On Logics of Perfect Paradefinite Algebras

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The present study shows how any De Morgan algebra may be enriched by a 'perfection operator' that allows one to express the Boolean properties of negation-consistency and negation-determinedness. The corresponding variety of 'perfect paradefinite algebras' (PP-algebras) is shown to be term-equivalent to the variety of involutive Stone algebras, introduced by R. Cignoli and M. Sagastume, and more recently studied from a logical perspective by M. Figallo and L. Cantú. Such equivalence then plays an important role in the investigation of the 1-assertional logic and also the order-preserving logic asssociated to the PP-algebras. The latter logic, which we call $\mathcal{PP}_{\leq}$, happens to be characterised by a single 6-valued matrix and consists very naturally in a Logic of Formal Inconsistency and Formal Undeterminedness. The logic $\mathcal{PP}_{\leq}$ is here axiomatised, by means of an analytic finite Hilbert-style calculus, and a related axiomatization procedure is presented that covers the logics of other classes of De Morgan algebras as well as super-Belnap logics enriched by a perfection connective.

1 Introduction

The variety of De Morgan algebras comprises all bounded distributive lattices equipped with a De Morgan negation, that is, an involutive unary primitive operation $\sim$ satisfying the well-known De Morgan laws. Involutive Stone algebras (henceforth referred to as IS-algebras) are De Morgan algebras endowed with a primitive unary operation ∇ that allows for the definition of a pseudo-complement operator $\neg$ satisfying the Stone equation $\neg x \vee \neg\neg x \approx \top$.

While the order-preserving logic canonically induced by De Morgan algebras, namely Dunn-Belnap's four-valued logic [6], has been extensively studied over the last decades, the logic so induced by IS-algebras, which we call $\mathcal{IS}_{\leq}$, has only recently attracted due attention [8, 9]. Some of the most prominent features of $\mathcal{IS}_{\leq}$ are the facts that it is paradefinite [3] (it is, indeed, at once $\sim$-paraconsistent and $\sim$-paracomplete, characteristics actually inherited from Dunn-Belnap logic), $\sim$-gently explosive and $\sim$-gently implosive [22]; in other words, it is a Logic of Formal Inconsistency (**LFI**) and a Logic of Formal

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Undeterminedness (**LFU**). All these features remain rather concealed in the presentation of IS-algebras in terms of ∇, an operator whose significance and philosophical motivations are at best unclear.[1]

In contrast, from a logical viewpoint, we note that *perfection operators* (in the sense of [23]) allow for the internalization of the very notions of negation-consistency and negation-determinedness at the object-language level, and logics with such an expressive capability also happen to have been extensively studied in the last two decades (cf. [5], for example, for the so-called 'classicality', 'restoration', 'recapture', or 'recovery' operators). In order to establish a fruitful dialogue with the logical study of negation, we propose in the present study an alternative presentation of IS-algebras, in terms of structures that we shall denominate 'perfect paradefinite algebras' (or simply PP-algebras), obtained by replacing ∇ with a primitive perfection operation $\circ$. The equational characterization we present for PP-algebras will not only guarantee that the corresponding variety is term-equivalent to the variety of IS-algebras but also highlight the paradefinite character of the order-preserving logic thereby induced ($\mathcal{PP}_{\leq}$); the latter will be shown more specifically to constitute a fully self-extensional and non-protoalgebraic member of the families of logics known as **C**-systems and **D**-systems. A procedure for constructing a PP-algebra using a De Morgan algebra as material is introduced and, akin to $\mathcal{IS}_{\leq}$, the logic $\mathcal{PP}_{\leq}$ will be seen to be characterizable by a single six-element logical matrix. At last, we are also to provide, here, a well-behaved symmetrical Hilbert-style calculus for the Set-Set logics induced by logical matrices based on De Morgan algebras enriched with $\circ$, as well as conventional Hilbert-style calculi for the Set-Fmla logics induced by logical matrices based on De Morgan algebras with prime filters enriched with $\circ$ — and, in particular, an analytical proof system for the logic $\mathcal{PP}_{\leq}$ itself.

2 Algebraic and logical preliminaries

A *propositional signature* is a family $\Sigma := \{\Sigma_k\}_{k\in\omega}$, where each Σ_k is a collection of k-ary *connectives*. A Σ-*algebra* is a structure $\mathbf{A} := \langle A, \cdot^{\mathbf{A}}\rangle$, where A is a non-empty set called the *carrier* of $\mathbf{A}$ and, for each $© \in \Sigma_k$, $©^{\mathbf{A}} : A^k \to A$ is the *interpretation* of $©$ in $\mathbf{A}$. Given a denumerable set $P \supseteq \{p,q,r,x,y\}$, the absolutely free algebra over Σ freely generated by P, or simply the *language* over Σ (generated by P), is denoted by $\mathbf{L}_\Sigma(P)$, and its members are called Σ-*formulas*. The collection of all propositional variables occurring in a formula $\varphi \in L_\Sigma(P)$ is denoted by $\mathsf{props}(\varphi)$, and we let $\mathsf{props}(\Phi) := \bigcup_{\varphi\in\Phi}\mathsf{props}(\varphi)$, for all $\Phi \subseteq L_\Sigma(P)$. Given $\Sigma' \subseteq \Sigma$ (that is, $\Sigma'_k \subseteq \Sigma_k$ for all $k \in \omega$), the Σ'-*reduct* of a Σ-algebra $\mathbf{A}$ is the Σ'-algebra over the same carrier of $\mathbf{A}$ that agrees with $\mathbf{A}$ on the interpretation of the connectives in Σ'. The collection of homomorphisms between two Σ-algebras $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathsf{Hom}(\mathbf{A},\mathbf{B})$, and the collection of mappings that are structure-preserving over $\Sigma' \subseteq \Sigma$ is denoted by $\mathsf{Hom}_{\Sigma'}(\mathbf{A},\mathbf{B})$. Furthermore, the set of endomorphisms on $\mathbf{A}$ is denoted by $\mathsf{End}(\mathbf{A})$ and each one of the members of $\sigma \in \mathsf{End}(\mathbf{L}_\Sigma(P))$ is called a *substitution*. In case $p_1,\dots,p_n$ are the only propositional variables ocurring in $\varphi \in L_\Sigma(P)$, we say that φ is n-ary and denote by $\varphi^{\mathbf{A}}$ the n-ary operation on A such that, for all $a_1,\dots,a_n \in A$, $\varphi^{\mathbf{A}}(a_1,\dots,a_n) = h(\varphi)$, for an $h \in \mathsf{Hom}(\mathbf{L}_\Sigma(P),\mathbf{A})$ with $h(p_i) = a_i$ for each $1 \leq i \leq n$. Also, if $\psi_1,\dots,\psi_n \in L_\Sigma(P)$, we let $\varphi(\psi_1,\dots,\psi_n)$ denote the formula $\varphi^{\mathbf{L}_\Sigma(P)}(\psi_1,\dots,\psi_n)$. A Σ-*equation* is a pair (φ,ψ) of Σ-formulas that we will denote by $\varphi \approx \psi$, and a Σ-algebra $\mathbf{A}$ is said to *satisfy* $\varphi \approx \psi$ if $h(\varphi) = h(\psi)$ for every $h \in \mathsf{Hom}(\mathbf{L}_\Sigma(P),\mathbf{A})$. We call Σ-*variety* the class of all Σ-algebras that satisfy the same given collection of Σ-equations; an equation is said to be *valid* in a given variety if it is satisfied by each algebra in this variety. The variety generated by a class K of Σ-algebras, denoted by $\mathbb{V}(\mathsf{K})$, is the closure of K under homomorphic images, subalgebras

