REPORT

Actions of Automorphisms on Some Classes of Fuzzy Bi-implications

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In a previous paper we have studied two classes of fuzzy bi-implications based on t-norms and r-implications, and shown that they constitute increasingly weaker subclasses of the Fodor-Roubens bi-implication. Now we prove that each of these three classes of bi-implications is closed under automorphisms.

1. Introduction

The collection of all automorphisms on a given mathematical object, i.e., isomorphisms from this object into itself, form a group with respect to the composition operator. Automorphisms have played an interesting role with respect to fuzzy connectives, given that when a class of fuzzy connectives is closed under automorphisms the action of the group of automorphisms establishes an equivalence relation among the connectives and therefore determines a partition on this class of connectives. Such partitions, in some cases, have characterized important subclasses of fuzzy connectives. For example, the class of strict t-norms is the equivalence class of the product t-norm [8], the class of nilpotent t-norms coincides with the equivalence class of the Łukasiewicz t-norm [8], the class of strong negations is the same as the equivalence class of the standard negation [9], and the class of implications which are both strong and residual is the equivalence class of the Łukasiewicz implication [1]. Concerning the bi-implication connective, in [5] we have studied the relation between the more well known definition proposed by Fodor and Roubens and other appealing definitions, old or new, of fuzzy operators that extend the interpretation of the classical bi-implication. It seems only reasonable then to study the action of automorphisms on fuzzy bi-implications. In the present paper we prove that each of the three classes of fuzzy bi-implications studied in [5] is closed under automorphisms.

2. Fuzzy extensions of conjunction and implication

All the following definitions concern the totally-ordered unit interval $\mathcal{U} = [0, 1]$.

Definition 9 *A triangular norm (in short, t-norm) is a binary operator T on $\mathcal{U}$ that: agrees with classical conjunction on the boolean inputs $\{0, 1\}$, is commutative, is associative, is increasing on both arguments, and has 1 as neutral element.*

Notions related to continuity are inherited from Analysis. In particular:

Definition 10 *A t-norm T is called left-continuous if for all non-decreasing sequences $(x_n)_{n\in\mathbb{N}}$ we have that $\lim_{n\to\infty} T(x_n, y) = T(\lim_{n\to\infty} x_n, y)$.*

It is also opportune to recall that a continuous function preserves both limits and suprema.

Definition 11 *A fuzzy implication is a binary operator I on $\mathcal{U}$ that: agrees with classical implication on boolean inputs, is decreasing on the first argument and is increasing on the second argument.*

Definition 12 *The residuum of a left-continuous t-norm T is the operation I such that $I(x, y) \geq z$ iff $T(z, x) \leq y$.*

It is easy to check that the residuum of a left-continuous t-norm is unique. A particularly interesting class of fuzzy implications is precisely the one based on residua:

Definition 13 *A binary operator I on $\mathcal{U}$ is called an r-implication if there is a t-norm T such that:*

$$I(x, y) = \sup\{z \in \mathcal{U} \mid T(x, z) \leq y\} \quad (125)$$

In such case we may also say that I is an r-implication based on T, and denote it by I^T. We will say that I^T is of type $\mathbb{LC}$ in case T is left-continuous. In the latter situation we also say that (T, I^T) forms an adjoint pair, or that I^T is the adjoint companion of T.

2.1 Automorphisms and their actions on the fuzzy connectives

Definition 14 *An automorphism ρ on $\mathcal{U}$ is a continuous strictly increasing unary function with boundary conditions $\rho(0) = 0$ and $\rho(1) = 1$.*

Recall that the inverse of a strictly increasing function on a totally ordered domain is also strictly increasing, and that continuous strictly increasing functions over closed intervals are bijective. From the above if follows that the inverse of an automorphism on the unit interval is strictly increasing, and in view of the boundary conditions is also follows that automorphisms are bijective. Moreover, since the inverse of an automorphism is also an automorphism and automorphisms are closed under composition, then $Aut(\mathcal{U})$, the set of automorphisms on $\mathcal{U}$, forms a group with respect to the composition operator. Thus, as usual in algebra (see for example [7]), we may entertain the action of members of the group $\langle Aut(\mathcal{U}), \circ\rangle$ on arbitrary representatives of a given collection of n-ary functions on $\mathcal{U}$.

