



Negative Modalities, Consistency and Determinedness

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Abstract

We study a modal language for negative operators —an intuitionistic-like negation and its paraconsistent dual—added to (bounded) distributive lattices. For each non-classical negation an extra operator is hereby adjoined in order to allow for standard logical inferences to be opportunely restored. We present abstract characterizations and exhibit the main properties of each kind of negative modality, as well as of the associated connectives that express consistency and determinedness at the object-language level. Appropriate sequent-style proof systems and adequate kripke semantics are also introduced, characterizing the minimal normal logic and a few other basic logics containing such negative modalities and their companions.

Keywords: Modal Logics, Paraconsistency, Paracompleteness, Derivability Adjustment.

1 Context

Negationless normal modal logics with box-like and diamond-like operators were studied by Dunn in [10], where the author obtains completeness results for the systems characterized by the class of all kripke frames and by a few specific subclasses thereof. In [7], Celani & Jansana extend that study so as to cover many other logics, and to that effect they consider kripke-style semantics based on frames containing two relations —one of them being a preorder, as in intuitionistic logic, allowing for the expression of appropriate heredity conditions. Systems containing analogous negative modalities were studied by Dunn & Zhou, who investigate in [11] modal logics with conjunction, disjunction, an impossibility operator intended to play the role of an intuitionistic-like negation and a non-necessity operator intended to play

¹ The research reported in this paper falls within the scope of the EU FP7 Marie Curie PIRSES-GA-2012-318986 project GeTFun: Generalizing Truth-Functionality. For their important feedback, the authors wish to thank in particular Cláudia Nalon, Carlos Caleiro, Arnon Avron, Ori Lahav and Yoni Zohar, as well as the audiences at IMLA/UNILOG and at SQIG/IT.

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the role of a paraconsistent negation. Restall, in [16], proposed a combination of positive and negative diamond-like modal operators; one of his aims was to use the resulting system to exhibit examples of modal logics that turn out to be undecidable even in the absence of classical negation. In the present paper we will study a logic that contains the already mentioned negative modal operators over a strictly positive propositional basis (on a fragment agreed upon by intuitionists and classical logicians) to which we add extra operators that express at the object-language level the very notions of consistency and determinedness that allows one to recover much of the standard logical reasoning even when neither classical negation nor classical implication are available. The mentioned extra modal ‘restoration’ connectives were first proposed in [14]. The basic universal logic apparatus used here is based on [13] and [15], and the proof-theoretical approach to the consistency operator is inherited from [3].

The structure of the paper is as follows: in Section 2 we present the Universal Logic background, including the formulation of properties characterizing negative modalities, and the properties that characterize connectives intended to express consistency and determinedness at the object-language level; in Section 3 a sequent system is used to define our main and most basic modal system, in which we include rules for introducing the restoration connectives and rules for the interaction between the non-classical negations; in Sections 4 and 5 the intended kripke semantics is presented for our full modal language and our deductive system is shown to be sound and complete with respect to this semantics; a few extensions of the basic system are then formulated in Section 6; in Section 7 we study how the inferences of more standard logic systems may be recovered with the use of our rich modal language, by way of appropriate Derivability Adjustment Theorems; last, in Section 8, we briefly comment upon some directions for future research.

2 Universal Logic perspective

Let \mathcal{L} be a standard propositional language. As customary, we shall use small Greek letters to denote arbitrary sentences, and capital Greek letters for sets of sentences of \mathcal{L} . A generalized consequence relation (gcr) will here be assumed to be a relation $\triangleright \subseteq \mathbf{2}^{\mathcal{L}} \times \mathbf{2}^{\mathcal{L}}$ that enjoys the following universal properties:

- (ovl) $\Gamma, \varphi \triangleright \varphi, \Delta$
- (mon) If $\Gamma_1 \triangleright \Delta_1$, then $\Gamma_2, \Gamma_1 \triangleright \Delta_1, \Delta_2$
- (trn) If $\Gamma_1, \varphi \triangleright \Delta_1$ and $\Gamma_2 \triangleright \varphi, \Delta_2$, then $\Gamma_1, \Gamma_2 \triangleright \Delta_1, \Delta_2$

In writing a statement such as $\Pi \cup \{\pi\} \triangleright \emptyset$ in the simplified form $\Pi, \pi \triangleright$ we are simply aligning with standard usage from the literature. Here we shall write $\Gamma \blacktriangleright \Delta$ to indicate that $\Gamma \triangleright \Delta$ fails, that is, that $\langle \Gamma, \Delta \rangle \notin \triangleright$. Furthermore, aiming at a structured outlook on the above properties and on proofs based on them, we shall employ the following graphical representation:

$$\frac{}{\Gamma, \varphi \triangleright \varphi, \Delta}^{(ovl)} \quad \frac{\Gamma_1 \triangleright \Delta_1}{\Gamma_2, \Gamma_1 \triangleright \Delta_1, \Delta_2}^{(mon)} \quad \frac{\Gamma_1, \varphi \triangleright \Delta_1 \quad \Gamma_2 \triangleright \varphi, \Delta_2}{\Gamma_1, \Gamma_2 \triangleright \Delta_1, \Delta_2}^{(trn)}$$

A set $\Sigma \subseteq \mathcal{L}$ will be called a \triangleright -**theory** if $\varphi \in \Sigma$ whenever $\Sigma \triangleright \varphi, \Delta$ for every $\Delta \subseteq \mathcal{L}$. Dually, the set of sentences Σ will be called a \triangleright -**cotheory** if $\varphi \in \Sigma$ whenever $\Gamma, \varphi \triangleright \Sigma$ for every $\Gamma \subseteq \mathcal{L}$. Taking such definitions into account, a \triangleright -**theory pair** will be any pair $\langle \Sigma_1, \Sigma_0 \rangle$ where Σ_1 is a \triangleright -theory and Σ_0 is a \triangleright -cotheory. Given two \triangleright -theory pairs $\Sigma = \langle \Sigma_1, \Sigma_0 \rangle$ and $\Pi = \langle \Pi_1, \Pi_0 \rangle$, we say that Π **extends** Σ if $\Sigma_1 \subseteq \Pi_1$ and $\Sigma_0 \subseteq \Pi_0$ — we denote this by $\Sigma \subseteq \Pi$. In addition, fixed a given gcr \triangleright , a theory pair $\Sigma = \langle \Sigma_1, \Sigma_0 \rangle$ is called **unconnected** if $\Sigma_1 \blacktriangleright \Sigma_0$, and is called **closed** if $\Sigma_1 \cup \Sigma_0 = \mathcal{L}$. A gcr is called **trivial** if it does not allow for any unconnected theory pair.

A gcr \triangleright is called **finitary** if it enjoys the following property:

(fin) If $\Gamma \triangleright \Delta$, then there are finite sets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \triangleright \Delta'$

For finitary gcrs, thus, a connected theory pair extends some *finite* connected theory pair.

We confirm next that finitary gcrs enjoy the following property — a version of the well-known *Lindenbaum-Asser Lemma* (cf. [17]):

Proposition 2.1 *Let \triangleright be a finitary gcr. Then every unconnected \triangleright -theory pair can be extended into a closed unconnected \triangleright -theory pair.*

Proof. Assume $\Gamma \blacktriangleright \Delta$. Let \mathcal{E} be the collection of unconnected extensions of $\langle \Gamma, \Delta \rangle$, partially ordered by inclusion, and let $\mathcal{C} = \{\langle \mathcal{C}_1^i, \mathcal{C}_0^i \rangle\}_{i \in I}$ be some chain (a totally ordered set) on \mathcal{E} . We claim that $\bigcup \mathcal{C} = \langle \bigcup_{i \in I} \mathcal{C}_1^i, \bigcup_{i \in I} \mathcal{C}_0^i \rangle$ is an upper bound for \mathcal{E} , i.e., we claim that $\Pi \subseteq \bigcup \mathcal{C}$ for every $\Pi \in \mathcal{E}$ (which is obvious) and also claim that (*) $\bigcup \mathcal{C} \in \mathcal{E}$.

We check (*). Where $\Lambda = \langle \Lambda_1, \Lambda_0 \rangle$ is a \triangleright -theory pair, let $\text{Fin}(\Lambda)$ denote the set of \triangleright -theory pairs $\langle \widehat{\Lambda}_1, \widehat{\Lambda}_0 \rangle$ where $\widehat{\Lambda}_1 \cup \widehat{\Lambda}_0$ is a finite set and Λ extends $\langle \widehat{\Lambda}_1, \widehat{\Lambda}_0 \rangle$. Consider an arbitrary $\Phi = \langle \Phi_1, \Phi_0 \rangle$ such that $\Phi \in \text{Fin}(\bigcup \mathcal{C})$. Then there is some $\mathcal{C}^k \in \mathcal{C}$ such that $\Phi \subseteq \mathcal{C}^k$. Once $\mathcal{C}_1^k \blacktriangleright \mathcal{C}_0^k$, by (mon) we conclude that $\Phi_1 \blacktriangleright \Phi_0$. By (fin) it follows that $\bigcup \mathcal{C}$ is unconnected. By Zorn’s Lemma, if every chain in a partially ordered set has an upper bound, then there is a maximal element in that set; so, we conclude that \mathcal{E} must have a maximal unconnected element $\langle \Gamma^*, \Delta^* \rangle \supseteq \langle \Gamma, \Delta \rangle$. To see that $\langle \Gamma^*, \Delta^* \rangle$ is indeed closed, suppose there is some $\varphi \in \mathcal{L}$ such that neither $\langle \Gamma^* \cup \{\varphi\}, \Delta^* \rangle$ nor $\langle \Gamma^*, \Delta^* \cup \{\varphi\} \rangle$ are unconnected. Then, by (trn), it would follow that $\Gamma^* \triangleright \Delta^*$. \square

From this point on we consider some language specifics, concerning connectives of \mathcal{L} . A binary connective \wedge in \mathcal{L} will be called a \triangleright -**ordinary conjunction** when it satisfies

$$(oC) \quad \Gamma, \varphi \wedge \psi \triangleright \Delta \quad \text{iff} \quad \Gamma, \varphi, \psi \triangleright \Delta$$

for arbitrary sentences $\varphi, \psi \in \mathcal{L}$ and arbitrary contexts $\Gamma, \Delta \subseteq \mathcal{L}$. In other words, to have a classic-like behavior, a conjunction will be expected to internalize, at the object-level, the meta-level commas that appear in the left-hand side of \triangleright . Dually,

a binary connective \vee is called a \triangleright -**ordinary disjunction** when it satisfies

$$(oD) \quad \Gamma \triangleright \varphi \vee \psi, \Delta \quad \text{iff} \quad \Gamma \triangleright \varphi, \psi, \Delta$$

In addition, a \triangleright -**ordinary top** and a \triangleright -**ordinary bottom** are 0-ary connectives \top and \perp satisfying

$$(oT) \quad \Gamma, \top \triangleright \Delta \quad \text{iff} \quad \Gamma \triangleright \Delta \qquad (oB) \quad \Gamma \triangleright \perp, \Delta \quad \text{iff} \quad \Gamma \triangleright \Delta$$

From such definitions one may easily check for instance that:

Proposition 2.2 *For any \triangleright -ordinary conjunction \wedge and any \triangleright -ordinary disjunction \vee , the following rule-statements may be shown to hold:*

$$\frac{\Gamma_1 \triangleright \alpha, \Delta_1 \quad \Gamma_2 \triangleright \beta, \Delta_2}{\Gamma_1, \Gamma_2 \triangleright \alpha \wedge \beta, \Delta_1, \Delta_2} (Cj1) \qquad \frac{\Gamma_1, \alpha \triangleright \Delta_1 \quad \Gamma_2, \beta \triangleright \Delta_2}{\Gamma_1, \Gamma_2, \alpha \vee \beta \triangleright \Delta_1, \Delta_2} (Dj1)$$

Proof. The proofs proceed as follows. We starting with rule-statement (Cj1):

| Proof using the properties of gr | Graphical representation of the proof |
|--|---|
| Assume (1) $\Gamma_1 \triangleright \alpha, \Delta_1$ and (2) $\Gamma_2 \triangleright \beta, \Delta_2$. By (ov1) we know that (3) $\alpha \wedge \beta \triangleright \alpha \wedge \beta$. By (3) and (oC) it follows that (4) $\alpha, \beta \triangleright \alpha \wedge \beta$. Using (trn) on (1) and (4) we obtain (5) $\Gamma_1, \beta \triangleright \alpha \wedge \beta, \Delta_1$. From (5), (2) and (trn) we conclude that $\Gamma_1, \Gamma_2 \triangleright \alpha \wedge \beta, \Delta_1, \Delta_2$. | $\frac{\frac{\frac{\alpha \wedge \beta \triangleright \alpha \wedge \beta}{\alpha, \beta \triangleright \alpha \wedge \beta}^{(ov1)}}{\Gamma_1, \beta \triangleright \alpha \wedge \beta, \Delta_1}^{(oC)} \quad \Gamma_1 \triangleright \alpha, \Delta_1}{\Gamma_1, \beta \triangleright \alpha \wedge \beta, \Delta_1}^{(trn)} \quad \Gamma_2 \triangleright \beta, \Delta_2}{\Gamma_1, \Gamma_2 \triangleright \alpha \wedge \beta, \Delta_1, \Delta_2}^{(trn)}$ |

