of K with finitely many instances of $G^{k,l,m,n}$, as done, e.g., in [6].

Notice that the completeness of the modal systems KT, KB and KD are direct consequences of Prop. 1, while the completeness of B, $S4$ and $S5$ follows from Prop. 6. For instance, here is how we may obtain completeness for $S5$.

Example 1. S5 is complete with respect all reflexive and euclidean fuzzy kripke models. *The modal system S5 is axiomatized by K, T and 5, i.e. $K+\langle 0,0,1,0\rangle+\langle 1,1,0,1\rangle$. But $\rightsquigarrow$ is $\langle 1,1,0,1\rangle$-confluent iff (by Definition 12) $\forall x\forall y\forall z((\mu_{\rightsquigarrow}(x,y) = 1 \wedge \mu_{\rightsquigarrow}(x,z) = 1) \rightarrow \exists w(y = w \wedge \mu_{\rightsquigarrow}(z,w) = 1))$ iff for arbitrary $x,y,z \in W$ we have that $(\mu_{\rightsquigarrow}(x,y) = 1 \wedge \mu_{\rightsquigarrow}(x,z) = 1) \rightarrow (\mu_{\rightsquigarrow}(z,y) = 1)$ iff $\forall x\forall y\forall z(\mu_{\rightsquigarrow}(x,y) = 1 \wedge \mu_{\rightsquigarrow}(x,z) = 1) \rightarrow (\mu_{\rightsquigarrow}(z,y) = 1))$ iff (by Definition 12) $\rightsquigarrow$ is euclidean. Furthermore, using a similar reasoning we note that $\rightsquigarrow$ is $\langle 0,0,1,0\rangle$-confluent iff $\forall x\forall y\forall z((x = y \wedge x = z) \rightarrow \exists w(\mu_{\rightsquigarrow}(y,w) = 1 \wedge z = w))$ iff $\forall x(\mu_{\rightsquigarrow}(x,x) = 1)$ iff $\rightsquigarrow$ is reflexive. So, by Theorem 6 follows the completeness of $K + \langle 0,0,1,0\rangle + \langle 1,1,0,1\rangle$ with respect all fuzzy kripke models that are reflexive and euclidean.*

6 Final Remarks

We believe it is possible to study a multimodal (diamond) version of the axiom of confluence by adding appropriate indices to the modalities, at the linguistic level, and adding corresponding fuzzy accessibility relations, at the semantic level (in such case, the initial case with iterated modalities will accordingly be reduced to distinct one-step modalities). Completeness should in this case be attainable, as in the case of normal modal logics extending classical logic, by adding appropriate interaction axioms.

We also conjecture that the above results on the axiom of confluence and its corresponding collection of frames may be extended to every Sahlqvist-definable frame class. This thread of investigation, however, shall be left as matter for future work.

References

1. Bedregal, B., Santiago, R.H.N., Benevides, M., Viana, P.: Characterization of fuzzy Kripke models for some usual normal modal logics. Preprint, 2011.
2. Bedregal, B.R.C., Cruz, A.P.: A characterization of classic-like fuzzy semantics. Logic Journal of the IGPL, 16:357–370, 2008.
3. Bedregal, B., Santiago, R.H.N., Benevides, M., Viana, P.: K, T and D-like fuzzy Kripke models. In Proceedings of the XXX North American Fuzzy Information Processing Society Annual Conference (NAFIPS 2011), held at El Paso / TX, US, 2011, 1–5. IEEE Computer Society, 2011.
4. Boolos,G.: The Logic of Provability. Cambridge, 1995.
5. Lemmon, J., Scott, D.S.: An Introduction to Modal Logic. Blackwell, 1977.
6. Pizzi, C., Carnielli, W. A.: Modalities and Multimodalities. Springer, 2008.