Definition 15 *The action of an automorphism ρ on a function $f : \mathcal{U}^n \to \mathcal{U}$ is the function $f^\rho : \mathcal{U}^n \to \mathcal{U}$ defined by*

$$f^\rho(x_1, \ldots, x_n) = \rho^{-1}(f(\rho(x_1), \ldots, \rho(x_n))) \qquad (126)$$

In such situation we refer to f^ρ as a conjugate of f. A set $\mathcal{F}$ of n-ary functions on $\mathcal{U}$ is said to be closed under automorphisms if it contains the conjugates of each of its elements.

Given a collection $\mathcal{F}$ of functions that turns out to be closed under automorphisms, the relation of 'being a conjugate of' is clearly an equivalence relation on $\mathcal{F}$. Indeed, if g is a conjugate of a function f, then f is a conjugate of g — assuming that $g = f^\rho$, then, given that $f = f^{\rho\circ\rho^{-1}} = (f^\rho)^{\rho^{-1}}$, it follows that $f = g^{\rho^{-1}}$. In addition, if f is conjugate of g and g is conjugate of h then f is a conjugate of h, and clearly each function is conjugate of itself. Consequently, an automorphism allows us to partition the collection $\mathcal{F}$.

It is well known that the sets of t-norms, s-norms, fuzzy negations and implications are each closed under automorphisms (check, e.g., [2, 3, 8]). In the following we will check that the subclasses of left-continuous t-norms and of the r-implications are closed under automorphisms.

Proposition 14 *Let T be a t-norm and ρ be an automorphism. Then T is left-continuous iff T^ρ is a left-continuous t-norm.*

Proof. ($\Rightarrow$) Let $(x_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence. Then, given that ρ is strictly increasing, $(\rho(x_n))_{n\in\mathbb{N}}$ also is a non-decreasing sequence. Thus:

$$\begin{aligned}
&\lim_{n\to\infty} T^\rho(x_n, y) = \lim_{n\to\infty} \rho^{-1}(T(\rho(x_n), \rho(y))) = \\
&\qquad \text{by Eq. (126)} \\
&= \rho^{-1}(\lim_{n\to\infty} T(\rho(x_n), \rho(y))) = \\
&\qquad \text{because } \rho^{-1} \text{ is continuous} \\
&= \rho^{-1}(T(\lim_{n\to\infty} \rho(x_n), \rho(y)) = \\
&\qquad \text{because } T \text{ is left-continuous} \\
&= \rho^{-1}(T(\rho(\lim_{n\to\infty} x_n), \rho(y)) = \\
&\qquad \text{because } \rho \text{ is continuous} \\
&= T^\rho(\lim_{n\to\infty} x_n, y) \\
&\qquad \text{by Eq. (126)}
\end{aligned}$$

($\Leftarrow$) Follows straightforwardly from ($\Rightarrow$) and the fact that $(T^\rho)^{\rho^{-1}} = T$.

Proposition 15 *Let T be a t-norm, (I^T) be its residuum, and let ρ be an automorphism. Then $(I^T)^\rho = I^{(T^\rho)}$.*