For rule-statement (Dj1) we rely directly on the corresponding tree-like presentation:

$$\frac{\frac{\frac{\alpha \vee \beta \triangleright \alpha \vee \beta}{\alpha \vee \beta \triangleright \alpha, \beta}^{(ov1)}}{\alpha \vee \beta \triangleright \alpha, \beta}^{(oD)} \quad \Gamma_1, \alpha \triangleright \Delta_1}{\Gamma_1, \alpha \vee \beta \triangleright \beta, \Delta_1}^{(trn)} \quad \Gamma_2, \beta \triangleright \Delta_2}{\Gamma_1, \Gamma_2, \alpha \vee \beta \triangleright \Delta_1, \Delta_2}^{(trn)}$$

□

An immediate offshoot of the above result is that theories are closed under ordinary conjunctions and cotheories are closed under ordinary disjunctions:

Corollary 2.3 *Let \wedge be a \triangleright -ordinary conjunction, \vee be a \triangleright -ordinary disjunction, \top be a \triangleright -ordinary top and \perp be a \triangleright -ordinary bottom. Consider a \triangleright -theory pair $\langle \Sigma_1, \Sigma_0 \rangle$. Then:*

- (i) If $\varphi \in \Sigma_1$ and $\psi \in \Sigma_1$, then $\varphi \wedge \psi \in \Sigma_1$
- (ii) If $\varphi \in \Sigma_0$ and $\psi \in \Sigma_0$, then $\varphi \vee \psi \in \Sigma_0$
- (iii) $\top \in \Sigma_1$
- (iv) $\perp \in \Sigma_0$

This will be very useful later on, in particular in Section 5. For closed theories, a further important result may be proven:

Proposition 2.4 *Let \wedge be a \triangleright -ordinary conjunction and \vee be a \triangleright -ordinary disjunction, and let $\langle \Sigma_1, \Sigma_0 \rangle$ be a closed unconnected \triangleright -theory pair. Then:*

- (i) *If $\varphi \vee \psi \in \Sigma_1$, then $\varphi \in \Sigma_1$ or $\psi \in \Sigma_1$*
- (ii) *If $\varphi \wedge \psi \in \Sigma_0$, then $\varphi \in \Sigma_0$ or $\psi \in \Sigma_0$*

Proof. For item (i), suppose by contraposition that $\varphi \notin \Sigma_1$ and $\psi \notin \Sigma_1$. By closure it follows that $\varphi \in \Sigma_0$ and $\psi \in \Sigma_0$. By the definition of cotheory, this means that $\Gamma, \varphi \triangleright \Sigma_0$ and $\Gamma, \psi \triangleright \Sigma_0$ for every $\Gamma \subseteq \mathcal{L}$. For any arbitrary such Γ it follows by (Dj1) that $\Gamma, \varphi \vee \psi \triangleright \Sigma_0$. So, given that Σ_0 is a cotheory, we have $\varphi \vee \psi \in \Sigma_0$, and by unconnectedness it follows that $\varphi \vee \psi \notin \Sigma_1$. The proof of item (ii) uses (Cj1). \square

Fix now a unary connective $\#$ in \mathcal{L} . We say that $\#$ is \triangleright -**preserving** if it satisfies

$$(Pvs\#) \quad \varphi \triangleright \psi \text{ implies } \#\varphi \triangleright \#\psi$$

and say that $\#$ is \triangleright -**reversing** if

$$(Rvs\#) \quad \varphi \triangleright \psi \text{ implies } \#\psi \triangleright \#\varphi$$

Following [15], the **minimal conditions** we will demand for calling $\#$ a **negation** consist on the existence of sentences φ and ψ such that $\#\varphi \blacktriangleright \varphi$ and $\psi \blacktriangleright \#\psi$, that is, such that the theory pairs $\langle \#\varphi, \varphi \rangle$ and $\langle \psi, \#\psi \rangle$ are unconnected. The underlying intuition is that negation should bring about some ‘inversion’ with respect to the underlying notion of consequence. It should be noticed that, in principle, a negation abiding to such minimal conditions need not be \triangleright -reversing —yet, (Rvs $\#$) is a typical and desirable property of *modal* negations such as the ones we will be studying in the present paper. The following result introduces some properties that will play an important role in what follows:

Proposition 2.5 *Assume \wedge to be a \triangleright -ordinary conjunction and \vee to be a \triangleright -ordinary disjunction. For any \triangleright -preserving connective $\#$, the following statements may then be checked to hold:*

$$(PM1.1\#) \quad \#(\varphi \wedge \psi) \triangleright \#\varphi \wedge \#\psi \qquad (PM2.1\#) \quad \#\varphi \vee \#\psi \triangleright \#(\varphi \vee \psi)$$

If $\#$ is \triangleright -reversing, the following alternative statements may be checked instead:

$$(DM1.1\#) \quad \#(\varphi \vee \psi) \triangleright \#\varphi \wedge \#\psi \qquad (DM2.1\#) \quad \#\varphi \vee \#\psi \triangleright \#(\varphi \wedge \psi)$$

Proof. The proofs below use rule-statements (Cj1) and (Dj1) from Prop. 2.2.

| | |
|---|---|
| $\frac{\frac{\frac{\overline{\varphi \triangleright \varphi}^{(ov1)}}{\varphi \triangleright \varphi}^{(mon)}}{\varphi, \psi \triangleright \varphi}^{(oC)}}{\#\varphi \wedge \psi \triangleright \#\varphi}^{(Pvs\#)} \quad \frac{\frac{\frac{\overline{\psi \triangleright \psi}^{(ov1)}}{\psi \triangleright \psi}^{(mon)}}{\varphi, \psi \triangleright \psi}^{(oC)}}{\varphi \wedge \psi \triangleright \psi}^{(Pvs\#)}}{\#\varphi \wedge \psi \triangleright \#\psi}^{(Cj1)}}{\#\varphi \wedge \psi \triangleright \#\varphi \wedge \#\psi}^{(Cj1)}$ | $\frac{\frac{\frac{\frac{\overline{\varphi \triangleright \varphi}^{(ov1)}}{\varphi \triangleright \varphi}^{(mon)}}{\varphi \triangleright \varphi, \psi}^{(oD)}}{\varphi \triangleright \varphi \vee \psi}^{(Rvs\#)}}{\#(\varphi \vee \psi) \triangleright \#\varphi}^{(Rvs\#)} \quad \frac{\frac{\frac{\frac{\overline{\psi \triangleright \psi}^{(ov1)}}{\psi \triangleright \psi}^{(mon)}}{\psi \triangleright \varphi, \psi}^{(oD)}}{\psi \triangleright \varphi \vee \psi}^{(Rvs\#)}}{\#(\varphi \vee \psi) \triangleright \#\psi}^{(Rvs\#)}}{\#(\varphi \vee \psi) \triangleright \#\varphi \wedge \#\psi}^{(Cj1)}$ |
| $\frac{\frac{\frac{\frac{\overline{\varphi \triangleright \varphi}^{(ov1)}}{\varphi \triangleright \varphi}^{(mon)}}{\varphi \triangleright \varphi, \psi}^{(oD)}}{\#\varphi \triangleright \#(\varphi \vee \psi)}^{(Pvs\#)}}{\#\varphi \triangleright \#(\varphi \vee \psi)}^{(Dj1)} \quad \frac{\frac{\frac{\frac{\overline{\psi \triangleright \psi}^{(ov1)}}{\psi \triangleright \psi}^{(mon)}}{\psi \triangleright \varphi, \psi}^{(oD)}}{\psi \triangleright \varphi \vee \psi}^{(Pvs\#)}}{\#\psi \triangleright \#(\varphi \vee \psi)}^{(Pvs\#)}}{\#\psi \triangleright \#(\varphi \vee \psi)}^{(Dj1)}}{\#\varphi \vee \#\psi \triangleright \#(\varphi \vee \psi)}^{(Dj1)}$ | $\frac{\frac{\frac{\frac{\overline{\varphi \triangleright \varphi}^{(ov1)}}{\varphi \triangleright \varphi}^{(mon)}}{\varphi, \psi \triangleright \varphi}^{(oC)}}{\varphi \wedge \psi \triangleright \varphi}^{(Rvs\#)}}{\#\varphi \triangleright \#(\varphi \wedge \psi)}^{(Rvs\#)} \quad \frac{\frac{\frac{\frac{\overline{\psi \triangleright \psi}^{(ov1)}}{\psi \triangleright \psi}^{(mon)}}{\varphi, \psi \triangleright \psi}^{(oC)}}{\varphi \wedge \psi \triangleright \psi}^{(Rvs\#)}}{\#\psi \triangleright \#(\varphi \wedge \psi)}^{(Rvs\#)}}{\#\psi \triangleright \#(\varphi \wedge \psi)}^{(Dj1)}}{\#\varphi \vee \#\psi \triangleright \#(\varphi \wedge \psi)}^{(Dj1)} \quad \square$ |

In what follows we shall say that $\#$ has **type** $\boxed{+}$ (read ‘box-plus’) if it respects

$$(PM1.2\#) \quad \#\varphi \wedge \#\psi \triangleright \#(\varphi \wedge \psi)$$

and say that $\#$ has **type** \diamondleftrightarrow (‘diamond-plus’) if it respects

$$(PM2.2\#) \quad \#(\varphi \vee \psi) \triangleright \#\varphi \vee \#\psi$$

When a \triangleright -ordinary top \top is available, we will expect a **full** type $\boxed{+}$ connective $\#$ to also respect

$$(PT\#) \quad \triangleright \#\top$$

Given an \triangleright -ordinary bottom \perp , a **full** type \diamondleftrightarrow connective $\#$ is also to respect

$$(PB\#) \quad \#\perp \triangleright$$

Dually, we shall say that $\#$ has **type** $\boxed{-}$ (read as ‘box-minus’) if it respects

$$(DM1.2\#) \quad \#\varphi \wedge \#\psi \triangleright \#(\varphi \vee \psi)$$

and say that $\#$ has **type** \diamondleftarrow (‘diamond-minus’) if it respects

$$(DM2.2\#) \quad \#(\varphi \wedge \psi) \triangleright \#\varphi \vee \#\psi$$

When a \triangleright -ordinary top \top or a \triangleright -ordinary bot \perp are available, a **full** type $\boxed{-}$ connective $\#$ will be expected to respect

$$(DB\#) \quad \triangleright \#\perp$$

and a **full** type \diamondleftarrow connective $\#$ will be expected to respect

$$(DT\#) \quad \#\top \triangleright$$

We now turn to properties induced by our main (non-classical) negations and use them to characterize the restoration connectives that will accompany them. Here, given some specific sentence φ , the gcr \triangleright will be called **$\#$ -consistent with respect to φ** in case it satisfies

$$(Cns\#\varphi) \quad \Gamma, \varphi, \#\varphi \triangleright \Delta$$

for any choice of contexts Γ and Δ . Dually, the gcr \triangleright will be called **$\#$ -determined with respect to φ** in case it satisfies

$$(Dtm\#\varphi) \quad \Gamma \triangleright \#\varphi, \varphi, \Delta$$

for any choice of contexts Γ and Δ . A gcr will be called **$\#$ -inconsistent** if there is some sentence φ with respect to which $(Cns\#\varphi)$ fails, and will be called **$\#$ -undetermined** if there is some sentence φ with respect to which $(Dtm\#\varphi)$ fails. If some negation $\#$ is available such that \triangleright turns out to be both $\#$ -consistent and $\#$ -determined with respect to all sentences, such $\#$ will be called a **\triangleright -ordinary negation**. For $\#$ -inconsistent and for $\#$ -undetermined gcrs it will often be useful to have a way of internalizing, at the object-level, the corresponding notions of consis-

tency and determinedness. To that effect, a \triangleright -**ordinary consistency** connective will be defined as a unary symbol \oplus satisfying

$$(GC\#) \quad \Gamma \triangleright \oplus\varphi, \Delta \text{ iff } \Gamma, \varphi, \#\varphi \triangleright \Delta$$

for any choice of contexts Γ and Δ and any sentence φ . Analogously, a \triangleright -**ordinary determinedness** connective will be defined as a unary symbol \oplus satisfying

$$(GD\#) \quad \Gamma, \oplus\varphi \triangleright \Delta \text{ iff } \Gamma \triangleright \#\varphi, \varphi, \Delta$$

A useful alternative abstract characterization of such new connectives is exhibited in what follows:

Proposition 2.6 *Let \triangleright be a gcr. Then:*

[EQ1] *Clause (GC#) is equivalent to the following three clauses taken together:*

$$(Cb\#) \quad \Gamma, \oplus\varphi, \#\varphi, \varphi \triangleright \Delta \quad (Ck1\#) \quad \Gamma \triangleright \varphi, \oplus\varphi, \Delta \quad (Ck2\#) \quad \Gamma \triangleright \#\varphi, \oplus\varphi, \Delta$$

[EQ2] *Clause (GD#) is equivalent to the following three clauses taken together:*

$$(Db\#) \quad \Gamma \triangleright \varphi, \#\varphi, \oplus\varphi, \Delta \quad (Dk1\#) \quad \Gamma, \oplus\varphi, \varphi \triangleright \Delta \quad (Dk2\#) \quad \Gamma, \oplus\varphi, \#\varphi \triangleright \Delta$$

Proof. Rule-statement (GC#) may be split in two halves, namely:

$$\frac{\Gamma, \varphi, \#\varphi \triangleright \Delta}{\Gamma \triangleright \oplus\varphi, \Delta} (GC1\#) \qquad \frac{\Gamma \triangleright \oplus\varphi, \Delta}{\Gamma, \varphi, \#\varphi \triangleright \Delta} (GC2\#)$$

These will help in attaining our goals below. We start by verifying [EQ1].