[1] For the 3-valued case, such a 'possibility' operator is known at least since [19], where J. Łukasiewicz notes it has been first defined during one of his 1921 seminars by a student called Tarski. The lack of a robust *modal* reading for such an operator, however, has caused it to have largely fallen by the wayside over the following decades.

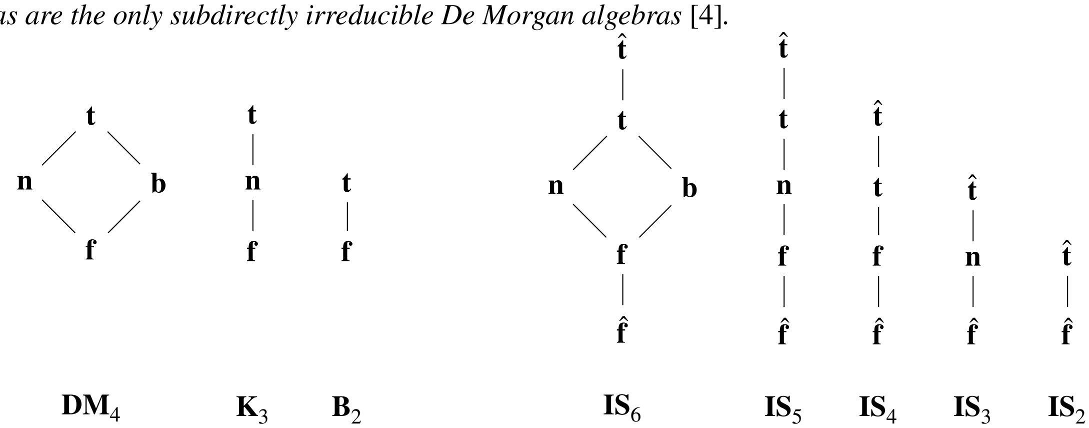
and direct products. We write $\mathsf{Cng}\,\mathbf{A}$ to refer to the collection of all congruence relations on $\mathbf{A}$, which is known to form a complete lattice under inclusion.

In what follows, we assume the reader is familiar with basic notations and terminology of lattice theory [12]. We denote by Σ^{bL} the signature containing but two binary connectives, $\wedge$ and $\vee$, and two nullary connectives $\top$ and $\bot$, and by Σ^{DM} the extension of the latter with a unary connective $\sim$. Moreover, we let Σ^{IS} and Σ^{PP} be the signatures obtained from Σ^{DM} by adding unary connectives ∇ and $\circ$, respectively. We provide below the definitions and some examples of De Morgan and of involutive Stone algebras.

Definition 2.1. *Given a Σ^{DM}-algebra whose Σ^{bL}-reduct is a bounded distributive lattice, we say that it constitutes a* De Morgan algebra *if it satisfies the equations:*

(DM1) ${\sim}{\sim}x \approx x$ **(DM2)** ${\sim}(x \wedge y) \approx {\sim}x \vee {\sim}y$

Example 2.2. *Let $\mathcal{V}_4 := \{\mathbf{t},\mathbf{b},\mathbf{n},\mathbf{f}\}$ and let $\mathbf{DM}_4 := \langle \mathcal{V}_4, \cdot^{\mathbf{DM}_4}\rangle$ be the Σ^{DM}-algebra known as the four-element Dunn-Belnap algebra, whose interpretations for the lattice connectives are those induced by the Hasse diagram in Figure* 1a*, and the interpretation for $\sim$ is such that ${\sim}^{\mathbf{DM}_4}\mathbf{f} := \mathbf{t}$, ${\sim}^{\mathbf{DM}_4}\mathbf{t} := \mathbf{f}$ and ${\sim}^{\mathbf{DM}_4}a := a$, for $a \in \{\mathbf{n},\mathbf{b}\}$; as expected, for the nullary connectives, we have $\top^{\mathbf{DM}_4} := \mathbf{t}$ and $\bot^{\mathbf{DM}_4} := \mathbf{f}$. In Figure* 1a*, besides depicting the lattice structure of $\mathbf{DM}_4$, we also show its subalgebras $\mathbf{K}_3$ and $\mathbf{B}_2$, which coincide with the three-element Kleene algebra and the two-element Boolean algebra. These three algebras are the only subdirectly irreducible De Morgan algebras* [4].



(a) The subdirectly irreducible De Morgan algebras. (b) The subdirectly irreducible IS-algebras.

Figure 1

Definition 2.3. *Given a Σ^{IS}-algebra whose Σ^{DM}-reduct is a De Morgan algebra, we say that it constitutes an* involutive Stone algebra *(*IS-algebra*) if it satisfies the equations:*

(IS1) $\nabla\bot \approx \bot$ **(IS2)** $x \wedge \nabla x \approx x$ **(IS3)** $\nabla(x \wedge y) \approx \nabla x \wedge \nabla y$ **(IS4)** ${\sim}\nabla x \wedge \nabla x \approx \bot$

Example 2.4. *Let $\mathcal{V}_6 := \mathcal{V}_4 \cup \{\hat{\mathbf{f}},\hat{\mathbf{t}}\}$ and let $\mathbf{IS}_6 := \langle \mathcal{V}_6, \cdot^{\mathbf{IS}_6}\rangle$ be the Σ^{IS}-algebra whose lattice structure is depicted in Figure* 1b *and interprets $\sim$ and ∇ as per the following:*

$$
{\sim}^{\mathbf{IS}_6} a := \begin{cases} {\sim}^{\mathbf{DM}_4} a & a \in \mathcal{V}_4 \\ \hat{\mathbf{f}} & a = \hat{\mathbf{t}} \\ \hat{\mathbf{t}} & a = \hat{\mathbf{f}} \end{cases} \qquad \nabla^{\mathbf{IS}_6} a := \begin{cases} \hat{\mathbf{t}} & a \in \mathcal{V}_6 \setminus \{\hat{\mathbf{f}}\} \\ \hat{\mathbf{f}} & a = \hat{\mathbf{f}} \end{cases}
$$

The subalgebras of $\mathbf{IS}_6$ exhibited in Figure 1b *constitute the only subdirectly irreducible IS-algebras* [11].

