Towards a Herbrand's Theorem for Hybrid Logic

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Abstract

The original version of Herbrand's theorem [7] for first-order logic provided the theoretical basis for theorem proving, by allowing a constructive method for associating with each first-order formula χ a sequence of quantifier-free formulas $\chi_1, \chi_2, \chi_3, \cdots$ so that χ has a first-order proof if and only if some χ_i is a tautology. Some other versions of Herbrand's theorem have been developed for classical logic, such as the one in [4], which states that a set of quantifier-free sentences is satisfiable if and only if it is propositionally satisfiable. The literature concerning versions of Herbrand's theorem proved in the context of non-classical logics is meager. We aim to investigate in this note a version of Herbrand's theorem for hybrid logic, which is an extension of modal logic that is expressive enough so as to allow reference to specific states, and to the accessibility relations and equality between states, thus being completely suitable to deal with relational structures [2]. This reports on work-in-progress. Our first result follows the arguments in [4] and states that a set of satisfaction statements is satisfiable if and only if it is propositionally satisfiable.

1 Introduction

Hybrid logics [2] are a brand of modal logics that provide appropriate syntax for referring to the possible worlds semantics through the use of nominals. In particular, it adds to the modal description of transition structures the ability to refer to specific states. If modal logics have been successfully used for specifying reactive systems, the hybrid component adds enough expressivity so as to refer to individual states and reason about the system's local behavior at each of them.

In hybrid logic we may express equality between states named by i and j ($@_i j$) or accessibility of the latter from the former through a modality ($@_i \Diamond j$) as well as we may make statement about a specific state ($@_i \varphi$). Moreover, hybrid logic is strictly more expressive than its modal fragment. For example, irreflexivity ($i \to \neg\Diamond i$), asymmetry ($i \to \neg\Diamond\Diamond i$) or antisymmetry ($i \to \Box(\Diamond i \to i)$) are properties of the underlying transition structure which fail to be definable in standard modal logic (see [3]). Note that, however, for the propositional case, the satisfiability problem for hybrid logics is still decidable.

Another important feature of hybrid logics that will have a central role in our approach is the fact that basic hybrid logic can specify Robinson Diagrams: namely, $@_i p$ affirms that the proposition p is true at the state named by i, while $\neg @_i p$ (logically equivalent to $@_i \neg p$) denies this. Furthermore, $@_i j$ affirms that the states named by i and j are identical, while $\neg @_i j$ (logically equivalent to $@_i \neg j$) affirms that they are distinct. Finally, $@i \Diamond j$ affirms that the state named by j is a successor of the state named by i, and $@i \Box \neg j$ denies this. Consequently, in hybrid logic we are able to completely describe models using the rich underlying syntax.

1.1 The Basic Hybrid Language

The simplest form of hybrid logic is based on the *basic hybrid language*, which adds nominals and the satisfaction operator to the language of propositional modal logic. This simple upgrade of the usual modal language (with only nominals and the satisfaction operator) carries great power in terms of expressivity.

Definition 1.1. Let $\mathcal{L} = \langle \text{Prop}, \text{Nom} \rangle$ be a *hybrid signature* where Prop is a set of *propositional symbols* and Nom is a set disjoint from Prop. We use $p, q, r, etc.$ to refer to the elements in Prop. The elements in Nom are called *nominals* and we typically write them as i, j, k, etc. The well-formed formulas over $\mathcal{L}$, which we denote by $\text{Form}_@(\mathcal{L})$, are defined by the following grammar:

$$WFF := \ i \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid @_i\varphi$$

The formulas with prefix @ are called *satisfaction statements*. The connectives $\vee$, $\rightarrow$, and $\Box$ are defined as usual.

Definition 1.2. Let $\mathcal{L} = \langle \text{Prop}, \text{Nom} \rangle$ be a hybrid similarity type. A *hybrid structure* $\mathcal{M}$ over $\mathcal{L}$ is a tuple (W, R, V). Here, W is a non-empty set called *domain* whose elements are called *states* or *worlds*, R is a binary relation such that $R \subseteq W \times W$ and is called the *accessibility relation*, and $V : \text{Prop} \cup \text{Nom} \rightarrow Pow(W)$ is a *hybrid valuation*, where $V(i)$ is a singleton for any $i \in \text{Nom}$. The pair (W, R) is called the *frame* underlying $\mathcal{M}$, and $\mathcal{M}$ is said to be a structure based on this frame.

The satisfaction relation, which is defined as follows, is a generalization of Kripke-style satisfaction.