[Part 1] Assume (GC1#) and (GC2#) to hold. Then notice that:

$$\frac{\frac{\Gamma, \oplus\varphi \triangleright \oplus\varphi, \Delta}{\Gamma, \oplus\varphi, \#\varphi, \varphi \triangleright \Delta} (GC2\#)}{\Gamma, \oplus\varphi \triangleright \oplus\varphi, \Delta} (ovl) \quad \left| \quad \frac{\frac{\Gamma, \varphi \triangleright \varphi, \Delta}{\Gamma, \varphi, \#\varphi \triangleright \varphi, \Delta} (mon)}{\Gamma \triangleright \varphi, \oplus\varphi, \Delta} (GC1\#)}{\Gamma, \oplus\varphi \triangleright \oplus\varphi, \Delta} (ovl) \quad \left| \quad \frac{\frac{\Gamma, \#\varphi \triangleright \#\varphi, \Delta}{\Gamma, \varphi, \#\varphi \triangleright \#\varphi, \Delta} (mon)}{\Gamma \triangleright \#\varphi, \oplus\varphi, \Delta} (GC1\#)}{\Gamma, \oplus\varphi \triangleright \oplus\varphi, \Delta} (ovl)$$

[Part 2] Assume (Cb#), (Ck1#) and (Ck2#) to hold. Then notice that:

$$\frac{\frac{\Gamma, \#\varphi \triangleright \oplus\varphi, \Delta}{\Gamma, \#\varphi \triangleright \oplus\varphi, \Delta} (Ck1\#)}{\Gamma \triangleright \oplus\varphi, \Delta} (trn) \quad \frac{\frac{\Gamma \triangleright \varphi, \oplus\varphi, \Delta}{\Gamma \triangleright \#\varphi, \oplus\varphi, \Delta} (Ck2\#)}{\Gamma \triangleright \oplus\varphi, \Delta} (trn) \quad \left| \quad \frac{\Gamma \triangleright \oplus\varphi, \Delta \quad \frac{\Gamma, \oplus\varphi, \#\varphi, \varphi \triangleright \Delta}{\Gamma, \varphi, \#\varphi \triangleright \Delta} (Cb\#)}{\Gamma, \varphi, \#\varphi \triangleright \Delta} (trn)$$

Verifying equivalence [EQ2], now, is an entirely analogous exercise, which we shall leave to the interested reader. \square

From this point on we shall fix a set of sentences \mathcal{L} inductively defined by:

$$\varphi ::= p \mid (\varphi_1 \wedge \varphi_2) \mid (\varphi_1 \vee \varphi_2) \mid (\neg\varphi) \mid (\sim\varphi) \mid (\odot\varphi) \mid (\ominus\varphi)$$

where p ranges over a denumerable set \mathcal{P} of propositional variables, both \neg and \sim are symbols intended to represent negation, and the symbols \ominus and \odot are intended to represent the restoration connectives that will be associated to the latter negation symbols. Fixed an arbitrary sentence φ , we will define \top as an abbreviation for $\varphi \vee \neg\varphi \vee \odot\varphi$ and will define \perp as short for $\varphi \wedge \sim\varphi \wedge \ominus\varphi$. In the following sections we shall introduce a convenient deductive system involving the above connectives, and provide subsequently a characteristic modal interpretation for them. Using

such proof system and such interpretation we will be able to easily classify each connective of \mathcal{L} with respect to the terminology introduced above.

It is worth adding a few words on the connection between inconsistency, undeterminedness and the perhaps more usual terms ‘paraconsistency’ and ‘paracompleteness’, now very common in the literature on non-classical negations. Suppose the language of a given gcr \triangleright contains a symbol $\#$ satisfying the minimal conditions to be called a negation. In that case, we say that \triangleright is **#-paraconsistent** if there are sentences φ and ψ such that $\varphi, \#\varphi \blacktriangleright \psi$, and say that \triangleright is **#-paracomplete** if there are sentences φ and ψ such that $\varphi \blacktriangleright \#\psi, \psi$. Obviously, in case a #-ordinary bottom \perp is available, #-paraconsistency simply coincides with #-inconsistency, and in case a #-ordinary top \top is available, #-paracompleteness coincides with #-undeterminedness. Paraconsistent logics equipped with ordinary consistency connectives constitute particularly interesting examples of the so-called **logics of formal inconsistency**, or more simply **LFIs** (check [6,5]). Their duals, paracomplete logics with ordinary determinedness connectives, are called **logics of formal undeterminedness**, or **LFUs**.

As an additional useful matter of notation for the next sections, given $T \subseteq \mathcal{L}$ and any unary connective \odot we shall by $\odot[T]$ denote the set $\{\odot\varphi : \varphi \in T\}$, and by $\odot^{-1}[T]$ denote the set $\{\varphi : \odot\varphi \in T\}$. By \overline{T} we will denote the complement of T relative to \mathcal{L} .

3 Proof-theoretical presentation

We will introduce in what follows our main sequent systems, namely, proof formalisms with each rule has the format $\frac{\{A_i \Rightarrow B_i : i \in I\}}{A \Rightarrow B}$ (rule) where each A_k and each B_k represents a finite sequence of sentences of \mathcal{L} and where I is a finite set of indices. As usual, given a collection \mathcal{R} of rules, a deductive system is associated to \mathcal{R} by defining $\Gamma \vdash \Delta$ to hold if there are finite sets $A \subseteq \Gamma$ and $B \subseteq \Delta$ such that $A \Rightarrow B$ is derivable from the rules in \mathcal{R} . In what follows we impose the standard structural rules, defining the system \mathcal{S} :

$$\frac{}{\varphi \Rightarrow \varphi} \text{ (id)} \qquad \frac{A_1, \varphi \Rightarrow B_1 \quad A_2 \Rightarrow \varphi, B_2}{A_1, A_2 \Rightarrow B_1, B_2} \text{ (cut)}$$

$$\frac{A \Rightarrow B}{A, \varphi \Rightarrow B} \text{ (w/)} \qquad \frac{A \Rightarrow B}{A \Rightarrow \varphi, B} \text{ (w)}$$

Such rules and the very definition of \vdash are obviously sufficient to guarantee that the corresponding deductive system is a finitary gcr. As is well-known, the system \mathcal{DL} for distributive lattices is obtained from \mathcal{S} by adding the standard rules for (classical) conjunction and disjunction:

$$\frac{A, \varphi, \psi \Rightarrow B}{A, \varphi \wedge \psi \Rightarrow B} \text{ (}\wedge\text{)} \qquad \frac{A \Rightarrow \varphi, B \quad A \Rightarrow \psi, B}{A \Rightarrow \varphi \wedge \psi, B} \text{ (}\wedge\text{)}$$

$$\frac{A, \varphi \Rightarrow B \quad A, \psi \Rightarrow B}{A, \varphi \vee \psi \Rightarrow B} \text{ (}\vee\text{)} \qquad \frac{A \Rightarrow \varphi, \psi, B}{A \Rightarrow \varphi \vee \psi, B} \text{ (}\vee\text{)}$$

Where \vdash_{dl} is the gcr associated to \mathcal{DL} , the interplay between the structural rules and the logical rules allows us to easily check that both \wedge and \vee are \vdash_{dl} -ordinary, as well as to derive the usual distributivity rules involving the connectives \wedge and \vee — namely, to derive both $\varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$ and $(\varphi \wedge \psi) \vee (\varphi \wedge \chi) \Rightarrow \varphi \wedge (\psi \vee \chi)$, as well as their duals, exchanging the roles of \wedge and \vee .

Our main system \mathcal{K}^n adds to \mathcal{DL} the following logical rules involving the remaining connectives of the language \mathcal{L} :

$$\frac{A \Rightarrow \varphi, B \quad A \Rightarrow \sphericalangle \varphi, B}{A, \ominus \varphi \Rightarrow B} (\ominus /) \quad \frac{A, \varphi, \sphericalangle \varphi \Rightarrow B}{A \Rightarrow \ominus \varphi, B} (/ \ominus)$$

$$\frac{A \Rightarrow \varphi, \neg \varphi, B}{A, \odot \varphi \Rightarrow B} (\odot /) \quad \frac{A, \varphi \Rightarrow B \quad A, \neg \varphi \Rightarrow B}{A \Rightarrow \odot \varphi, B} (/ \odot)$$

$$\frac{A \Rightarrow \varphi, B}{\neg[B], \sphericalangle \varphi \Rightarrow \sphericalangle[A]} (\sphericalangle \sim) \quad \frac{A, \varphi \Rightarrow B}{\neg[B] \Rightarrow \neg \varphi, \sphericalangle[A]} (\sphericalangle \sim)$$

Using the structural rules and the rules for \ominus and \odot , it easily follows that (as in Prop. 2.6):

Proposition 3.1 *The following sequents are derivable in \mathcal{K}^n :*

| | |
|---|--|
| <p>(GCb\sphericalangle) $\ominus \varphi, \sphericalangle \varphi, \varphi \Rightarrow$</p> <p>(GCK1$\sphericalangle$) $\Rightarrow \varphi, \ominus \varphi$</p> <p>(GCK2$\sphericalangle$) $\Rightarrow \sphericalangle \varphi, \ominus \varphi$</p> | <p>(GDb$\neg$) $\Rightarrow \varphi, \neg \varphi, \odot \varphi$</p> <p>(GDK1$\neg$) $\odot \varphi, \varphi \Rightarrow$</p> <p>(GDK2$\neg$) $\odot \varphi, \neg \varphi \Rightarrow$</p> |
|---|--|

Recalling the appropriate definitions from Section 2 and substituting \vdash_n for \triangleright , one may easily check in \mathcal{K}^n the following assertions as derived rules:

Proposition 3.2

- (i) \perp is a \vdash_n -ordinary bottom and \top is a \vdash_n -ordinary top
- (ii) \neg is a full type $[-]$ \vdash_n -reversing connective, and \sphericalangle is a full type $\langle \rangle$ \vdash_n -reversing connective
- (iii) \odot is a \vdash_n -ordinary determinedness connective, and \ominus is a \vdash_n -ordinary consistency connective

Proof. [\perp is a \vdash_n -ordinary bottom]

$$\frac{A \Rightarrow B}{A \Rightarrow \perp, B} (/w) \quad \frac{A \Rightarrow \perp, B}{A \Rightarrow \varphi \wedge \sphericalangle \varphi \wedge \ominus \varphi, B} (\text{def. } \perp) \quad \frac{\frac{\frac{}{\varphi, \sphericalangle \varphi, \ominus \varphi \Rightarrow} (\text{GCb}\sphericalangle)}{\varphi \wedge \sphericalangle \varphi, \ominus \varphi \Rightarrow} (\wedge /)}{\varphi \wedge \sphericalangle \varphi \wedge \ominus \varphi \Rightarrow} (\wedge /)}{A \Rightarrow B} (\text{cut})$$

[\top is a \vdash_n -ordinary top]

$$\frac{A \Rightarrow B}{A, \top \Rightarrow B} (w /) \quad \frac{\frac{A, \top \Rightarrow B}{A, \varphi \vee \neg \varphi \vee \odot \varphi \Rightarrow B} (\text{def. } \top)}{\frac{\frac{\frac{}{\Rightarrow \varphi, \neg \varphi, \odot \varphi} (\text{GDb}\neg)}{\Rightarrow \varphi \vee \neg \varphi, \odot \varphi} (\vee /)}{\Rightarrow \varphi \vee \neg \varphi \vee \odot \varphi} (\vee /)}}{A \Rightarrow B} (\text{cut})$$