We denote by $\mathbb{IS}$ the variety of IS-algebras. The next result lists some equations satisfied by IS-algebras, which will be useful for proving the results in the next section.

Lemma 2.5. *The following equations are satisfied by IS-algebras:*

1. $x \vee \nabla{\sim}x \approx \top$
2. $x \wedge {\sim}\nabla x \approx \bot$
3. ${\sim}\nabla(x \wedge {\sim}x) \wedge {\sim}x \approx {\sim}\nabla x$
4. $\nabla\nabla x \approx \nabla x$
5. $\nabla{\sim}\nabla x \approx {\sim}\nabla x$
6. ${\sim}\nabla{\sim}(x \wedge y) \approx {\sim}\nabla{\sim}x \wedge {\sim}\nabla{\sim}y$

Proof. Equation 3 may be proved by using the usual De Morgan algebra equations together with $\nabla x \vee x \approx \nabla x$, an equation that is easily derivable from **(IS2)**. All other equations follow from Lemma 3.2 in [9]. □

Here, a SET-FMLA *logic (over* Σ*)* is a consequence relation $\vdash$ on $L_\Sigma(P)$ and a SET-SET *logic (over* Σ*)* is a generalised consequence relation $\rhd$ on $L_\Sigma(P)$ [17]. We will write $\Phi \lhd\rhd \Psi$ when $\Phi \rhd \Psi$ and $\Psi \rhd \Phi$. The complement of a given SET-SET logic $\rhd$ will be denoted by $\blacktriangleright$. We say that $\vdash'$ *extends* $\vdash$ when $\vdash' \supseteq \vdash$. It is worth recalling that the collection of all extensions of a given logic forms a complete lattice under inclusion. Given $\Sigma \subseteq \Sigma'$, a logic $\vdash'$ over Σ' is a *conservative expansion* of a logic $\vdash$ over Σ when $\vdash'$ extends $\vdash$ and, for all $\Phi \cup \{\psi\} \subseteq L_\Sigma(P)$, we have $\Phi \vdash' \psi$ iff $\Phi \vdash \psi$. These concepts may be extended to the SET-SET framework in the obvious way. We say, in addition, that a SET-FMLA Σ-logic $\vdash$ *has a disjunction* provided that $\Phi, \varphi \vee \psi \vdash \phi$ iff $\Phi, \varphi \vdash \phi$ and $\Phi, \psi \vdash \phi$ (for $\vee$ a binary connective in Σ).

A *(logical)* Σ*-matrix* $\mathfrak{M}$ is a structure $\langle \mathbf{A}, D\rangle$ where $\mathbf{A}$ is a Σ-algebra and the members of $D \subseteq A$ are called *designated values*. We will write $\overline{D}$ to refer to $A \backslash D$. Provided that $\mathbf{A}$ has a lattice structure with underlying order $\leq$, we will often employ the notation ${\uparrow}a := \{b \in A \mid a \leq b\}$ when specifying sets of designated values. The mappings in $\mathsf{Hom}(\mathbf{L}_\Sigma(P), \mathbf{A})$ are called $\mathfrak{M}$*-valuations*. Every Σ-matrix induces a SET-SET logic $\rhd_{\mathfrak{M}}$ such that $\Phi \rhd_{\mathfrak{M}} \Psi$ iff $h(\Phi) \cap \overline{D} \neq \varnothing$ or $h(\Psi) \cap D \neq \varnothing$ as well as a SET-FMLA logic $\vdash_{\mathfrak{M}}$ with $\Phi \vdash_{\mathfrak{M}} \psi$ iff $\Phi \rhd_{\mathfrak{M}} \{\psi\}$. Given a SET-SET logic $\rhd$ (resp. a SET-FMLA logic $\vdash$), if $\rhd \subseteq \rhd_{\mathfrak{M}}$ (resp. $\vdash \subseteq \vdash_{\mathfrak{M}}$), we shall say that $\mathfrak{M}$ *is a model of* $\rhd$ (resp. $\vdash$), and if the converse also holds we shall say that $\mathfrak{M}$ *characterises* $\rhd$ (resp. $\vdash$). The SET-SET (resp. SET-FMLA) logic induced by a class $\mathcal{M}$ of Σ-matrices is given by $\bigcap\{\rhd_{\mathfrak{M}} \mid \mathfrak{M} \in \mathcal{M}\}$ (resp. $\bigcap\{\vdash_{\mathfrak{M}} \mid \mathfrak{M} \in \mathcal{M}\}$). At a few occasions, below, we shall prefer to write $\mathsf{Log}\,\mathcal{M}$ for $\vdash_{\mathcal{M}}$.

Example 2.6. *The* Σ^{DM}*-matrix* $\langle \mathbf{DM}_4, {\uparrow}\mathbf{b}\rangle$ *induces the logic known as the four-valued Dunn-Belnap logic, or First-Degree Entailment (FDE)* [6]*, which we hereby denote by* $\mathcal{B}$*. Extensions of* $\mathcal{B}$ *are known as* super-Belnap logics [26].

Example 2.7. *Classical Logic, hereby denoted by* $\mathcal{CL}$*, is induced by the* Σ^{DM}*-matrix* $\langle \mathbf{B}_2, \{\mathbf{t}\}\rangle$*.*

Given a Σ-matrix $\mathfrak{M} = \langle \mathbf{A}, D\rangle$, a congruence $\theta \in \mathsf{Cng}\,\mathbf{A}$ is said to be *compatible with* $\mathfrak{M}$ when $b \in D$ whenever both $a \in D$ and $a\theta b$, for all $a, b \in A$. We denote by $\Omega^{\mathfrak{M}}$ the *Leibniz congruence* associated to $\mathfrak{M}$, namely the greatest congruence of $\mathbf{A}$ compatible with $\mathfrak{M}$. The matrix $\mathfrak{M}^* = \langle \mathbf{A}/\Omega^{\mathfrak{M}}, D/\Omega^{\mathfrak{M}}\rangle$ is the *reduced version* of $\mathfrak{M}$. We say that $\mathfrak{M}$ is *reduced* when its Leibniz congruence is the identity relation on A. It is well known that $\rhd_{\mathfrak{M}} = \rhd_{\mathfrak{M}^*}$ (and thus $\vdash_{\mathfrak{M}} = \vdash_{\mathfrak{M}^*}$) and, since every logic is determined by a class of matrix models, we have that every logic coincides with the logic determined by its reduced matrix models. The class of all reduced matrix models for a logic $\vdash$ is denoted by $\mathsf{Mat}^*(\vdash)$.