Proof. Assume T be a t-norm with residuum (I^T), and notice that:

$$\begin{aligned}
&(I^T)^\rho(x, y) = \rho^{-1}(I^T(\rho(x), \rho(y))) = \\
&\qquad \text{by Eq. (126)} \\
&= \rho^{-1}(\sup\{z \in \mathcal{U} \mid T(\rho(x), z) \le \rho(y)\}) = \\
&\qquad \text{by Eq. (125)} \\
&= \rho^{-1}(\sup\{z \in \mathcal{U} \mid \rho^{-1}(T(\rho(x), z)) \le \rho^{-1}(\rho(y))\}) = \\
&\qquad \rho^{-1} \text{ is strictly increasing} \\
&= \rho^{-1}(\sup\{z \in \mathcal{U} \mid \rho^{-1}(T(\rho(x), \rho(\rho^{-1}(z)))) \le y\}) = \\
&\qquad \rho^{-1} \text{ is the inverse of } \rho \\
&= \sup\{\rho^{-1}(z) \in \mathcal{U} \mid \rho^{-1}(T(\rho(x), \rho(\rho^{-1}(z)))) \le y\} \\
&\qquad \rho^{-1} \text{ is continuous} \\
&= \sup\{\rho^{-1}(z) \in \mathcal{U} \mid T^\rho(x, \rho^{-1}(z)) \le y\} \\
&\qquad \text{by Eq. (126)} \\
&= I^{(T^\rho)}(x, y) \\
&\qquad \text{by Eq. (125)}
\end{aligned}$$

Corollary 1 *Consider a mapping $I : \mathcal{U}^2 \to \mathcal{U}$ and an automorphism ρ. Then I is an r-implication of type $\mathbb{LC}$ iff I^ρ is an r-implication of type $\mathbb{LC}$.*

Proof. Straightforward from Propositions 14 and 15.

3. Fuzzy bi-implication and automorphisms

3.1 Automorphisms on an axiomatized class of fuzzy bi-implications Fodor and Roubens have introduced an important class of fuzzy bi-implications [6]:

Definition 16 *The class of f-bi-implications contains all binary operators B on the unit interval $\mathcal{U}$ respecting the following axioms:*

(B1) $B(x, y) = B(y, x)$

(B2) $B(x, x) = 1$

(B3) $B(0, 1) = 0$

(B4) *If $w \le x \le y \le z$, then $B(w, z) \le B(x, y)$*

In view of (B1), (B2) and (B3), it is easy to see that any f-fuzzy bi-implication is bound to agree with classical bi-implication on boolean inputs.

The following are some examples of f-bi-implications:

Example 14

1. $B_M(x, y) = \begin{cases} 1 & \text{if } x = y \\ \min(x, y) & \text{otherwise} \end{cases}$

2. $B_P(x, y) = \begin{cases} 1 & \text{if } x = y \\ \frac{\min(x,y)}{\max(x,y)} & \text{otherwise} \end{cases}$

3. $B_L(x, y) = 1 - |x - y|$

4. $B_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 1 & \text{otherwise} \end{cases}$

5. $B_B^{TI1}(x, y) = \begin{cases} 1 & \text{if } x = y \text{ or } \max(x, y) \neq 1 \\ 0 & \text{otherwise} \end{cases}$

Proposition 16 *Consider a mapping $B : \mathcal{U}^2 \to \mathcal{U}$ and an automorphism ρ. Then B satisfies (Bi) iff B^ρ satisfies (Bi), for $i = 1, \ldots, 4$.*

Proof. (⇒) If B satisfies (Bi) for $i = 1, 2, 3$ then, from Eq. (126) and the fact that $\rho(1) = 1$ and $\rho(0) = 0$, trivially B^ρ satisfies (Bi). Moreover, if $w \le x \le y \le z$, then because ρ is strictly increasing, $\rho(w) \le \rho(x) \le \rho(y) \le \rho(z)$ and so, since B satisfies (B4), we have that $B(\rho(w), \rho(z)) \le B(\rho(x), \rho(y))$. Therefore, because ρ^{-1} is strictly increasing, $\rho^{-1}(B(\rho(w), \rho(z))) \le \rho^{-1}(B(\rho(x), \rho(y)))$, i.e., $B^\rho(w, z) \le B^\rho(x, y)$ and so B^ρ satisfies (B4).
(⇐) Follows straightforwardly from (⇒) and the fact that $(B^\rho)^{\rho^{-1}} = B$.