[\neg is a full type \neg] \vdash_n -reversing connective]

$$\frac{\frac{\frac{}{\varphi, \sim\varphi, \ominus\varphi \Rightarrow} \text{(GCb}\sim)}{\varphi \wedge \sim\varphi \wedge \ominus\varphi \Rightarrow} (\wedge/) \times 2}{\frac{\perp \Rightarrow}{\Rightarrow \neg\perp} (\sim)}}{\frac{}{\varphi \wedge \sim\varphi \wedge \ominus\varphi \Rightarrow} \text{(def. } \perp)} \quad \frac{\frac{\frac{}{\varphi \Rightarrow \varphi} \text{(id)} \quad \frac{}{\psi \Rightarrow \psi} \text{(id)}}{\varphi \vee \psi \Rightarrow \varphi, \psi} (\vee/)}{\frac{\neg\varphi, \neg\psi \Rightarrow \neg(\varphi \vee \psi)}{\neg\varphi \wedge \neg\psi \Rightarrow \neg(\varphi \vee \psi)} (\wedge/)} \quad \frac{\varphi \Rightarrow \psi}{\neg\psi \Rightarrow \neg\varphi} (\sim)$$

[\sim is a full type \leftrightarrow] \vdash_n -reversing connective]

$$\frac{\frac{\frac{}{\Rightarrow \varphi, \neg\varphi, \ominus\varphi} \text{(GDb}\neg)}{\Rightarrow \varphi \vee \neg\varphi \vee \ominus\varphi} (\vee/) \times 2}{\Rightarrow \top} \text{(def. } \top)}{\sim\top \Rightarrow} (\sim\sim) \quad \frac{\frac{\frac{}{\varphi \Rightarrow \varphi} \text{(id)} \quad \frac{}{\psi \Rightarrow \psi} \text{(id)}}{\varphi, \psi \Rightarrow \varphi \wedge \psi} (\wedge/)}{\sim(\varphi \wedge \psi) \Rightarrow \sim\varphi, \sim\psi} (\sim\wedge)}{\sim(\varphi \wedge \psi) \Rightarrow \sim\varphi \vee \sim\psi} (\vee\sim)} \quad \frac{\varphi \Rightarrow \psi}{\sim\psi \Rightarrow \sim\varphi} (\sim\sim)$$

[\ominus is a \vdash_n -ordinary determinedness connective and \ominus is a \vdash_n -ordinary consistency connective] To check this, note first that rules $(\ominus/)$ and $(/ \ominus)$ already do half of the job. As for the other half:

$$\frac{A, \ominus\varphi \Rightarrow B}{A \Rightarrow \varphi, \neg\varphi, B} \text{(cut)} \quad \frac{\frac{}{\Rightarrow \varphi, \neg\varphi, \ominus\varphi} \text{(GDb}\neg)}{A \Rightarrow \ominus\varphi, B} \text{(cut)} \quad \frac{A \Rightarrow \ominus\varphi, B}{A, \varphi, \sim\varphi \Rightarrow B} \text{(cut)} \quad \square$$

The following simple observation follows from Prop. 3.2(ii) and rules $(\ominus/)$ and $(/ \ominus)$, as may be easily checked by the reader:

Proposition 3.3 *The sequents $\Rightarrow \ominus\top$ and $\ominus\perp \Rightarrow$ are derivable in \mathcal{K}^n .*

The following result concerns \vdash -theory pairs, and will play an important role in Section 5:

Proposition 3.4 *Let $\langle \Sigma_1, \Sigma_0 \rangle$ be an unconnected \vdash -theory pair. Then, the derivability of the nonempty sequent $A \Rightarrow B$ implies that either $\alpha \notin \Sigma_1$ for some $\alpha \in A$, or $\beta \notin \Sigma_0$ for some $\beta \in B$.*

Proof. Consider a derivable sequent of the form $\alpha_1, \alpha_2, \dots, \alpha_m \Rightarrow \beta_1, \beta_2, \dots, \beta_n$ where $m + n > 0$. Using rules $(\wedge/)$ and $(/ \vee)$ we may derive the sequent $\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_m \Rightarrow \beta_1 \vee \beta_2 \vee \dots \vee \beta_n$. Call the latter sequent *Seq*. Suppose $\alpha_i \in \Sigma_1$ for every $1 \leq i \leq m$, and $\beta_j \in \Sigma_0$ for every $1 \leq j \leq n$. By Corol. 2.3 and in view of the fact that \wedge is a \vdash -ordinary conjunction and that \vee is a \vdash -ordinary disjunction, it follows that (i) $\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_m \in \Sigma_1$ and that (ii) $\beta_1 \vee \beta_2 \vee \dots \vee \beta_n \in \Sigma_0$. Given that Σ_1 is a \vdash -theory, from (i) and the derivability of *Seq* it follows that (iii) $\beta_1 \vee \beta_2 \vee \dots \vee \beta_n \in \Sigma_1$; given that Σ_0 is a \vdash -cotheory, from (ii) and the derivability of *Seq* it follows that (iv) $\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_m \in \Sigma_0$. Now one may use the sequent axiom (id) to conclude from (i) and (iv), in case $m \neq 0$, that the pair $\langle \Sigma_1, \Sigma_0 \rangle$ is not unconnected; the same may be concluded from (ii) and (iii) in case $n \neq 0$. \square

The next section will introduce an adequate kripke semantics for \mathcal{K}^n .

4 Kripke semantics

Here, as usual, a **frame** $\mathcal{F} = \langle W, R \rangle$ will be a structure containing a nonempty set W and a relation $R \subseteq W \times W$ — members of W are often called **worlds** and R

is said to be an **accessibility relation** between these worlds. A **state-of-affairs** s on the frame \mathcal{F} is a mapping $s : \mathcal{P} \rightarrow \mathbf{2}^W$. A **valuation** is defined as the recursive extension of a given state-of-affairs s into a mapping $V^s : \mathcal{L} \rightarrow \mathbf{2}^W$, as follows:

$$\begin{aligned} V^s(p) &= s(p), \text{ where } p \in \mathcal{P} \\ V^s(\varphi_1 \wedge \varphi_2) &= V^s(\varphi_1) \cap V^s(\varphi_2) \\ V^s(\varphi_1 \vee \varphi_2) &= V^s(\varphi_1) \cup V^s(\varphi_2) \\ V^s(\neg\varphi) &= \{w \in W : \forall v \in W (wRv \text{ implies } v \notin V^s(\varphi))\} \\ V^s(\sim\varphi) &= \{w \in W : \exists v \in W (wRv \text{ and } v \notin V^s(\varphi))\} \\ V^s(\odot\varphi) &= \{w \in W : w \notin V^s(\varphi) \text{ and } w \notin V^s(\neg\varphi)\} \\ V^s(\ominus\varphi) &= \{w \in W : w \notin V^s(\varphi) \text{ or } w \notin V^s(\sim\varphi)\} \end{aligned}$$

As there is thus a unique valuation V^s associated to each given state-of-affairs s , we will in what follows simply omit the index s from V^s . It is helpful to fix at this point the reading of the statement ‘ $w \in V(\odot\varphi)$ ’ as guaranteeing the consistency of φ at w , and to fix the reading of the statement ‘ $w \notin V(\odot\varphi)$ ’ as guaranteeing the determinedness of φ at w .

Given the definitions of \perp and \top as abbreviations (Section 2), it is easy to check from the above notion of valuation that $V(\perp) = \emptyset$ and $V(\top) = W$. A **model** $\mathcal{M} = \langle \mathcal{F}, V \rangle$ is a structure where \mathcal{F} is a frame and V is a valuation on \mathcal{F} . Given a class of frames \mathbb{F} , with the above definitions we may immediately consider the class \mathbb{M} of all models based on such frames. We say that $\varphi \in \mathcal{L}$ is **satisfied** at a state $w \in W$ of a model $\mathcal{M} = \langle W, R, V \rangle$ if $w \in V(\varphi)$; this is denoted by $\mathcal{M}, w \Vdash \varphi$. When $w \notin V(\varphi)$ we write $\mathcal{M}, w \not\Vdash \varphi$ and say that \mathcal{M} **falsifies** φ at w . Given two sets of sentences, Γ and Δ , we say that Γ **entails** Δ , denoted by $\Gamma \models \Delta$, when at every world of every model either some sentence in Γ is falsified or some sentence in Δ is satisfied; sometimes this definition is relativized to some given class of frames on which the relevant models are to be based. It is not hard to check that \models is a gcr. As usual, the failure of $\Gamma \models \Delta$ will be denoted by $\Gamma \not\models \Delta$. When φ is satisfied at all states of all models of a given frame \mathcal{F} we say that φ is **valid** in \mathcal{F} , in symbols $\mathcal{F} \Vdash \varphi$. The definition of satisfaction is extended to sequents by writing $\mathcal{M}, w \Vdash A \Rightarrow B$ if \mathcal{M} falsifies some $\varphi \in A$ at w or \mathcal{M} satisfies some $\varphi \in B$ at w . Moreover, on what concerns the other definitions, for any given model \mathcal{M} and any given frame \mathcal{F} we write $\mathcal{M} \Vdash A \Rightarrow B$ to say that $\mathcal{M}, w \Vdash A \Rightarrow B$ at every state w in \mathcal{M} , and write $\mathcal{F} \Vdash A \Rightarrow B$ to say that $\mathcal{M} \Vdash A \Rightarrow B$ for every model \mathcal{M} of \mathcal{F} .

Using the above semantics, and taking into account the definitions in Section 2, is not hard to check that:

Proposition 4.1 *Both \neg and \sim enjoy the minimal conditions expected of a negation. Indeed, for any atomic variables p and q :*

- | | |
|----------------------------|----------------------------|
| (1) $\neg p \not\models p$ | (2) $q \not\models \neg q$ |
| (3) $\sim p \not\models p$ | (4) $q \not\models \sim q$ |

Proof. Consider the frame in which $W = \{w\}$, $R = \emptyset$ and, based on this frame, consider a model such that $V(p) = \emptyset$ and $V(q) = W$. It is easy to see that this is a counter-model that bears witness to (1) and (4). From that a counter-model witnessing assertions (2) and (3) is built by simply replacing $R = \emptyset$ by its complement $R = W \times W$. \square

Proposition 4.2 *The entailment \models is \neg -undetermined as well as \sim -inconsistent.*

Proof. Consider a frame \mathcal{F} such that $W = \{u, v\}$ and R is the total relation $W \times W$, and consider a model \mathcal{M} such that $V(p) = \{u\}$ for an atomic variable p . It follows that $V(\sim p) = \{u\}$ and $V(\neg p) = \emptyset$, thus both p and $\sim p$ are satisfied at u , and both p and $\neg p$ are falsified at v . \square

Our present semantical framework allows us also to provide straightforward verifications for many inferences which would give rise to long derivations. The following statements that guarantee that consistency propagates through conjunction and that determinedness propagates through disjunction may indeed very easily be verified by the reader.

Proposition 4.3 $\odot\varphi, \odot\psi \models \odot(\varphi \wedge \psi)$ and $\odot(\varphi \vee \psi) \models \odot\varphi, \odot\psi$

More importantly, the usual inductive reasoning allows us to establish that any derivable inference can be checked semantically:

Proposition 4.4 [Soundness] *All rules of \mathcal{K}^n are sound for frame validity, for arbitrary frames, that is, the conclusion of each given rule is valid on all frames that validate the premisses of that rule.*

Proof. Let \mathcal{F} be some fixed arbitrary frame. We will skip the proof of frame validity for the standard structural rules and for the standard rules for conjunction and disjunction, and concentrate below on the distinctive rules of \mathcal{K}^n .

Rule (\odot/\odot): Assume that (a) $\mathcal{F} \Vdash (A \Rightarrow \varphi, B)$ and (b) $\mathcal{F} \Vdash (A \Rightarrow \sim\varphi, B)$. Suppose that $\mathcal{F} \not\Vdash A, \odot\varphi \Rightarrow B$. Then, there are a model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ and a world w in \mathcal{M} such that $\mathcal{M}, w \not\Vdash A, \odot\varphi \Rightarrow B$. From this we have that (c) $\mathcal{M}, w \Vdash \alpha$ for every $\alpha \in A$, (d) $\mathcal{M}, w \Vdash \odot\varphi$ and (e) $\mathcal{M}, w \not\Vdash \beta$ for every $\beta \in B$. By (c), (e) and (a) it follows that (f) $\mathcal{M}, w \Vdash \varphi$. Now from (c), (e) and (b) it follows that (g) $\mathcal{M}, w \Vdash \sim\varphi$. By the definition of valuation, (f) and (g), we conclude that $\mathcal{M}, w \not\Vdash \odot\varphi$. This contradicts (d).

Rule ($/\odot$): Assume that (a) $\mathcal{F} \Vdash (A, \varphi, \sim\varphi \Rightarrow B)$ and suppose that (b) $\mathcal{F} \not\Vdash (A \Rightarrow \odot\varphi, B)$. By (b) there are a model \mathcal{M} and a world w such that (c) $\mathcal{M}, w \Vdash \alpha$ for every $\alpha \in A$, (d) $\mathcal{M}, w \not\Vdash \odot\varphi$ and (e) $\mathcal{M}, w \not\Vdash \beta$ for every $\beta \in B$. By (a), (c) and (e) we have that $\mathcal{M}, w \not\Vdash \varphi$ or $\mathcal{M}, w \not\Vdash \sim\varphi$, and so $\mathcal{M}, w \Vdash \odot\varphi$, which contradicts (d).