Every Σ-variety K such that each $\mathbf{A} \in \mathsf{K}$ has a $\bigwedge$-semilattice reduct with top element $\top$ induces a SET-FMLA *order-preserving logic* $\vdash^{\leq}_{\mathsf{K}}$ according to which ψ follows from Φ iff (i) $\Phi = \varnothing$ and $\psi \approx \top$ is valid in K or (ii) there are $\varphi_1, \ldots, \varphi_n \subseteq \Phi$ ($n \geq 1$) such that the equation $\bigwedge_i \varphi_i \approx \bigwedge_i \varphi_i \wedge \psi$ is valid in K. A *lattice filter* of a $\bigwedge$-semilattice $\mathbf{A}$ with a top element $\top$ is a subset $D \subseteq A$ with $\top^{\mathbf{A}} \in D$ and closed under $\wedge^{\mathbf{A}}$; moreover, D is a *proper* lattice filter of $\mathbf{A}$ when $D \neq A$. If $\mathbf{A}$ is a $\bigvee$-semilattice, a *prime filter* of $\mathbf{A}$ is a proper lattice filter D of $\mathbf{A}$ such that $a \vee b \in D$ iff $a \in D$ or $b \in D$, for all $a, b \in A$. In case each $\mathbf{A} \in \mathsf{K}$ has a bounded distributive lattice reduct, as all varieties treated in the present work do, it follows that its order-preserving logic coincides with the logic determined by the class of matrices $\{\langle \mathbf{A}, D\rangle \mid \mathbf{A} \in \mathsf{K}, D \subseteq A$ is a non-empty lattice filter of $\mathbf{A}\}$. Furthermore, we associate to K the 1-*assertional logics* $\rhd^{\top}_{\mathsf{K}}$ and $\vdash^{\top}_{\mathsf{K}}$ corresponding respectively to the SET-SET and SET-FMLA logics induced by the class of Σ-matrices $\{\langle \mathbf{A}, \{\top^{\mathbf{A}}\}\rangle \mid \mathbf{A} \in \mathsf{K}\}$.

Based on [27, 7], we define a *symmetrical* (*Hilbert-style*) *calculus* R as a collection of pairs $(\Phi, \Psi) \in \wp L_\Sigma(P) \times \wp L_\Sigma(P)$, denoted by $\frac{\Phi}{\Psi}$ and called (*symmetrical*) *inference rules*, where Φ is the *antecedent* and Ψ is the *succedent* of the said rule. We will adopt the convention of omitting curly braces when writing sets of formulas and leaving a blank space instead of writing $\varnothing$ when presenting inference rules and statements involving (generalised) consequence relations. We proceed to define what constitutes a proof in such calculi.

A *bounded rooted tree* t is a poset $\langle \mathsf{nds}(t), \leq^t\rangle$ with a single minimal element $\mathsf{rt}(t)$, the *root* of t, such that, for each *node* $n \in \mathsf{nds}(t)$, the set $\{n' \in \mathsf{nds}(t) \mid n' \leq^t n\}$ of *ancestors* of n (or the *branch* up to n) is well-ordered under $\leq^t$, and every branch of t has a maximal element (a *leaf* of t). We may assign a label $l^t(n) \in \wp L_\Sigma(P) \cup \{*\}$ to each node n of t, in which case t is said to be *labelled*. Given $\Psi \subseteq L_\Sigma(P)$, a leaf n is Ψ-*closed* in t when $l^t(n) = *$ or $l^t(n) \cap \Psi \neq \varnothing$. The tree t itself is Ψ-*closed* when all of its leaves are Ψ-closed. The immediate successors of a node n with respect to $\leq^t$ are called the *children* of n in t.

Let R be a symmetrical calculus. An R-*derivation* is a labelled bounded rooted tree such that for every non-leaf node n of t there exists a rule of inference $\mathsf{r} = \frac{\Pi}{\Theta} \in \mathsf{R}$ and a substitution σ such that $\sigma(\Pi) \subseteq l^t(n)$, and the set of the children of n is either (i) $\{n^\varphi \mid \varphi \in \sigma(\Theta)\}$, in case $\Theta \neq \varnothing$, where n^φ is a node labelled with $l^t(n) \cup \{\varphi\}$, or (ii) a singleton $\{n^*\}$ with $l^t(n) = *$, in case $\Theta = \varnothing$. We say that $\Phi \rhd_{\mathsf{R}} \Psi$ whenever there is a Ψ-closed derivation t such that $\Phi \supseteq \mathsf{rt}(t)$; such a tree consists in a *proof* that Ψ follows from Φ in R. As a matter of simplification when drawing such trees, we usually avoid copying the formulas inherited from the parent nodes (see Example 2.8 below). The relation $\rhd_{\mathsf{R}}$ so defined is a SET-SET logic and, when $\rhd_{\mathsf{R}} = \rhd_{\mathfrak{M}}$, we say that R *axiomatises* $\mathfrak{M}$. A rule $\frac{\Phi}{\Psi}$ is *sound* with respect to $\mathfrak{M}$ when $\Phi \rhd_{\mathfrak{M}} \Psi$. It should be pointed out that such deductive formalism generalises the conventional (SET-FMLA) Hilbert-style calculi: the latter corresponds to symmetrical calculi whose rules have, each, a finite antecedent and a singleton as succedent. Given $\Lambda \subseteq L_\Sigma(P)$, we write $\Phi \rhd^{\Lambda}_{\mathsf{R}} \Psi$ whenever there is a proof of Ψ from Φ using only formulas in Λ. We say that R is Ξ-*analytic* when, for all $\Phi, \Psi \subseteq L_\Sigma(P)$, whenever $\Phi \rhd_{\mathsf{R}} \Psi$, we have $\Phi \rhd^{\Upsilon^\Xi}_{\mathsf{R}} \Psi$, with $\Upsilon^\Xi := \mathsf{sub}(\Phi \cup \Psi) \cup \{\sigma(\varphi) \mid \varphi \in \Xi \text{ and } \sigma : P \to \mathsf{sub}(\Phi \cup \Psi)\}$, which we shall dub the *generalised subformulas* of (Φ, Ψ). Intuitively, it means that a proof in R that Ψ follows from Φ may only use subformulas of $\Phi \cup \Psi$ or substitution instances of the formulas in Ξ built with those same subformulas.