Corollary 2 *Consider a mapping $B : \mathcal{U}^2 \to \mathcal{U}$ and an automorphism ρ. Then B is an f-bi-implication iff B^ρ is also an f-bi-implication.*

Proof. Straightforward from previous proposition.

Example 15 *Let ρ be the following automorphism: $\rho(x) = x^2$ for each $x \in \mathcal{U}$. Then:*

1. $B_M(x, y) = B^\rho_M(x, y)$
2. $B_P(x, y) = B^\rho_P(x, y)$
3. $B^\rho_L(x, y) = \sqrt{1 - |x^2 - y^2|}$
4. $B_D(x, y) = B^\rho_D(x, y)$.
5. $B^{TI1}_B(x, y) = (B^{TI1}_B)^\rho(x, y)$

Note that equations 1, 4 and 5 from the previous example hold good in fact for any choice of automorphism ρ.

Definition 17 *An f-bi-implication B is said to satisfy the diagonal principle if $B(x, y) \ne 1$ whenever $x \ne y$.*

Proposition 17 *Let B be an f-bi-implication and ρ be an automorphism. Then B satisfies the diagonal principle iff its conjugate B^ρ satisfies this same principle.*

Proof. (⇒) If $x \ne y$ then because ρ is injective, $\rho(x) \ne \rho(y)$ and so, because B satisfies the diagonal principle, $B(\rho(x), \rho(y)) \ne 1$. Thus, because ρ^{-1} is injective, then $\rho^{-1}(B(\rho(x), \rho(y))) \ne \rho^{-1}(1)$, i.e., $B^\rho(x, y) \ne 1$.
(⇐) Follows straightforwardly from (⇒) and the fact that $(B^\rho)^{\rho^{-1}} = B$.

As is well known, t-norms, s-norms and fuzzy implications, with the help of the truth constants 0 and 1, induce 'natural' classes of fuzzy negations [2]. In the case of a (f-)bi-implication B, the natural negation is the function $N_B : \mathcal{U} \to \mathcal{U}$ defined by

$$N_B(x) = B(x, 0) \quad (127)$$

In the following result we check that the conjugate of the natural fuzzy negation induced by a fuzzy bi-implication coincides with the natural negation induced by the conjugate (with respect to the same automorphism) of the given bi-implication.

Proposition 18 *Let ρ be an automorphism and B be an f-bi-implication. Then the equation $(N_B)^\rho = N_{B^\rho}$ holds.*

Proof. Let $x \in \mathcal{U}$. Then, by equations (126) and (127), we have that $N^\rho_B(x) = \rho^{-1}(N_B(\rho(x))) = \rho^{-1}(B(\rho(x), 0)) = \rho^{-1}(B(\rho(x), \rho(0))) = B^\rho(x, 0) = N_{B^\rho}(x)$.

3.2 Automorphisms on classes of fuzzy bi-implications based on a defining standard involving t-norms and fuzzy implications In the definitions that follow, we assume fuzzy bi-implication B to be presented through the so-called TI *defining standard*, according to which $B(x, y) = T(I(x, y), I(y, x))$, where T is a t-norm and I an r-implication.

Definition 18 ([5]) *The class of a-bi-implications contains all binary operators B on $\mathcal{U}$ following the TI defining standard and based on an arbitrary t-norm T and on its residuum I^T, that is, operators defined by setting*

$$B(x, y) = T(I^T(x, y), I^T(y, x)) \quad (128)$$

Proposition 19 *Consider a mapping $B : \mathcal{U}^2 \to \mathcal{U}$ and an automorphism ρ. Then B is an a-bi-implication iff B^ρ is an a-bi-implication.*