Rule (\odot/\neg): Assume that (a) $\mathcal{F} \Vdash (A \Rightarrow \varphi, \neg\varphi, B)$ and suppose that (b) $\mathcal{F} \not\Vdash (A, \odot\varphi \Rightarrow B)$. By (b) there are a model \mathcal{M} and a world w such that (c) $\mathcal{M}, w \Vdash \alpha$ for every $\alpha \in A$, (d) $\mathcal{M}, w \Vdash \odot\varphi$ and (e) $\mathcal{M}, w \not\Vdash \beta$ for every $\beta \in B$. By (a), (c) and (e) we have that $\mathcal{M}, w \Vdash \varphi$ or $\mathcal{M}, w \Vdash \neg\varphi$, and so $\mathcal{M}, w \not\Vdash \odot\varphi$, contradicting (d).

Rule ($/\odot$) : Assume that (a) $\mathcal{F} \Vdash (A, \varphi \Rightarrow B)$ and (b) $\mathcal{F} \Vdash (A, \wedge\varphi \Rightarrow B)$. Suppose that $\overline{\mathcal{F}} \not\Vdash A \Rightarrow \odot\varphi, B$. Then, there are a model \mathcal{M} and a world w in \mathcal{M} such that $\mathcal{M}, w \not\Vdash A \Rightarrow \odot\varphi, B$. From this we have that (c) $\mathcal{M}, w \Vdash \alpha$ for every $\alpha \in A$, (d) $\mathcal{M}, w \not\Vdash \odot\varphi$ and (e) $\mathcal{M}, w \not\Vdash \beta$ for every $\beta \in B$. By (c), (e) and (a) we have that (f) $\mathcal{M}, w \not\Vdash \varphi$, and from (c), (e) and (b) it follows that (g) $\mathcal{M}, w \not\Vdash \wedge\varphi$. By (f) and (g) we have $\mathcal{M}, w \Vdash \odot\varphi$, which contradicts (d).

Rule ($\smile\wedge$) : By contraposition assume that $\mathcal{F} \not\Vdash (\wedge[B], \smile\varphi \Rightarrow \smile[A])$. Then, there are a model \mathcal{M} and a state u in \mathcal{M} such that (i) $\mathcal{M}, u \Vdash \wedge\beta$ for every $\beta \in B$, (ii) $\mathcal{M}, u \Vdash \smile\varphi$ and (iii) $\mathcal{M}, u \not\Vdash \smile\alpha$ for every $\alpha \in A$. By (ii) there exists a world v in \mathcal{M} such that (iv) uRv and (v) $\mathcal{M}, v \not\Vdash \varphi$. It follows, by (iii) and (iv), that (vi) $\mathcal{M}, v \Vdash \alpha$ for every $\alpha \in A$. From (i) and (iv) it follows that (vii) $\mathcal{M}, v \not\Vdash \beta$ for every $\beta \in B$. By (v), (vi) and (vii) we conclude that $\mathcal{M}, v \not\Vdash A \Rightarrow \varphi, B$, therefore $\mathcal{F} \not\Vdash A \Rightarrow \varphi, B$.

Rule ($\wedge\smile$) : By contraposition assume that $\mathcal{F} \not\Vdash (\wedge[B] \Rightarrow \wedge\varphi, \smile[A])$. Then, there are a model \mathcal{M} and a state w in \mathcal{M} such that (i) $\mathcal{M}, w \Vdash \wedge\beta$ for every $\beta \in B$, (ii) $\mathcal{M}, w \not\Vdash \wedge\varphi$ and (iii) $\mathcal{M}, w \not\Vdash \smile\alpha$ for every $\alpha \in A$. By (ii) there exists a world z in \mathcal{M} such that (iv) wRz and (v) $\mathcal{M}, z \not\Vdash \varphi$. It follows, by (iii) and (iv), that $\mathcal{M}, z \Vdash \alpha$ for every $\alpha \in A$. From (i) and (iv) it follows that (vii) $\mathcal{M}, z \not\Vdash \beta$ for each $\beta \in B$. By (v), (vi) and (vii) we conclude that $\mathcal{M}, z \not\Vdash A, \varphi \Rightarrow B$, therefore $\mathcal{F} \not\Vdash A, \varphi \Rightarrow B$. \square

As an immediate application of the above soundness result, we may transfer the results in Prop. 4.2 to our sequent system, and conclude that the consequence relation \vdash associated to \mathcal{K}^g is in fact \wedge -undetermined and \smile -inconsistent. For the same reason, the results in Prop. 3.1 may be transferred to our semantics. With little effort, results analogous to those in Prop. 3.2 concerning the \triangleright -ordinary connectives originally characterized by way of our sequent system may also be restated in our present modal semantical framework, in which those connectives are conveniently interpreted. The connections between the two previous approaches will in fact be strengthened by the completeness result to be proven in the next section.

5 Completeness

Recall from Section 2 that a theory Σ_1 and a cotheory Σ_0 define a closed theory pair if $\Sigma_1 \cup \Sigma_0 = \mathcal{L}$. For closed theory pairs it will often be simpler thus to refer to the cotheory Σ_0 as $\overline{\Sigma_1}$, and we shall follow such policy from this point on, calling the single theory Π **saturated** if $\langle \Pi, \overline{\Pi} \rangle$ forms a closed (and obviously unconnected) theory pair. Following the definition of gcr from Section 3, we will concentrate below on the gcr \vdash defined by the deductive system for \mathcal{K}^n . Given a set of sentences Ψ , by $\lceil \Psi \rceil$ we will denote the theory $\{\psi : \Psi \vdash \psi\}$, and by $\lfloor \Psi \rfloor$ we will denote the cotheory $\{\psi : \psi \vdash \Psi\}$.

The interaction rules of our system \mathcal{K}^n allow us to prove some useful properties of saturated theories:

Lemma 5.1 *For any saturated theory Σ :*

- (i) $\overline{\overline{\lceil \Sigma \rceil}}$ is a theory
- (ii) $\wedge^{-1}[\Sigma]$ is a cotheory

Proof. [Item (i)] Assume that $\overline{\neg^{-1}[\Sigma]} \vdash \varphi$ and suppose by reductio that $\varphi \notin \overline{\neg^{-1}[\Sigma]}$, that is, $\neg\varphi \in \Sigma$. By the assumption we know that there is some derivable sequent $\varphi_1, \varphi_2, \dots, \varphi_n \Rightarrow \varphi$ in \mathcal{K}^n where $\{\varphi_1, \varphi_2, \dots, \varphi_n\} \subseteq \overline{\neg^{-1}[\Sigma]}$. From this sequent, using rule $(\neg\wedge)$ it follows that $\neg\varphi \Rightarrow \neg\varphi_1, \neg\varphi_2, \dots, \neg\varphi_n$ is derivable in \mathcal{K}^n . Given that Σ is a theory, \vee is \vdash -ordinary, and $\neg\varphi \in \Sigma$, then $\neg\varphi_1 \vee \neg\varphi_2 \vee \dots \vee \neg\varphi_n \in \Sigma$. But Σ is also saturated, thus $\neg\varphi_i \in \Sigma$ for some $1 \leq i \leq n$, by Prop. 2.4(i). It follows that $\varphi_i \notin \overline{\neg^{-1}[\Sigma]}$. Absurd.

[Item (ii)] This is analogous to the previous item, but we now use rule $(\neg\sim)$, the fact that \wedge is \vdash -ordinary and Prop. 2.4(ii). Details are safely left to the reader. \square

Let W_S be the set of all saturated theories of \mathcal{K}^n . Define over W_S the following binary relation R_S :

$$\Gamma R_S \Delta \text{ iff } \overline{\neg^{-1}[\Gamma]} \subseteq \Delta \subseteq \overline{\neg^{-1}[\Gamma]}$$

The **canonical frame** is defined as the structure $\mathcal{F}_S = \langle W_S, R_S \rangle$.

The two following auxiliary results will be helpful in establishing the proof of the Canonical Model Lemma, further on.

Lemma 5.2 *Let Σ be a saturated theory. Then $\neg\varphi \in \Sigma$ if and only if there is a saturated theory Π such that $\Sigma R_S \Pi$ and $\varphi \notin \Pi$.*

Proof. Assume first that there is some Π such that $\Sigma R_S \Pi$ and $\varphi \notin \Pi$. Since $\varphi \notin \Pi$ and $\overline{\neg^{-1}[\Sigma]} \subseteq \Pi$ it follows that $\varphi \notin \overline{\neg^{-1}[\Sigma]}$. From this we conclude that $\neg\varphi \in \Sigma$.

Conversely, assume that $\neg\varphi \in \Sigma$. Suppose $\alpha \in \overline{\neg^{-1}[\Sigma]}$ and $\beta \notin \overline{\neg^{-1}[\Sigma]}$, and thus $\neg\alpha \notin \Sigma$ and $\neg\beta \in \Sigma$. Recall, by Lemma 5.1, that $\overline{\neg^{-1}[\Sigma]}$ is a theory and $\neg^{-1}[\Sigma]$ is a cotheory. We need to show that there is some saturated theory Π such that $\alpha \in \Pi$ and $\beta \in \overline{\Pi}$, from which it will follow that $\Sigma R_S \Pi$, and such that $\varphi \notin \Pi$. We claim that the pair $P = \langle P_1, P_0 \rangle = \langle \overline{\neg^{-1}[\Sigma]}, [\neg^{-1}[\Sigma] \cup \{\varphi\}] \rangle$ is \vdash -unconnected.

Suppose instead, by reductio, that $\overline{\neg^{-1}[\Sigma]} \vdash \varphi, \neg^{-1}[\Sigma]$. It follows that there are finite sequences of sentences $\alpha_1, \dots, \alpha_m \notin \overline{\neg^{-1}[\Sigma]}$ and $\beta_1, \dots, \beta_n \in \neg^{-1}[\Sigma]$ such that $\alpha_1, \dots, \alpha_m \Rightarrow \varphi, \beta_1, \dots, \beta_n$ is derivable. Call such sequent *Seq*. Notice that $\alpha_i \notin \overline{\neg^{-1}[\Sigma]}$ means that (a) $\neg\alpha_i \in \Sigma$, for every $1 \leq i \leq m$, and $\beta_1, \dots, \beta_n \in \neg^{-1}[\Sigma]$ means that (b) $\neg\beta_j \in \Sigma$, for every $1 \leq j \leq n$. From *Seq*, using rule $(\neg\wedge)$ it follows that $\neg\beta_1, \dots, \neg\beta_n, \neg\varphi \Rightarrow \neg\alpha_1, \dots, \neg\alpha_m$ is also derivable. In view of Prop. 3.4, from the latter sequent and facts (a) and (b) we may conclude that $\neg\varphi \notin \Sigma$, which conflicts with our initial assumption.

Now that we know that the pair P is unconnected, we may use Prop. 2.1 to extend it to a saturated unconnected pair $P^* = \langle \Pi, \overline{\Pi} \rangle$. By construction, $\alpha \in P_1$ and $\beta, \varphi \in P_0$, so it follows that $\alpha \in \Pi$ and $\beta \in \overline{\Pi}$, and also that $\varphi \notin \Pi$. \square

Lemma 5.3 *Let Σ be a saturated theory. Then $\neg\varphi \in \Sigma$ if and only if $\varphi \notin \Pi$ for every saturated theory Π such that $\Sigma R_S \Pi$.*

Proof. Runs as in the previous result, with the obvious adaptations. \square

We define the canonical model \mathcal{M}_S as the structure $\langle \mathcal{F}_S, V_S \rangle$ where \mathcal{F}_S is the canonical frame and V_S is the valuation defined by:

$$V_S(p) = \{\Gamma \in W_S : p \in \Gamma\}$$

In the proof of the following result it will help to work with the non-canonical measure of sentence complexity given by the function $\ell : \mathcal{L} \rightarrow \mathbb{N}$, recursively defined as follows:

$$\begin{aligned} \ell(p) &= 0 && \text{if } p \in \mathcal{P} \\ \ell(\varphi * \psi) &= 1 + \max\{\ell(\varphi), \ell(\psi)\} && \text{if } * \in \{\wedge, \vee\} \\ \ell(\# \varphi) &= 1 + \ell(\varphi) && \text{if } \# \in \{\sim, \neg\} \\ \ell(\oplus \varphi) &= 2 + \ell(\varphi) && \text{if } \oplus \in \{\ominus, \odot\} \end{aligned}$$

Lemma 5.4 [Canonical Model]

In the canonical model, for any saturated theory Γ and any sentence φ :

$$\mathcal{M}_S, \Gamma \Vdash \varphi \text{ if and only if } \varphi \in \Gamma$$

Proof. The proof is an induction over $\ell(\varphi)$. The base case ($\ell(\varphi) = 0$) is trivial, using (id) and the definition of the canonical model. Assume now, by Induction Hypothesis, that $\mathcal{M}_S, \Gamma \Vdash \varphi$ iff $\varphi \in \Gamma$, for any saturated theory Γ and for every sentence φ such that $\ell(\varphi) < k$. We will detail below the ‘non-local’ cases involving one of the modal negations and one of the restoration connectives.