A general method is introduced in [7, 21] for obtaining analytic calculi (in the sense of analyticity introduced in the above paragraph) for logics given by a Σ-matrix $\langle \mathbf{A}, D\rangle$ whenever a certain expressiveness requirement (called 'monadicity' in [27]) is met: for every $a, b \in A$, there is a single-variable formula S (a so-called *separator*) such that $\mathrm{S}^{\mathbf{A}}(a) \in D$ and $\mathrm{S}^{\mathbf{A}}(b) \notin D$ or vice-versa. The next example illustrates a symmetrical calculus for $\mathcal{B}$ generated by this method, as well as some proofs in this calculus.

Example 2.8. *The matrix $\langle \mathbf{DM}_4, \uparrow\mathbf{b}\rangle$ fulfills the above expressiveness requirement, with the following set of separators: $\mathcal{S} := \{p, {\sim}p\}$. We may therefore apply the method introduced in* [21] *to obtain for $\mathcal{B}$ the following $\mathcal{S}$-analytic axiomatization we call $\mathsf{R}_{\mathcal{B}}$:*

$$\frac{}{\top}\,\mathsf{r}_1 \qquad \frac{{\sim}\top}{}\,\mathsf{r}_2 \qquad \frac{}{{\sim}\bot}\,\mathsf{r}_3 \qquad \frac{\bot}{}\,\mathsf{r}_4 \qquad \frac{p}{{\sim}{\sim}p}\,\mathsf{r}_5 \qquad \frac{{\sim}{\sim}p}{p}\,\mathsf{r}_6$$

$$\frac{p \land q}{p}\,\mathsf{r}_7 \qquad \frac{p \land q}{q}\,\mathsf{r}_8 \qquad \frac{p, q}{p \land q}\,\mathsf{r}_9 \qquad \frac{{\sim}p}{{\sim}(p \land q)}\,\mathsf{r}_{10} \qquad \frac{{\sim}q}{{\sim}(p \land q)}\,\mathsf{r}_{11} \qquad \frac{{\sim}(p \land q)}{{\sim}p, {\sim}q}\,\mathsf{r}_{12}$$

$$\frac{p}{p \lor q}\,\mathsf{r}_{13} \qquad \frac{q}{p \lor q}\,\mathsf{r}_{14} \qquad \frac{p \lor q}{p, q}\,\mathsf{r}_{15} \qquad \frac{{\sim}p, {\sim}q}{{\sim}(p \lor q)}\,\mathsf{r}_{16} \qquad \frac{{\sim}(p \lor q)}{{\sim}p}\,\mathsf{r}_{17} \qquad \frac{{\sim}(p \lor q)}{{\sim}q}\,\mathsf{r}_{18}$$

Figure 2 *illustrates some derivations in* $\mathsf{R}_{\mathcal{B}}$.

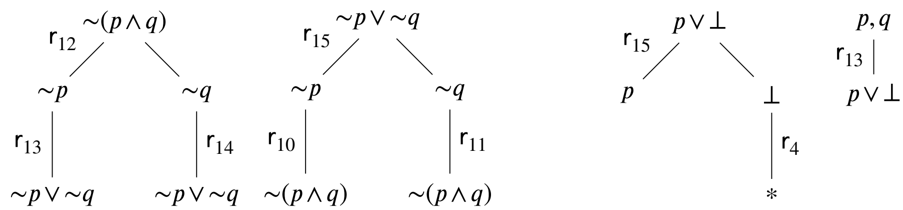


Figure 2: Proofs in $\mathsf{R}_{\mathcal{B}}$ witnessing that ${\sim}(p \land q) \lhd\!\rhd_{\mathcal{B}} {\sim}p \lor {\sim}q$ and $p \lor \bot \lhd\!\rhd_{\mathcal{B}} p, q$.

Let Σ be any signature containing a unary connective $\sim$. A SET-SET logic $\rhd$ over Σ is said to be *$\sim$-paraconsistent* when we have $p, {\sim}p \blacktriangleright q$, and *$\sim$-paracomplete* when we have $q \blacktriangleright p, {\sim}p$, with $p, q \in \boldsymbol{P}$. Moreover, $\rhd$ is *$\sim$-gently explosive* in case there is a collection $\bigcirc(p) \subseteq L_\Sigma(\boldsymbol{P})$ of formulas on a single variable such that, for some $\varphi \in L_\Sigma(\boldsymbol{P})$, we have $\bigcirc(\varphi), \varphi \blacktriangleright \varnothing$ and $\bigcirc(\varphi), {\sim}\varphi \blacktriangleright \varnothing$, and, for all $\psi \in L_\Sigma(\boldsymbol{P})$, we have $\bigcirc(\psi), \psi, {\sim}\psi \rhd \varnothing$. Dually, $\rhd$ is *$\sim$-gently implosive* in case there is a collection of formulas $\star(p) \subseteq L_\Sigma(\boldsymbol{P})$ on a single variable such that, for some $\varphi \in L_\Sigma(\boldsymbol{P})$, we have $\varnothing \blacktriangleright \varphi, \star(\varphi)$ and $\varnothing \blacktriangleright {\sim}\varphi, \star(\varphi)$, and, for all $\psi \in L_\Sigma(\boldsymbol{P})$, we have $\rhd {\sim}\psi, \psi, \star(\psi)$. A SET-SET logic is *$\sim$-paradefinite* when it is both $\sim$-paraconsistent and $\sim$-paracomplete; is a *logic of formal inconsistency (**LFI**)* when it is $\sim$-paraconsistent yet $\sim$-gently explosive; and is a *logic of formal undeterminedness (**LFU**)* when it is $\sim$-paracomplete yet $\sim$-gently implosive. Furthermore, if $\rhd_1$ and $\rhd_2$ are logics over $\Sigma_1 \supseteq \Sigma$ and $\Sigma_2 \supseteq \Sigma$ respectively, we say that $\rhd_1$ is a ***C**-system based on* $\rhd_2$ with respect to $\sim$ (or simply a ***C**-system*) when it is an **LFI** that agrees with $\rhd_2$ on statements involving formulas without $\sim$, and $\bigcirc(p) = \{\circ p\}$, for $\circ$ a composite *(consistency) connective* in the language of $\rhd_1$. We may dually define the notion of ***D**-system* [23].