Definition 1.3 (Satisfaction)**.** The local satisfaction relation $\Vdash$ between a hybrid structure $\mathcal{M} = (W, R, V)$, a state $w \in W$, and a hybrid formula is recursively defined by:

1. $\mathcal{M}, w \Vdash i$ iff $\{w\} = V(i)$;
2. $\mathcal{M}, w \Vdash p$ iff $w \in V(p)$;
3. $\mathcal{M}, w \Vdash \neg\varphi$ iff not $\mathcal{M}, w \Vdash \varphi$;
4. $\mathcal{M}, w \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \psi$;
5. $\mathcal{M}, w \Vdash \Diamond\varphi$ iff $\exists w' \in W(wRw'$ and $\mathcal{M}, w' \Vdash \varphi)$;
6. $\mathcal{M}, w \Vdash @_i\varphi$ iff $\mathcal{M}, w' \Vdash \varphi$, where $\{w'\} = V(i)$;

If $\mathcal{M}, w \Vdash \varphi$ we say that φ is satisfied in $\mathcal{M}$ at w. If φ is satisfied at all states in a structure $\mathcal{M}$, we write $\mathcal{M} \Vdash \varphi$. If φ is satisfied at all states in all structures based on a frame $\mathcal{F}$, then we say that φ is valid on $\mathcal{F}$ and we write $\mathcal{F} \Vdash \varphi$. If φ is valid on all frames, then we simply say that φ is valid and we write $\Vdash \varphi$. We say that a set Φ of hybrid formulas is satisfiable if there is a model $\mathcal{M}$ and a world $w \in W$ such that $\mathcal{M}, w \Vdash \Phi$, i.e., for all $\phi \in \Phi$, $\mathcal{M}, w \Vdash \phi$.

For $\Delta \subseteq \text{Form}_@(\mathcal{L})$, we say that $\mathcal{M}$ is a *model of* Δ if $\mathcal{M} \Vdash \delta$ for all $\delta \in \Delta$.

2 A Herbrand's Theorem for Hybrid Logic

Herbrand's theorem is a fundamental result of mathematical logic. It essentially allows a certain kind of reduction of first-order logic to propositional logic. Several versions of Herbrand's theorem are now available for classical logic; here we present our version of it for (the non-classical) hybrid logic.

Let $\mathcal{L}$ be a hybrid signature. The set $\mathrm{At}(\mathcal{L})$ of *atomic satisfaction statements over* $\mathcal{L}$ is the set of $\mathcal{L}$-formulas of the forms $@_i p, @_i\Diamond j, @_i j$ for $i, j \in \mathrm{Nom}$ and $p \in \mathrm{Prop}$, i.e., $\mathrm{At}(\mathcal{L}) = \{@_i p,\ @_i\Diamond j,\ @_i j \mid i, j \in \mathrm{Nom},\ p \in \mathrm{Prop}\}$. We use $\mathrm{BCAt}(\mathcal{L})$ to denote the set of all (finite) Boolean combinations of atomic satisfaction statements over $\mathcal{L}$, i.e., $\mathrm{BCAt}(\mathcal{L})$ is the smallest set containing $\mathrm{At}(\mathcal{L})$ and closed under $\wedge$ and $\neg$.

An $\mathcal{L}$-*truth assignment* is a mapping $f : \mathrm{At}(\mathcal{L}) \to \{T, F\}$. Given an assignment f, we extend it to $\bar{f} : \mathrm{BCAt}(\mathcal{L}) \to \{T, F\}$ by the usual rules for propositional connectives.

Definition 2.1. Let $\Phi \subseteq \mathrm{BCAt}(\mathcal{L})$. We say that Φ is *propositionally satisfiable* if there is a truth assignment that simultaneously satisfies every member of Φ. We say that Φ is *propositionally unsatisfiable* if there is no such assignment.

We have now the basis to state the first part of a Herbrand's theorem for hybrid logic:

Theorem 2.1. *Let* $\Phi \subseteq \mathrm{BCAt}(\mathcal{L})$*. If* Φ *is propositionally unsatisfiable then* Φ *is unsatisfiable.*

Proof. Suppose that Φ is satisfiable. Then there is a model $\mathcal{M}$ and a world $w \in W$ such that $\mathcal{M}, w \Vdash \Phi$. Define $f^{\mathcal{M}} : \mathrm{At}(\mathcal{L}) \to \{T, F\}$ by setting $f^{\mathcal{M}}(\varphi) = T$ iff $\mathcal{M}, w \Vdash \varphi$. Clearly $\overline{f^{\mathcal{M}}}(\varphi) = T$, for any $\varphi \in \Phi$. □

The converse of the previous theorem is not true in general. As in the case of first-order logic with equality, we have to consider the equality axioms.

In hybrid logic we do not have a explicit symbol of equality in the language; however, there are hybrid formulas that express the equality axioms over nominals:

- Reflexivity: $@_i i$, for $i \in \mathrm{Nom}$;
- Symmetry: $@_i j \to @_j i$, for $i, j \in \mathrm{Nom}$;
- Transitivity: $(@_i j \wedge @_j k) \to @_i k$, for $i, j, k \in \mathrm{Nom}$;
- Congruence: $(@_i j \wedge @_k n) \to (@_i\Diamond k \leftrightarrow @_j\Diamond n)$, for $i, j, k, n \in \mathrm{Nom}$.