[$\ell(\sim\psi) = k$] By the definition of satisfaction, we have $\mathcal{M}_S, \Gamma \Vdash \sim\psi$ iff $\mathcal{M}_S, \Delta \not\Vdash \psi$ for some saturated theory Δ such that $\Gamma R_S \Delta$. Since $\ell(\sim\psi) = \ell(\psi) + 1$, then $\ell(\psi) < k$, thus the Induction Hypothesis applies and allows us to conclude that $\mathcal{M}_S, \Delta \not\Vdash \psi$ iff $\psi \notin \Delta$. From Lemma 5.2 we know that there is a saturated theory Δ such that $\Gamma R_S \Delta$ and $\psi \notin \Delta$ if and only if $\sim\psi \in \Gamma$. Summing up, we may conclude that $\mathcal{M}_S, \Gamma \Vdash \sim\psi$ iff $\sim\psi \in \Gamma$.

[$\ell(\ominus\psi) = k$] Suppose first that $\ominus\psi \in \Gamma$. In view of the derivability of $\ominus\psi, \sim\psi, \psi \Rightarrow$ (Prop. 3.1), from Prop. 3.4 we conclude that either $\psi \notin \Gamma$ or $\sim\psi \notin \Gamma$. Given that both $\ell(\psi) < \ell(\ominus\psi)$ and $\ell(\sim\psi) < \ell(\ominus\psi)$, the Induction Hypothesis guarantees that $\psi \notin \Gamma$ iff $\mathcal{M}_S, \Gamma \not\Vdash \psi$, and also that $\sim\psi \notin \Gamma$ iff $\mathcal{M}_S, \Gamma \not\Vdash \sim\psi$. By the definition of satisfaction we know that $\mathcal{M}_S, \Gamma \not\Vdash \psi$ or $\mathcal{M}_S, \Gamma \not\Vdash \sim\psi$ if and only if $\mathcal{M}_S, \Gamma \Vdash \ominus\psi$. It follows from $\ominus\psi \in \Gamma$, thus, that $\mathcal{M}_S, \Gamma \Vdash \ominus\psi$. For the converse, suppose now that $\mathcal{M}_S, \Gamma \Vdash \ominus\psi$. By the definition of satisfaction, the definition of ℓ and the Induction Hypothesis, we know that (a) $\sim\psi \notin \Gamma$ or (b) $\psi \notin \Gamma$. In case (b), in view of the derivability of $\Rightarrow \psi, \ominus\psi$ (Prop. 3.1), from Prop. 3.4 we conclude that $\ominus\psi \notin \bar{\Gamma}$, that is, $\ominus\psi \in \Gamma$; in case (a) the same conclusion follows in view of the derivability of $\Rightarrow \sim\psi, \ominus\psi$. □

As usual, from the above lemma we immediately conclude the following:

Proposition 5.5 [Completeness] *If $\Gamma \models \Delta$ then $\Gamma \vdash \Delta$.*

Proof. Suppose by contraposition that $\Gamma \not\models \Delta$. By Prop. 2.1 there is a closed unconnected pair $\langle \Gamma^*, \Delta^* \rangle$ that extends $\langle [\Gamma], [\Delta] \rangle$. It follows that Γ^* is a saturated theory and that $\Delta^* = \bar{\Gamma}^*$. By the Canonical Model Lemma, we have $\mathcal{M}_S, \Gamma^* \Vdash \varphi$ iff $\varphi \in \Gamma^*$. Thus, we conclude that $\Gamma^* \not\models \Delta^*$, and by monotonicity it follows that $\Gamma \not\models \Delta$. □

6 Extensions of \mathcal{K}^n

In the literature it is common to find the minimal system of normal modal logic extended by adding new axioms and to see the resulting system shown to be sound and complete with respect to a class of frames in which the accessibility relation enjoys certain appropriate properties. As an illustration of how this strategy may be applied to our systems with negative modalities, we introduce in this section two systems that extend \mathcal{K}^n . The system \mathcal{T}^n extends \mathcal{K}^n by adding the dual axiomatic rules $\frac{}{\Rightarrow \varphi, \cup \varphi}$ ^(rf1) and $\frac{}{\wedge \varphi, \varphi \Rightarrow}$ ^(rf2). Adding the axiomatic rules $\frac{}{\cup \cup \varphi \Rightarrow \varphi}$ ^(sm1) and $\frac{}{\varphi \Rightarrow \wedge \wedge \varphi}$ ^(sm2) to \mathcal{K}^n we define the system \mathcal{B}^n . The gcrs $\vdash_n^{\mathcal{T}}$ and $\vdash_n^{\mathcal{B}}$ correspond, respectively, to the deductive systems associated to \mathcal{T}^n and \mathcal{B}^n .

Recall that a binary relation R is called **reflexive** if xRx holds for every x , and is called **symmetric** if xRy implies yRx . In what follows we will show that \mathcal{T}^n is sound and complete with respect to the class of reflexive frames (i.e., the class of frames with a reflexive accessibility relation), and similarly for \mathcal{B}^n and the class of symmetric frames.

Proposition 6.1 [Correspondence] *Let $\mathcal{F} = \langle W, R \rangle$ be a frame. Then:*

- (1.1) R is reflexive only if $\mathcal{F} \Vdash \Rightarrow \varphi, \cup \varphi$ and $\mathcal{F} \Vdash \wedge \varphi, \varphi \Rightarrow$
- (1.2) R is reflexive if $\mathcal{F} \Vdash \Rightarrow \varphi, \cup \varphi$ or $\mathcal{F} \Vdash \wedge \varphi, \varphi \Rightarrow$
- (2.1) R is symmetric only if $\mathcal{F} \Vdash \cup \cup \varphi \Rightarrow \varphi$ and $\mathcal{F} \Vdash \varphi \Rightarrow \wedge \wedge \varphi$
- (2.2) R is symmetric if $\mathcal{F} \Vdash \cup \cup \varphi \Rightarrow \varphi$ or $\mathcal{F} \Vdash \varphi \Rightarrow \wedge \wedge \varphi$

Proof. (1.1) Assume that (i) R is reflexive and suppose that (ii) $\mathcal{F} \not\Vdash \Rightarrow \varphi, \cup \varphi$, for some sentence φ . It follows from (ii) that (iii) $\mathcal{M}_0, w \not\Vdash \varphi$ and (iv) $\mathcal{M}_0, w \not\Vdash \cup \varphi$ for some model \mathcal{M}_0 of \mathcal{F} and some w in \mathcal{M}_0 . From (i) and (iv), the definition of valuation gives us (v) $\mathcal{M}_0, w \Vdash \varphi$. This contradicts (iii). Suppose now that $\mathcal{F} \not\Vdash \wedge \varphi, \varphi \Rightarrow$. Then, there are a model \mathcal{M}_1 and a world u such that $\mathcal{M}_1, u \Vdash \varphi$ and $\mathcal{M}_1, u \not\Vdash \wedge \varphi$. From the latter, invoking the reflexivity of R , we conclude that $\mathcal{M}_1, u \not\Vdash \varphi$. Contradiction.

(1.2) Suppose that \mathcal{F} is not reflexive. Then, there is a world m such that $\langle m, m \rangle \notin R$. Let C be the set $\{z : \langle m, z \rangle \in R\}$. Let p be a propositional variable and let $\mathcal{M}_2 = \langle \mathcal{F}, V \rangle$ be a model such that $V(p) = C$. Obviously $m \notin V(p)$, thus \mathcal{M}_2 falsifies p at m . Moreover, by construction of C , we have $x \in V(p)$ for every x such that $\langle m, x \rangle \in R$. By the definition of valuation, \mathcal{M}_2 falsifies $\cup p$ at m . Thus, \mathcal{M}_2 falsifies $\Rightarrow p, \cup p$ at m . If we enrich \mathcal{M}_2 by a propositional variable q such that $V(q) = \overline{C}$, we see that \mathcal{M}_2 falsifies $\wedge q, q \Rightarrow$ at m .

(2.1) Assume that R is symmetric. Suppose that $\mathcal{F} \not\Vdash \cup \cup \varphi \Rightarrow \varphi$. There is thus a model \mathcal{M}_0 and a state w such that (i) $\mathcal{M}_0, w \Vdash \cup \cup \varphi$ and (ii) $\mathcal{M}_0, w \not\Vdash \varphi$. From (i), there must be some z such that (iii) $\langle w, z \rangle \in R$ and (iv) $\mathcal{M}_0, z \not\Vdash \cup \varphi$. The symmetry of R allows us to conclude (v) $\langle z, w \rangle \in R$ from (iii), and from (iv) and (v) it follows that $\mathcal{M}_0, w \not\Vdash \varphi$. This contradicts (ii). If we suppose that $\mathcal{F} \not\Vdash \varphi \Rightarrow \wedge \wedge \varphi$ we reach a contradiction through a similar line of reasoning.

(2.2) Suppose that R is not symmetric. Then there are m, n such that $\langle m, n \rangle \in R$ yet $\langle n, m \rangle \notin R$. Let C be the set $\{z : \langle n, z \rangle \in R\}$. Let p, q be propositional variables and let $\mathcal{M}_2 = \langle \mathcal{F}, V \rangle$ be a model where $V(p) = C$ and $V(q) = \overline{C}$. Since $m \notin V(p)$, then \mathcal{M}_2 falsifies p at m . Given that, for arbitrary z , we have that $\langle n, z \rangle \in R$ implies $z \in V(p)$, we conclude by the definition of valuation that \mathcal{M}_2 falsifies $\smile p$ at n . Once $\langle m, n \rangle \in R$, then \mathcal{M}_2 satisfies $\smile p$ at m . Thus, \mathcal{M}_2 falsifies $\smile p \Rightarrow p$ at m . Similarly, \mathcal{M}_2 also falsifies $q \Rightarrow \frown q$ at m . \square

Soundness of \mathcal{T}^n and \mathcal{B}^n are corollaries of the ‘only-if’ part of Prop. 6.1. To illustrate some differences between those systems we invite the reader to use rules (sm1) and (sm2) of \mathcal{B}^n , on the one hand, and the soundness of \mathcal{T}^n , on the other hand, to check that:

Proposition 6.2 *Sequents $\ominus\varphi \Rightarrow \ominus\smile\varphi$ and $\ominus\frown\varphi \Rightarrow \ominus\varphi$ are derivable in \mathcal{B}^n but not in \mathcal{T}^n .*

It might be interesting to contrast the latter result concerning the propagation of consistency through the paraconsistent negation and the dual propagation of determinedness through the paracomplete negation to the earlier general propagation results in Prop. 4.3.

Completeness will be attained next with the help of the following auxiliary results.

Lemma 6.3 *Assume the theories Γ_a and Γ_b to be closed with respect to $\vdash_n^{\mathcal{T}}$ and $\vdash_n^{\mathcal{B}}$. Then:*

- (i) *In \mathcal{T}^n we have that $\varphi \vee \smile\varphi \in \Gamma_a$.*
- (ii) *In \mathcal{T}^n we have that $\varphi \wedge \frown\varphi \notin \Gamma_a$.*
- (iii) *In \mathcal{B}^n we have that $\smile\smile\varphi \in \Gamma_b$ implies $\varphi \in \Gamma_b$.*
- (iv) *In \mathcal{B}^n we have that $\varphi \in \Gamma_b$ implies $\frown\varphi \in \Gamma_b$.*

Proof. The first two facts follow from closure of Γ_a and the obvious derivability of $\Rightarrow \varphi \vee \smile\varphi$ and $\varphi \wedge \frown\varphi \Rightarrow$ in \mathcal{T}^n , in view of axiomatic rules (rf1) and (rf2). The remaining facts are easy consequences of closure of Γ_b and the axiomatic rules (sm1) and (sm2). \square

We should guarantee that the canonical construction yields the appropriate properties:

Proposition 6.4 [Canonical Systems] *The systems \mathcal{T}^n and \mathcal{B}^n are canonical.*

Proof. For \mathcal{T}^n we have to show, for the canonical frame \mathcal{F}_S , that $\langle \Gamma, \Gamma \rangle \in R_S$ for all $\Gamma \in W_S$, that is, $\overline{\smile^{-1}[\Gamma]} \subseteq \Gamma \subseteq \overline{\frown^{-1}[\Gamma]}$. Suppose that $\varphi \in \overline{\smile^{-1}[\Gamma]}$. Then, $\smile\varphi \notin \Gamma$. Since Γ is a closed theory and, by Lemma 6.3(i) and Prop. 2.4(i), $\varphi \vee \smile\varphi \in \Gamma$, it follows that $\varphi \in \Gamma$. To show that $\Gamma \subseteq \overline{\frown^{-1}[\Gamma]}$ the reasoning is similar, in view of Lemma 6.3(ii) and Prop. 2.4(ii).