Example 2.9. *By exploiting the fact that* $\mathbf{n}, \mathbf{b} \in \mathcal{V}_4$ *are fixpoints of* ${\sim}^{\mathbf{DM}_4}$*, one may easily notice that* $\mathcal{B}$ *is $\sim$-paraconsistent and $\sim$-paracomplete (thus $\sim$-paradefinite).*

3 Perfect paradefinite algebras and their logics

We propose in this section to extend De Morgan algebras with a perfection operator $\circ$, which will allow us to recover the classical properties of $\sim$-consistency and $\sim$-determinedness. In the sequel, we will prove that the variety of such algebras is term-equivalent to the variety of IS-algebras.

Definition 3.1. *Given a* Σ^{PP}*-algebra whose* Σ^{DM}*-reduct is a De Morgan algebra, we say that it constitutes a* perfect paradefinite algebra *(*PP-algebra*) if it satisfies the equations:*

(PP1) $\circ\circ x \approx \top$ **(PP2)** $\circ x \approx \circ{\sim}x$ **(PP3)** $\circ\top \approx \top$ **(PP4)** $\circ x \land ({\sim}x \land x) \approx \bot$
(PP5) $\circ(x \land y) \approx (\circ x \lor \circ y) \land (\circ x \lor {\sim}y) \land (\circ y \lor {\sim}x)$

Example 3.2. *An example of PP-algebra is* $\mathbf{PP}_6 := \langle \mathcal{V}_6, \cdot^{\mathbf{PP}_6} \rangle$*, the* Σ^{PP}*-algebra defined as* $\mathbf{IS}_6$ *in Example* 2.4*, differing only in that, instead of containing an interpretation for* ∇*, it interprets* $\circ$ *as follows:*

$$\circ^{\mathbf{PP}_6} a := \begin{cases} \hat{\mathbf{f}} & a \in \mathcal{V}_6 \setminus \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\} \\ \hat{\mathbf{t}} & a \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\} \end{cases}$$

Other examples are the algebras $\mathbf{PP}_i$*, for* $2 \leq i \leq 5$*, the subalgebras of* $\mathbf{PP}_6$ *having, respectively, the same lattice structures of the algebras* $\mathbf{IS}_i$ *exhibited in Figure* 1b*.*

As it occurs with IS-algebras, in the language of PP-algebras we may easily define, by setting $\neg x := \circ x \wedge {\sim} x$, a pseudo-complement satisfying the Stone equation. We denote by $\mathbb{PP}$ the variety of PP-algebras. The following result illustrates some useful equations satisfied by the members of $\mathbb{PP}$.

Lemma 3.3. *Every PP-algebra satisfies:*

1. ${\sim}\circ x \vee (x \vee {\sim} x) \approx \top$ *2.* $\circ x \wedge {\sim}\circ x \approx \bot$ *3.* $\circ x \approx \circ x \wedge (x \vee {\sim} x)$

Proof. Notice that 1 is a straightforward consequence of **(PP4)**, and 2 is a consequence of **(PP4)** using $\circ x$ in place of x and invoking **(PP1)**. Finally, 3 may be easily proved using 1 and 2. $\square$

Given $\varphi \in L_{\Sigma^{\mathsf{IS}}}(P)$ (resp. $\varphi \in L_{\Sigma^{\mathsf{PP}}}(P)$), let $\varphi^\circ \in L_{\Sigma^{\mathsf{PP}}}(P)$ (resp. $\varphi^\nabla \in L_{\Sigma^{\mathsf{IS}}}(P)$) be the result of applying the definition of $\circ$ (resp. of ∇) given below, in Theorem 3.4 (resp. Theorem 3.5), over φ. Extend this notion to sets of formulas in the usual way. The subsequent results establish the term-equivalence between the varieties of involutive Stone algebras and of perfect paradefinite algebras.

Theorem 3.4. *Let* $\mathbf{A} \in \mathbb{IS}$*. Then the* Σ^{PP}*-algebra* $\mathbf{A}^\circ$ *having the same* Σ^{DM}*-reduct of* $\mathbf{A}$ *and with* $\circ^{\mathbf{A}^\circ}$ *being the operation induced by* ${\sim}\nabla(x \wedge {\sim} x)$ *on* $\mathbf{A}$ *is a PP-algebra.*

Proof. We must show that $\mathbf{A}^\circ$ satisfies each of the characteristic equations of PP-algebras:

(PP1) $\circ\circ x \approx_{def} {\sim}\nabla(({\sim}\nabla(x \wedge {\sim}x)) \wedge {\sim}({\sim}\nabla(x \wedge {\sim}x))) \approx_{\mathbf{(IS3)}} {\sim}\nabla{\sim}\nabla(x \wedge {\sim}x) \vee {\sim}\nabla{\sim}{\sim}\nabla(x \wedge {\sim}x) \approx_{2.5.5} {\sim}{\sim}\nabla(x \wedge {\sim}x) \vee {\sim}\nabla{\sim}{\sim}\nabla(x \wedge {\sim}x) \approx_{\mathbf{(DM1)}} \nabla(x \wedge {\sim}x) \vee {\sim}\nabla\nabla(x \wedge {\sim}x) \approx_{2.5.4} \nabla(x \wedge {\sim}x) \vee {\sim}\nabla(x \wedge {\sim}x) \approx_{\mathbf{(IS4)}} \top$.

(PP2) $\circ x \approx_{def} {\sim}\nabla(x \wedge {\sim}x) \approx_{\mathbf{(DM1)}} {\sim}\nabla({\sim}{\sim}x \wedge {\sim}x) \approx_{def} \circ{\sim}x$.

(PP3) $\circ\top \approx_{def} {\sim}\nabla(\top \wedge {\sim}\top) \approx {\sim}\nabla(\top \wedge \bot) \approx {\sim}\nabla\bot \approx_{\mathbf{(IS1)}} {\sim}\bot \approx \top$.

(PP4) $\circ x \wedge ({\sim}x \wedge x) \approx_{def} {\sim}\nabla(x \wedge {\sim}x) \wedge ({\sim}x \wedge x) \approx ({\sim}\nabla(x \wedge {\sim}x) \wedge {\sim}x) \wedge x \approx_{2.5.3} {\sim}\nabla x \wedge x \approx_{2.5.2} \bot$.