Proof. (⇒) Assume B an a-bi-implication, and notice that:

$$\begin{aligned}
B^\rho(x, y) &= \rho^{-1}(B(\rho(x), \rho(y))) = \\
&\quad \text{by Eq. (126)} \\
&= \rho^{-1}(T(I^T(\rho(x), \rho(y)), I^T(\rho(y), \rho(x)))) = \\
&\quad \text{by Eq. (128)} \\
&= \rho^{-1}(T(\rho \circ \rho^{-1}(I^T(\rho(x), \rho(y))), \rho \circ \rho^{-1}(I^T(\rho(y), \rho(x))))) = \\
&= T^\rho((I^T)^\rho(x, y), (I^T)^\rho(y, x)) = \\
&\quad \text{by Eq. (126)} \\
&= T^\rho(I^{(T^\rho)})(x, y), I^{(T^\rho)}(y, x)) = \\
&\quad \text{by Prop. 15}
\end{aligned}$$

Therefore, also B^ρ is an a-bi-implication.
(⇐) Follows straightforwardly from (⇒) and the fact that $(B^\rho)^{\rho^{-1}} = B$.

Corollary 3 *Consider a mapping $B : \mathcal{U}^2 \to \mathcal{U}$ and an automorphism ρ. Then B is an a-bi-implication and not an f-bi-implication iff B^ρ is an a-bi-implication and not an f-bi-implication.*

Proof. Straightforward from Proposition 19, Corollary 2 and the fact that $(B^\rho)^{\rho^{-1}} = B$.

Definition 19 ([5]) *The class of ℓ-bi-implications contains all a-bi-implications based on left-continuous t-norms and their corresponding residua.*

Proposition 20 *Consider a mapping $B : \mathcal{U}^2 \to \mathcal{U}$ and an automorphism ρ. Then B is an ℓ-bi-implication iff B^ρ is an ℓ-bi-implication.*

Proof. Straightforward from Proposition 19 and Corollary 1.

Corollary 4 *Consider a mapping $B : \mathcal{U}^2 \to \mathcal{U}$ and an automorphism ρ. Then B is an a-bi-implication and not an ℓ-bi-implication iff B^ρ is an a-bi-implication and not an ℓ-bi-implication.*

Proof. Straightforward from Propositions 19 and 20 and the fact that $(B^{\rho})^{\rho^{-1}} = B$.

4. Conclusion

In this paper we considered the action of the group of automorphisms on the three classes of fuzzy bi-implications that were studied in [5], namely, f-bi-implications, a-bi-implications and ℓ-bi-implications. More specifically, we proved that all three classes are closed under automorphisms and therefore the actions of automorphisms induce partitions of these classes. For example, the equivalence class of the fuzzy bi-implication B_M is the singleton set $\{B_M\}$, and analogously for the fuzzy bi-implications B_D and B_B^{TI1} and the singleton sets $\{B_D\}$ and $\{B_B^{TI1}\}$, yet the equivalence class of B_L is not a countable set (to see that, in Ex. 15 it is sufficient to replace the automorphism $\rho(x) = x^2$ by $\rho(x) = x^r$, with r a positive real number).

This is a preliminary study, and several other aspects of the actions of automorphisms on bi-implications rest to be investigated. On the one hand, it would seem only natural to extend the present development to cover other classes of fuzzy bi-implications, characterized by other defining standards, as for instance the IST defining standard based on the classical equivalence in between $\alpha \Leftrightarrow \beta$ and $(\alpha \vee \beta) \Rightarrow (\alpha \wedge \beta)$. In general, the classes of fuzzy bi-implications that follow the IST defining standard are not subclasses of the class of f-bi-implications. On the other hand, it would also be interesting to extend the present study to cover other particularly interesting subclasses of the class of f-bi-implications, such as the so-called restricted equivalence functions introduced in [4].

Acknowledgments The research reported in this paper falls within the scope of the EU FP7 Marie Curie PIRSES-GA-2012-318986 project GeTFun: Generalizing Truth-Functionality. The authors acknowledge partial support by Brazilian funding agency CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico), Edital Universal 14/2011 — Faixa B, under the Proc. No. 480832/2011-0, project ConDE: Conjuntos Difusos Estendidos.

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