For \mathcal{B}^n assume that Γ, Δ are closed theories such that $\langle \Gamma, \Delta \rangle \in R_S$, that is, $\overline{\smile^{-1}[\Gamma]} \subseteq \Delta \subseteq \overline{\frown^{-1}[\Gamma]}$. If $\varphi \in \overline{\smile^{-1}[\Delta]}$, then $\smile\varphi \notin \Delta$. From this we have that

$\sim\varphi \notin \overline{\sim^{-1}[\Gamma]}$, therefore $\sim\varphi \in \Gamma$. By Lemma 6.3(iii) we conclude that $\varphi \in \Gamma$. Assume now that $\varphi \in \Gamma$. By Lemma 6.3(iv), $\sim\varphi \in \Gamma$. It follows that $\sim\varphi \notin \overline{\sim^{-1}[\Gamma]}$. Since $\Delta \subseteq \overline{\sim^{-1}[\Gamma]}$, then $\sim\varphi \notin \Delta$, that is, $\varphi \in \overline{\sim^{-1}[\Delta]}$. Thus, $\overline{\sim^{-1}[\Delta]} \subseteq \Gamma \subseteq \overline{\sim^{-1}[\Delta]}$, that is $\langle \Delta, \Gamma \rangle \in R_S$. \square

Let $\models^{\mathcal{T}}$ be the entailment relation defined with respect to the class of all reflexive frames, and $\models^{\mathcal{B}}$ be defined for the class of all symmetric frames. An immediate consequence of Prop. 6.4, proven exactly as in Prop. 5.5, is:

Corollary 6.5 [Completeness for $\mathcal{X} \in \{\mathcal{T}, \mathcal{B}\}$] *For every $\Gamma \cup \Delta \subseteq \mathcal{L}$:*

$$\Gamma \models^{\mathcal{X}} \Delta \text{ implies } \Gamma \vdash_n^{\mathcal{X}} \Delta$$

Having established, for both \mathcal{T}^n and \mathcal{B}^n , that all inferences verified semantically are also derivable in the next section we will study the role of these stronger modal systems in helping to more naturally restore inferences of some standard logical systems by means of Derivability Adjustment Theorems.

7 Recovering the lost perfection

Let \sim be a unary negation symbol. Some standard rules for negation that could be added to the system \mathcal{DL} are:

$$\frac{A, \varphi \Rightarrow B}{A \Rightarrow \sim\varphi, B} (\wedge\sim) \qquad \frac{A \Rightarrow \varphi, B}{A, \sim\varphi \Rightarrow B} (\sim/)$$

It is easy to see that such rules would characterize \sim as what we have, in Section 2, called an ordinary negation, respecting both statements (Cns) and (Dtm). Legitimate non-classical negations, nonetheless, while obviously failing either consistency or determinedness, may still respect other typical rules of negation. We list below, in particular, some standard sequent rules involving negation and the standard connectives modeled by a bounded distributive lattice:

$$\frac{A, \sim\varphi, \sim\psi \Rightarrow B}{A, \sim(\varphi \vee \psi) \Rightarrow B} \text{ (dm1.1)} \qquad \frac{A \Rightarrow \sim\varphi, B \quad A \Rightarrow \sim\psi, B}{A \Rightarrow \sim(\varphi \vee \psi), B} \text{ (dm1.2)}$$

$$\frac{A \Rightarrow \sim\varphi, \sim\psi, B}{A \Rightarrow \sim(\varphi \wedge \psi), B} \text{ (dm2.1)} \qquad \frac{A, \sim\varphi \Rightarrow B \quad A, \sim\psi \Rightarrow B}{A, \sim(\varphi \wedge \psi) \Rightarrow B} \text{ (dm2.2)}$$

$$\frac{A \Rightarrow \varphi, B}{A \Rightarrow \sim\sim\varphi, B} \text{ (dm3.1)} \qquad \frac{A, \varphi \Rightarrow B}{A, \sim\sim\varphi \Rightarrow B} \text{ (dm3.2)}$$

$$\frac{}{A, \sim\top \Rightarrow B} \text{ (dm4.1)} \qquad \frac{}{A \Rightarrow \sim\perp, B} \text{ (dm4.2)}$$

We will discuss in this section which of the above rules are derivable and which of them may be somehow *recovered* from the viewpoint of each of the sequent systems studied in the previous sections.

Recall that our language \mathcal{L} contains two indigenous symbols for negation, namely, \sim and \smile . For those negations it is not hard to check that:

Proposition 7.1 *In \mathcal{K}^n :*

- (i) *Rules (dm1.1) and (dm2.1) are derivable for both \wedge and \smile .*
- (ii) *Rules (dm1.2) and (dm4.1) are derivable for \wedge .*
- (iii) *Rules (dm2.2) and (dm4.2) are derivable for \smile .*
- (iv) *Rule (dm1.2) fails for \smile , and rule (dm2.2) fails for \wedge .*
- (v) *Rules (dm3.1) fails for \wedge and rule (dm3.2) fails for \smile .*

In \mathcal{T}^n :

- (vi) *Rule ($/\sim$) is derivable for \smile and rule ($\sim/$) is derivable for \wedge .*

In \mathcal{B}^n :

- (vii) *Rule (dm3.1) is derivable for \wedge and rule (dm3.2) is derivable for \smile .*

Moreover, in either \mathcal{T}^n or \mathcal{B}^n (thus, also in \mathcal{K}^n):

- (viii) *Rule (dm3.2) fails for \wedge and rule (dm3.1) fails for \smile .*
- (ix) *Rule ($\sim/$) fails for \smile and rule ($/\sim$) fails for \wedge .*

Proof. Items (i), (ii) and (iii) follow directly from Prop. 3.2(ii) and the second half of Prop. 2.5. Items (vi) and (vii) follow from the characterizing axioms of \mathcal{T}^n and \mathcal{B}^n .

To check the remaining items the completeness results in Prop. 5.5 and Corol. 6.5 come in handy. A simple strategy to show that some instance of a given schematic rule must fail involves falsifying some sequent that is derivable from that rule. On what concerns item (viii), for example, notice that $\sim\smile p \Rightarrow p$ would obviously be derivable from (dm3.2), for any atomic sentence p . Yet, to falsify the sequent $\wedge\smile p \Rightarrow p$ it suffices to consider a frame \mathcal{F}_1 such that $W_1 = \{u, v\}$ and R_1 is the total (thus reflexive and symmetric) relation $W_1 \times W_1$, and consider a model \mathcal{M} such that $V(p) = \{v\}$: note indeed that $\mathcal{M}, v \Vdash p$, and uR_1v and vR_1v imply $\mathcal{M}, u \not\Vdash \smile p$ and $\mathcal{M}, v \not\Vdash \wedge p$, and thus $\mathcal{M}, u \Vdash \wedge\smile p$ given that uR_1x implies $x \in \{u, v\}$, while obviously $\mathcal{M}, u \not\Vdash p$. Analogously, $p \Rightarrow \sim\smile p$ would be derivable from (dm3.1), yet in the model just considered we have $\mathcal{M}, v \Vdash p$ and $\mathcal{M}, v \not\Vdash \smile\smile p$, thus falsifying the sequent $p \Rightarrow \smile\smile p$.

For item (iv), consider a frame \mathcal{F}_2 where $W_2 = \{u, v, w\}$ and $R_2 = \{\langle u, v \rangle, \langle u, w \rangle\}$, and a model \mathcal{M}' in which $V'(p) = \{v\}$ and $V'(q) = \{w\}$, for atomic sentences p and q ; this is indeed a model that witnesses the failure of $\wedge(p \wedge q) \Vdash (\wedge p \vee \wedge q)$ and the failure of $\smile p \wedge \smile q \Vdash \smile(p \vee q)$. For item (v) one might consider a frame \mathcal{F}_2 such that $W_2 = \{u, v\}$ and $R_2 = \{\langle u, v \rangle\}$, and consider a model \mathcal{M}'' such that $V''(p) = \{u\}$ and $V''(q) = \emptyset$.

At last, on what concerns item (ix), note that the proof of Prop. 4.2 still applies unchanged. □

The result in Prop. 7.1(ix) should come as no surprise: As shown in [14], with the exception of degenerate cases, normal modal logics based on \wedge are paracomplete and modal logics based on \smile are paraconsistent. It is interesting to call attention,

though, to a particular byproduct of the proof of Prop. 7.1(viii): the counter-models presented to $\neg p \models p$ and to $p \models \sim p$ are based on equivalence relations, and so one should not expect these two inferences to be valid for any of the usual classes of frames characterizing modal logics weaker than $S5$ — in other words, one might say that the intuitionistic-like negation has indeed a good reason to fail double negation elimination, and analogously the paraconsistent negation may reasonably be expected to fail double negation introduction.

Notice now that the rules that are shown to fail in the previous proposition may often be restored in one way or another, with the help of the connectives expressing consistency and determinedness in our rich modal language. If, for instance, the following restored versions of our missing sequent rules turn out to be derivable, this will help us in finding conditions under which one can recover some of the lost inferences:

$$\frac{A, \varphi \Rightarrow B}{A \Rightarrow \neg\varphi, \odot\varphi, B} (/ \neg)^\circ \qquad \frac{A \Rightarrow \varphi, B}{A, \odot\varphi, \sim\varphi \Rightarrow B} (\sim /)^\circ$$

$$\frac{A \Rightarrow \sim\varphi, B \quad A \Rightarrow \sim\psi, B}{A, \odot\varphi, \odot\psi \Rightarrow \sim(\varphi \vee \psi), \odot(\varphi \vee \psi), B} (\text{dm1.2})^\circ$$

$$\frac{A, \neg\varphi \Rightarrow B \quad A, \neg\psi \Rightarrow B}{A, \odot(\varphi \wedge \psi), \neg(\varphi \wedge \psi) \Rightarrow \odot\varphi, \odot\psi, B} (\text{dm2.2})^\circ$$

$$\frac{A \Rightarrow \varphi, B}{A, \odot\varphi \Rightarrow \neg\neg\varphi, \odot\neg\varphi, B} (\text{dm3.1})^\circ \qquad \frac{A, \varphi \Rightarrow B}{A, \odot\sim\varphi, \sim\varphi \Rightarrow \odot\varphi, B} (\text{dm3.2})^\circ$$

Rules $(/ \neg)^\circ$ and $(\sim /)^\circ$ are obviously derivable from the basic rules $(/ \odot)$ and $(\odot /)$. The remaining rules above may be checked with the help of the following sequents:

Proposition 7.2 *In \mathcal{K}^\square the following are derivable:*

- (SD12) $\odot\varphi, \odot\psi, \sim\varphi, \sim\psi \Rightarrow \sim(\varphi \vee \psi), \odot(\varphi \vee \psi)$
- (SD22) $\odot(\varphi \wedge \psi), \neg(\varphi \wedge \psi) \Rightarrow \neg\varphi, \neg\psi, \odot\varphi, \odot\psi$
- (SD31) $\odot\varphi, \varphi \Rightarrow \neg\neg\varphi, \odot\neg\varphi$
- (SD32) $\odot\sim\varphi, \sim\varphi \Rightarrow \varphi, \odot\varphi$

Proof. For (SD12), suppose by *reductio* that there is a model \mathcal{M} with a world w in which $\odot\varphi, \odot\psi, \sim\varphi, \sim\psi$ are all satisfied and $\sim(\varphi \vee \psi), \odot(\varphi \vee \psi)$ are both falsified. It follows from the joint satisfaction of $\odot\varphi$ and $\sim\varphi$ at w that φ must be falsified at w . The same reasoning applies to ψ , and thus we may conclude that $\varphi \vee \psi$ is falsified at w . From the latter, given that $\odot(\varphi \vee \psi)$ is also falsified at w , we conclude that $\varphi \vee \psi$ is falsified indeed at every world accessible to w . Note now that the satisfaction of $\sim\varphi$ at w demands in particular the existence of a world w' accessible to w . Given that $\sim(\varphi \vee \psi)$ is falsified at w , we must also conclude that $\varphi \vee \psi$ is satisfied at w' . We reach thus a contradiction.

For (SD31), suppose by *reductio* that in the world w of a model \mathcal{M} the sentences $\odot\varphi$ and φ are both satisfied (forcing thereby φ to be satisfied at any world accessible

to w), while the sentences $\sim\varphi$ and $\ominus\varphi$ are both falsified (forcing φ to be falsified at any world accessible to w). But to falsify $\sim\varphi$ at w there must first of all exist some world w' accessible from w . Contradiction.