(PP5) $\circ(x \wedge y) \approx_{def} {\sim}\nabla((x \wedge y) \wedge {\sim}(x \wedge y)) \approx_{\mathbf{(IS3)}} {\sim}\nabla(x \wedge y) \vee {\sim}\nabla{\sim}(x \wedge y) \approx_{\mathbf{(IS3)}} ({\sim}\nabla x \vee {\sim}\nabla y) \vee {\sim}\nabla{\sim}(x \wedge y) \approx_{2.5.6} ({\sim}\nabla x \vee {\sim}\nabla y) \vee ({\sim}\nabla{\sim}x \wedge {\sim}\nabla{\sim}y) \approx ({\sim}\nabla x \vee {\sim}\nabla y \vee {\sim}\nabla{\sim}x) \wedge ({\sim}\nabla x \vee {\sim}\nabla y \vee {\sim}\nabla{\sim}y) \approx_{2.5.3} ({\sim}\nabla x \vee ({\sim}\nabla(y \wedge {\sim}y) \wedge {\sim}y) \vee {\sim}\nabla{\sim}x) \wedge ({\sim}\nabla y \vee ({\sim}\nabla(x \wedge {\sim}x) \wedge {\sim}x) \vee {\sim}\nabla{\sim}y) \approx_{\mathbf{(IS3)}} ({\sim}\nabla(x \wedge {\sim}x) \vee ({\sim}\nabla(y \wedge {\sim}y) \wedge {\sim}y)) \wedge ({\sim}\nabla(y \wedge {\sim}y) \vee ({\sim}\nabla(x \wedge {\sim}x) \wedge {\sim}x)) \approx_{def} (\circ x \vee (\circ y \wedge \neg y)) \wedge (\circ y \vee (\circ x \wedge \neg x)) \approx (\circ x \vee \circ y) \wedge (\circ x \vee {\sim}y) \wedge (\circ y \vee {\sim}x)$. $\square$

Theorem 3.5. *Let* $\mathbf{A} \in \mathbb{PP}$*. Then the* Σ^{IS}*-algebra* $\mathbf{A}^\nabla$ *having the same* Σ^{DM}*-reduct of* $\mathbf{A}$ *and with* $\nabla^{\mathbf{A}^\nabla}$ *being the operation induced by* ${\sim}\circ x \vee x$ *on* $\mathbf{A}$ *is an IS-algebra.*

Proof. We must show that $\mathbf{A}^\nabla$ satisfies each of the characteristic equations of IS-algebras:

(IS1) $\nabla\bot \approx_{def} {\sim}\circ\bot \vee \bot \approx {\sim}\circ\bot \approx {\sim}\circ{\sim}\top \approx_{\mathbf{(PP2)}} {\sim}\circ\top \approx_{\mathbf{(PP3)}} {\sim}\top \approx \bot$.

(IS2) By absorption and commutativity of $\vee$, we have $x \wedge \nabla x \approx_{def} x \wedge ({\sim}\circ x \vee x) \approx x$.

(IS3) $\nabla(x \wedge y) \approx_{def} {\sim}\circ(x \wedge y) \vee (x \wedge y) \approx_{\mathbf{(PP5)}} ({\sim}\circ x \wedge {\sim}\circ y) \vee ({\sim}\circ x \wedge y) \vee ({\sim}\circ y \wedge x) \vee (x \wedge y) \approx ({\sim}\circ x \vee x) \wedge ({\sim}\circ y \vee y) \approx_{def} \nabla x \wedge \nabla y$.

(IS4) $\sim\nabla x\wedge\nabla x\approx_{def}\sim(\sim\circ x\vee x)\wedge(\sim\circ x\vee x)\approx_{\textbf{(DM2)}}(\circ x\wedge\sim x)\wedge(\sim\circ x\vee x)\approx(\circ x\wedge\sim x\wedge\sim\circ x)\vee(\circ x\wedge\sim x\wedge x)\approx_{\textbf{(PP4)}}(\circ x\wedge\sim x\wedge\sim\circ x)\vee\bot\approx\circ x\wedge\sim x\wedge\sim\circ x\approx\circ x\wedge\sim x\wedge\sim\circ x\wedge\top\approx_{\textbf{(PP1)}}\circ x\wedge\sim x\wedge\sim\circ x\wedge\circ\circ x\approx_{\textbf{(PP4)}}\bot\wedge\sim x\approx\bot$. □

Theorem 3.6. *Given $\mathbf{A}\in\mathbb{IS}$ and $\mathbf{B}\in\mathbb{PP}$, we have $(\mathbf{A}^\circ)^\nabla=\mathbf{A}$ and $\left(\mathbf{B}^\nabla\right)^\circ=\mathbf{B}$.*

Proof. In order to prove that $(\mathbf{A}^\circ)^\nabla=\mathbf{A}$, it is enough to show that $\sim(\sim\nabla(x\wedge\sim x))\vee x\approx\nabla x$ holds in $\mathbf{A}$, that is, the operation induced by the term $((\nabla x)^\circ)^\nabla$ coincides with the interpretation of ∇. By the fact that $\nabla x\vee x\approx\nabla x$, we have $\sim(\sim\nabla(x\wedge\sim x))\vee x\approx_{\textbf{(DM1)}}\nabla(x\wedge\sim x)\vee x\approx_{\textbf{(IS3)}}(\nabla x\wedge\nabla\sim x)\vee x\approx(\nabla x\vee x)\wedge(\nabla\sim x\vee x)\approx_{\text{Lemma 2.5.1}}(\nabla x\vee x)\wedge\top\approx\nabla x\vee x\approx\nabla x$. Similarly, for proving $\left(\mathbf{B}^\nabla\right)^\circ=\mathbf{B}$, it is enough to show that $\left((\circ x)^\nabla\right)^\circ$ induces an operation that coincides with the interpretation of $\circ$, which amounts to proving that $\sim(\sim\circ(x\wedge\sim x)\vee(x\wedge\sim x))\approx\circ x$ holds in $\mathbf{B}$. Then, we have $\sim(\sim\circ(x\wedge\sim x)\vee(x\wedge\sim x))\approx_{\textbf{(DM2)}}\circ(x\wedge\sim x)\wedge(\sim x\vee x)\approx_{\textbf{(PP5)}}(\circ x\vee\circ\sim x)\wedge(\circ x\vee x)\wedge(\circ\sim x\vee\sim x)\wedge(\sim x\vee x)\approx_{\textbf{(PP2)}}(\circ x\vee\circ x)\wedge(\circ x\vee x)\wedge(\circ x\vee\sim x)\wedge(\sim x\vee x)\approx\circ x\wedge(\sim x\vee x)\approx_{\text{Lemma 3.3.3}}\circ x$. □

By inspecting the interpretation induced by the definition of $\circ$ in terms of ∇ given in Theorem 3.4, one may easily check the following result.

Proposition 3.7. $\mathbf{PP}_i=\mathbf{IS}_i^\circ$*, for all* $2\leq i\leq 6$.

From the equivalence just presented and a similar result for IS-algebras [20], we may now conclude that the variety of PP-algebras is generated by $\mathbf{PP_6}$:

Proposition 3.8. $\mathbb{PP}=\mathbb{V}(\{\mathbf{PP_6}\})$.