The set of all the latter formulas is denoted by $\mathrm{Eq}(\mathcal{L})$. They are all valid formulas in hybrid logic and, moreover, they are Boolean combinations of atomic satisfaction statements.

Theorem 2.2. *Let* $\Phi \subseteq \mathrm{BCAt}(\mathcal{L})$ *such that* $\mathrm{Eq}(\mathcal{L}) \subseteq \Phi$*. If* Φ *is unsatisfiable then* Φ *is propositionally unsatisfiable.*

Proof. Suppose that Φ is propositionally satisfiable. Let $f : \mathrm{At}(\mathcal{L}) \to \{T, F\}$ be such that $\overline{f}(\varphi) = T$ for any $\varphi \in \Phi$. We may assume that Nom is non-empty (if this were not the case, then $\mathrm{At}(\mathcal{L})$ would be empty and the theorem would trivially hold). Let $W = \mathrm{Nom}$ and define the binary relation $\sim$ on W by setting $i \sim j$ iff $@_i j \in \Phi$. This is an equivalence relation, as reflexivity, symmetry and transitivity are guaranteed by the formulas in $\mathrm{Eq}(\mathcal{L})$.

We define the hybrid structure $\mathcal{M} = (W_\Phi, R_\Phi, V_\Phi)$ such that:

- $W_\Phi = W/\sim$;
- $[i] R_\Phi [j]$ iff $@_i\Diamond j \in \Phi$, for $i, j \in \mathrm{Nom}$;
- $V_\Phi(i) = \{[i]\}$, for $i \in \mathrm{Nom}$; and
- $[i] \in V_\Phi(p)$ iff $@_i p \in \Phi$, $i \in \mathrm{Nom}$, for $p \in \mathrm{Prop}$.

Thus $\mathcal{M} \Vdash \Phi$ and clearly Φ is satisfiable. □

2.1 Generalization for any Satisfaction Statement

Let φ be any satisfaction statement. We can transform φ by applying the following rules to the subformulas of φ in order to obtain a semantically equivalent formula $\varphi^\circ \in \mathrm{BCAt}(\mathcal{L}^*)$, where $\mathcal{L}^*$ is an extension of $\mathcal{L}$ that adds new nominals.

Rules:

$$\frac{@_i\neg\varphi}{\neg @_i\varphi} \qquad \frac{@_i(\varphi\wedge\psi)}{@_i\varphi\wedge @_i\psi} \qquad \frac{@_i\Diamond\varphi}{@_i\Diamond k\wedge @_k\varphi}(*)$$

$(*)$ k is a fresh nominal

As an example we have $(@_i\Diamond(p\wedge\neg q))^\circ = @_i\Diamond k\wedge(@_k p\wedge\neg @_k q)$. Note that the new formula is in the extended language with the new nominal k.

Theorem 2.3. *Let Φ be a set of satisfaction statements such that $\mathrm{Eq}(\mathcal{L})\subseteq\Phi$. Then Φ is propositionally unsatisfiable iff Φ is unsatisfiable.*

Proof. We transform Φ into $\Phi^\circ := \{\varphi^\circ : \varphi\in\Phi\}\cup\mathrm{Eq}(\mathcal{L}^*)$, which contains only formulas in $\mathrm{BCAt}(\mathcal{L}^*)$, and then we apply the previous theorems. □

3 Conclusion

We have proposed a version of Herbrand's theorem in the context of Hybrid logic, with a restriction to satisfaction statements, by making use of rules which transform each satisfaction statement into a boolean combination of atomic satisfaction statements, and making use also of the fact that each model can be described by its diagram [1]. We managed to prove that a set of satisfaction statements is propositionally unsatisfiable if and only if it is unsatisfiable. Thus we paved the way for a detailed study of Herbrand's theorem for hybrid logic, regarding arbitrary hybrid formulas.

Formulas with quantifiers (over worlds or objects) constitute a challenge. Fitting proposes in [6] a version of Herbrand's theorem for the modal logic K. Following the classical steps, when passing through skolemization, one gets for some formulas non-rigid designators, which bring problems, as the act of designation and the act of passing to an alternate world do not commute. In order to overcome those problems, Fitting came up with the concepts of predicate abstraction and validity functional form. By combining those two, and with recourse to a modal Herbrand transform and a calculus for modal logic, it is possible to check the validity of a formula. We are confident that for the hybrid scenario, something similar is to be done. To deal with quantifiers over world variables we intend to follow the approach in [8], that adds function symbols with interpretations as functions on the set of worlds. This will allow Skolemization in a standard way.

In the future we would also like to address the issue of Herbrand's theorem for a paraconsistent version of hybrid logic. The basic version, quantifier-free, does not seem to give rise to any problem because in Quasi-hybrid logic, [5], it is still possible to count on Robinson diagrams.

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