Items (SD22) and (SD32) are proved similarly. In all cases, completeness may be used in the end to transfer the semantically verified results to facts about the proof formalism. \square

It is instructive to contrast the latter result to what we had learned from items (iv) and (v) from Prop. 7.1.

Instead of axiomatizing Classical Logic (\mathcal{CL}) simply by adding rules ($/\sim$) and ($\sim/$) to \mathcal{DL} , we will here axiomatize it in the language \mathcal{L} by adding the restored rules ($/\wedge$) $^\circ$ and ($\vee/$) $^\circ$ to \mathcal{DL} , plus the two following rules: $\overline{\Rightarrow \ominus\varphi}^{(\text{cns})}$ and $\overline{\ominus\varphi \Rightarrow}^{(\text{dtm})}$. The associated gcr will be referred to as \vdash_{cl} . The intuition behind such system is precisely that \mathcal{CL} is to be obtained by explicitly imposing a universal consistency assumption as well as a universal determinedness assumption.

At this point we can finally state:

Proposition 7.3 [Derivability Adjustment Theorem] *Let $\Pi^\#$ be the result of uniformly substituting each occurrence of the symbol \sim in each sentence of Π by an occurrence of a unary symbol $\# \in \{\wedge, \vee\}$. Then, inferences from \mathcal{CL} may be recovered from \mathcal{T}^n in the following way:*

$$\Gamma^\# \vdash_{\text{cl}} \Delta^\# \text{ iff there are finite sets } \Sigma_c, \Sigma_d \subseteq \mathcal{L} \text{ such that } \ominus[\Sigma_c], \Gamma \vdash_n^T \Delta, \ominus[\Sigma_d]$$

Furthermore, Σ_c may be constrained above to a finite collection of sub-sentences of Γ , and Σ_d may be constrained to a finite collection of sub-sentences of Δ .

Proof. For the right-to-left direction, first one should notice that all the rules of \mathcal{T}^n are classically valid. Any derivation constructed in \mathcal{T}^n may then in principle be reproduced as a derivation associated to the gcr \vdash_{cl} , any occurrence of a sentence of the form $\ominus\varphi$ on the left-hand side of a given sequent may be eliminated by cut using the axiomatic rule (cns), and any occurrence of a sentence of the form $\ominus\varphi$ on the right-hand side of a given sequent may be eliminated by cut using the axiomatic rule (dtm).

For the left-to-right direction, one may proceed by induction on the structure of the derivations. The base case (0-step derivations) is trivial, and it suffices to take $\Sigma_c = \Sigma_d = \emptyset$. The idea for the remainder of the construction is to collect consistency assumptions and determinedness assumptions on the fly: for each further step of a \mathcal{CL} -derivation intended to witness the fact that $A^\# \vdash_{\text{cl}} B^\#$, for appropriate finite sets $A \subseteq \Gamma$ and $B \subseteq \Delta$, check whether a rule has been used that does not belong to the common core of the sequent systems for \mathcal{CL} and for \mathcal{T}^n , in that case, construct the corresponding step in the \mathcal{T}^n -derivation by using the qualified versions of the same rules (taking into account Prop. 7.1 and the rules derived with the help of Prop. 7.2). For a bit more of detail, suppose the construction of the classical derivation has proceeded by applying rules (dm1.1) or rule (dm2.1) at a given derivation step. Then, according to Prop. 7.1(i), exactly the same derivation

step may be taken in \mathcal{K}^n (thus also in \mathcal{T}^n). Similarly, according to items (ii) and (iii) of Prop. 7.1, the same steps may be taken in \mathcal{K}^n (or in \mathcal{T}^n in case (dm1.2) is used with respect to \wedge or in case (dm2.2) is used with respect to \vee). Now, if (dm1.2) is expected to be used with respect to \vee , then (dm1.2) $^\circ$ should be used instead, and if (dm2.2) is expected to be used with respect to \wedge , then (dm2.2) $^\circ$ should be used instead — notice that in both cases there will be consistency and determinedness assumptions added to the contexts at the root of the derivation, that is, there will be sentences added to Σ_c and to Σ_d . Finally, notice that any derivation step using rule ($/\sim$) in a classical derivation may still be taken in \mathcal{T}^n with respect to \vee , in view of Prop. 7.1(vi); with respect to \wedge one should use the derivable rule ($/\wedge$) $^\circ$ instead — and in this case an appropriate sentence will be added to Σ_d . Dually, any classical derivation step using rule ($\sim/$) may be reproduced in \mathcal{T}^n with respect to \wedge , or be replaced, with respect to \vee , by a step making use of rule ($\sim/$) $^\circ$, demanding the addition of an appropriate sentence to Σ_c . \square

The above result could alternatively be checked by using the appropriate consistency and determinedness assumptions to semantically constrain the \mathcal{T}^n -models in order to emulate the corresponding \mathcal{CL} -models.

In the case of our basic system \mathcal{K}^n , a counterpart for the above result would not try to recover all classical inferences. The natural candidate, in that case, would be a weaker system, which we briefly mention. Let \mathcal{DM} be the system obtained by adding rules (dm1.1), (dm1.2), (dm2.1) and (dm2.2) to system \mathcal{DL} , let \mathcal{DM}^i be \mathcal{DM} plus (dm3.1), and let \mathcal{DM}^e be \mathcal{DM} plus (dm3.2). Adding both (dm3.1) and (dm3.2) to \mathcal{DM} characterizes the so-called De Morgan Logic (cf. [12]). Now, other Derivability Adjustment Theorems are to be expected if we fix our attention on the relation between \mathcal{DM}^i and the paracomplete fragment of \mathcal{K}^n , or on the relation between \mathcal{DM}^e and the paraconsistent fragment of \mathcal{K}^n . Furthermore, if \mathcal{B}^n is used instead of \mathcal{T}^n then less consistency and determinedness assumptions will need to be collected, as iterated negation is more well-behaved by the very design of \mathcal{B}^n .

A fully detailed exploration of the latter results on derivability adjustment is left as matter for a future study.

8 Closing remarks

We have started our study from the logic underlying bounded distributive lattices and investigated in this paper the logic \mathcal{K}^n that upgrades the former by adding a modal paraconsistent negation and a modal paracomplete negation, and also adds modal operators internalizing appropriate notions of consistency and determinedness into the object-language level. We have characterized the properties of our connectives from an abstract viewpoint, proposed a sequent-style proof formalism for the minimal normal system enjoying such properties in our chosen language, and proven its completeness with respect to the expected standard kripke-like semantics. We have also considered two extensions of our basic system, adding axioms connected to versions of excluded middle, pseudo-scotus and forms of double negation manipulation, and we have discussed how these systems allow one to re-

cover the inferences of some logics lying in between De Morgan Logic and Classical Logic. Studying other extensions should be instigating inasmuch as they are attained by adding axioms that express intuitively important properties of *negation*, such as the ‘controllable forms’ of consistency and of determinedness expressed by $\smile\varphi, \smile(\smile\varphi) \Rightarrow$ and $\Rightarrow \smile(\smile\varphi), \smile\varphi$, which are valid in euclidean frames. Axioms that involve the interaction between the two non-classical negations are also attractive, such as $\smile\varphi \Rightarrow \smile\varphi$, valid in functional frames, or as $\smile\varphi, \smile(\smile\varphi) \Rightarrow$ and $\Rightarrow \smile(\smile\varphi), \smile\varphi$, valid in transitive frames, or as $\smile\smile\varphi \Rightarrow \smile\smile\varphi$, valid in confluent (a.k.a. Church-Rosser) frames.

Nonetheless, in producing deductive extensions of the basic system without extending its language, the standard kripke semantics which we have employed has a somewhat serious shortcoming. Indeed, even though we have thought of our para-complete negation as independent of our paraconsistent negation, both \mathcal{T}^n and \mathcal{B}^n were built by adding not just one but two ‘dual’ axioms. It would have seemed more appropriate, however, to devise complete systems in which each one of those axioms could be introduced in separate. An obvious alternative to deal with such difficulty related to frame incompleteness is simply to change the semantical framework. Such a strategy is common in the literature on systems of intuitionistic modal logics, in which a second relation (a quasi ordering) is added to the frame, coupled with the consideration of truth-increasing valuations. This seems very well-motivated, and would allow one to prove in particular that truth is hereditarily preserved towards the future, according to the order introduced by the second relation, and falsity is hereditarily preserved towards the past, according to the same order (for the positive case, cf. [7]; for an application to the case of our modal negations, cf. [11]). The additional advantage of this alternative framework, besides bringing the heredity conditions to the fore, is that it allows one to add each axiom in separate, and continue thinking thus about the two non-classical negations as really independent of each other. However, that strategy cannot be extended without modification to our richer language. The reason is simple: the restoration connectives were in a sense designed to *fail* the heredity conditions, as they allow one to recover standard classic-like models when they are applied to sentences of a given theory. It rests as a challenge, thus, to identify the right semantic framework in which the study of extensions of our system \mathcal{K}^n should be done. For one thing, from [14] we already know that if we add a classical implication connective \rightarrow to \mathcal{K}^n , *any* normal modal logic may be rewritten in the minimal language containing just such \rightarrow and the paraconsistent negation \smile ; in this case indeed the usual classical connectives, the usual box-plus and diamond-plus connectives, the dual paracomplete negation \smile , and the restoration connectives dealing with \smile -consistency and with \smile -determinedness may all be explicitly *defined*. There is also a rich literature (important references include [9,18]) concerning the systems obtained by the addition of an intuitionistic implication instead of a classical implication — for those systems it is customary to consider interpretation structures containing two accessibility relations, one to deal with implication and another one to deal with the non-classical negations. In the present study we have concentrated however on the implicationless fragment of

these logics, to which the restoration connectives were explicitly added in order to internalize the corresponding useful meta-theoretical concepts.

Another line of research that we see as potentially fruitful is the investigation of matters related to variegated versions of our Derivability Adjustment Theorems, especially from a semantical perspective. We note that there is a modular way of connecting ‘quasi canonical sequent rules’ such as the main ones we have proposed in this paper to restrictions concerning the so-called ‘non-deterministic semantics’ (cf. [1,2]). From that viewpoint, one may see how De Morgan Logic gets associated to four truth-values, where conjunction and disjunction are interpreted as in Dunn-Belnap matrices, and its negation (both paraconsistent and paracomplete) is defined according to the so-called truth-order. Furthermore, by adding rule ($/\sim$) a further determinization is produced, and only three truth-values are left, as negation ceases to be paracomplete; an analogous phenomenon happens if ($\sim/$) is added, and negation ceases to be paraconsistent; if both rules are added, Classical Logic is obtained. Now, if one considers \mathcal{DM} from the start, a four-valued semantics is still available, but negation is non-deterministic: there are two possibilities of output for each of the four inputs. Such negation may be partially determinized by adding rules (dm3.1) or (dm3.2); adding both rules would result in the full determinization that corresponds to De Morgan Logic. Our non-classical negations go the other way round, by deleting some De Morgan rules, (dm1.2) or (dm2.2). The result of performing this deletion over \mathcal{DM} is that disjunction will also start to behave non-deterministically. Such modular approach may be easily extended to include the consistency and the determinedness operators, which will also be (non-deterministically) interpretable over the already mentioned four truth-values (an automated mechanism for uncovering the semantic aspects of such paraconsistent fragments of \mathcal{DM} was launched in [8]). Our Derivability Adjustment Theorems could then be thought of as ways of taming non-classicality and controlling non-determinism from a logical viewpoint.

Some of the sequent rules that we have studied are more important than others. Such is the case of the interaction rules ($\wedge\vee$) and ($\vee\wedge$), which could be thought of as a sort of multiple-conclusion sequent calculus contextual generalization of the so-called ‘Becker’s Rule’, from the traditional modal literature, adapted to the case of negative normal modalities. To the best of our knowledge, they seem not to have been proposed before. It is worth noting that by the addition of the usual sequent rules for classical implication, our system \mathcal{K}^n is upgraded into a modal version of the logic of formal inconsistency \mathcal{BK} (see [3]), obtained precisely by the addition of the already mentioned interaction rules (so, to be sure, \mathcal{K}^n plus classical implication coincides with \mathcal{BK} plus interaction rules). Such interaction rules are indeed absolutely instrumental in warranting the modal character of our systems (and, in particular, in guaranteeing that we are dealing with systems respecting the standard *replacement property*), and it seems worth studying the classes of paraconsistent and paracomplete logics that lend themselves in a natural way to reasonable extensions obtained by the addition of such rules. In a future study we will also show how sequent systems such as those studied in the present paper may be seen as particular examples of the ‘Basic Sequent Systems’ studied in [4].

In that paper, the authors have shown how to provide kripke semantics to such kinds of systems in a way so as to allow one to semantically obtain confirmations of important proof-theoretic properties such as cut-admissibility and analyticity. In showing that the mentioned approach indeed applies to our systems, we will guarantee that one can count on such proof-theoretic properties, provide alternative completeness proofs and allow for a smoother extension of our systems to normal systems characterized by other important classes of frames.

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