Proposition 3.9. *For all $\Phi\cup\{\psi\}\subseteq L_{\Sigma^{\mathsf{DM}}}(P)$, $\Phi\vdash_{\mathcal{CL}}\psi$ iff $\Phi\vdash^{\top}_{\mathbf{PP}_2}\psi$.*

Proof. Follows from the clear isomorphism between $\mathbf{PP}_2$ and $\mathbf{B}_2$. □

Let $\mathcal{PP}_\leq$ be the order-preserving logic induced by $\mathbb{PP}$. We will use the following auxiliary results together with an analogous result for $\mathcal{IS}_\leq$ [20] to prove that $\mathcal{PP}_\leq$ is characterised by a single 6-valued logical matrix.

Lemma 3.10. *Given $\mathbf{A}\in\mathbb{IS}$ and $\mathbf{B}\in\mathbb{PP}$,*

1. *if $h\in\mathsf{Hom}(\mathbf{L}_{\Sigma^{\mathrm{IS}}}(P),\mathbf{A})$, then $h\left((\varphi^\circ)^\nabla\right)=h(\varphi)$ for all $\varphi\in L_{\Sigma^{\mathrm{IS}}}(P)$;*
2. *if $h\in\mathsf{Hom}(\mathbf{L}_{\Sigma^{\mathsf{PP}}}(P),\mathbf{B})$, then $h\left(\left(\varphi^\nabla\right)^\circ\right)=h(\varphi)$ for all $\varphi\in L_{\Sigma^{\mathsf{PP}}}(P)$;*
3. *if $h\in\mathsf{Hom}(\mathbf{L}_{\Sigma^{\mathrm{IS}}}(P),\mathbf{A})$, then the mapping $h^\circ\in\mathsf{Hom}(\mathbf{L}_{\Sigma^{\mathsf{PP}}}(P),\mathbf{A}^\circ)$ such that $h^\circ(p)=h(p)$ for all $p\in P$ satisfies $h^\circ(\varphi^\circ)=h(\varphi)$ for all $\varphi\in L_{\Sigma^{\mathrm{IS}}}(P)$;*
4. *if $h\in\mathsf{Hom}(\mathbf{L}_{\Sigma^{\mathsf{PP}}}(P),\mathbf{B})$, then the mapping $h^\nabla\in\mathsf{Hom}(\mathbf{L}_{\Sigma^{\mathrm{IS}}}(P),\mathbf{B}^\nabla)$ such that $h^\nabla(p)=h(p)$ for all $p\in P$ satisfies $h^\nabla(\varphi^\nabla)=h(\varphi)$ for all $\varphi\in L_{\Sigma^{\mathsf{PP}}}(P)$.*

Proof. We will first discuss the proofs of items 1 and 3, which may then be easily adapted, respectively, for proving items 2 and 4. Both proofs are by structural induction on the set of formulas. Starting with 1, when $\varphi\in P$, the result trivially holds, as propositional variables are not affected by translations. In case $\varphi=\nabla\psi$, if $h((\psi^\circ)^\nabla)=h(\psi)$, we will have $h((\varphi^\circ)^\nabla))=h(((\nabla\psi)^\circ)^\nabla)$. From the argument in the proof of Theorem 3.6, we know that $((\nabla\psi)^\circ)^\nabla$ and $\nabla\psi$ induce the same operation on $\mathbf{A}$, thus $h(((\nabla\psi)^\circ)^\nabla)=\nabla(h(\psi))=h(\varphi)$. The proof is analogous for the cases of $\wedge,\vee,\sim,\top$ and $\bot$. Now, for item 3, the base case is again obvious, and, in case $\varphi=\nabla\psi$, we have $h^\circ((\nabla\psi)^\circ)=h^\circ(\sim\circ\psi^\circ\vee\psi^\circ)=\sim^{\mathbf{A}^\circ}\circ^{\mathbf{A}^\circ}h^\circ(\psi^\circ)\vee^{\mathbf{A}^\circ}h^\circ(\psi^\circ)$, and, by the induction hypothesis, the latter is equal to $\sim^{\mathbf{A}^\circ}\circ^{\mathbf{A}^\circ}h(\psi)\vee^{\mathbf{A}^\circ}h(\psi)$; this is the same as $h(\nabla\psi)$ in $(\mathbf{A}^\circ)^\nabla$, which coincides with $\mathbf{A}$ by Theorem 3.6. The proof is again analogous for $\wedge,\vee,\sim,\top$ and $\bot$. □

Proposition 3.11. *In what follows, let* $\mathbf{A} \in \mathbb{PP}$. *Then,*

1. $\Phi \vdash_{\langle \mathbf{A}, D\rangle} \psi$ *iff* $\Phi^\nabla \vdash_{\langle \mathbf{A}^\nabla, D\rangle} \psi^\nabla$*, where* $\langle \mathbf{A}, D\rangle$ *is a* Σ^{PP}*-matrix*
2. $\Phi \vdash_{\mathcal{M}} \psi$ *iff* $\Phi^\nabla \vdash_{\mathcal{M}^\nabla} \psi^\nabla$*, for* $\mathcal{M} = \{\langle \mathbf{A}, D\rangle \mid \mathbf{A} \in \mathbb{PP}\}$
3. $\Phi \vdash_{\mathcal{PP}_\leq} \psi$ *iff* $\Phi^\nabla \vdash_{\mathcal{IS}_\leq} \psi^\nabla$
4. $\Phi \vdash^{\top}_{\mathbb{PP}} \psi$ *iff* $\Phi^\nabla \vdash^{\top}_{\mathbb{IS}} \psi^\nabla$

Proof. We start by proving item 1. From the left to the right, suppose that there is a valuation $h \in \mathsf{Hom}\left(\mathbf{L}_{\Sigma^{\mathsf{IS}}}(P), \mathbf{A}^\nabla\right)$ such that $h(\Phi^\nabla) \subseteq D$ while $h(\psi^\nabla) \notin D$. By items 2 and 3 of Lemma 3.10, there is a valuation $h^\circ \in \mathsf{Hom}(\mathbf{L}_{\Sigma^{\mathsf{PP}}}(P), (\mathbf{A}^\nabla)^\circ) = \mathsf{Hom}(\mathbf{L}_{\Sigma^{\mathsf{PP}}}(P), \mathbf{A})$ such that $h^\circ((\Phi^\nabla)^\circ) = h^\circ(\Phi)$ and $h^\circ((\Phi^\nabla)^\circ) = h(\Phi^\nabla)$, thus $h^\circ(\Phi) = h(\Phi^\nabla) \subseteq D$. Similarly, we may conclude that $h^\circ(\psi) \notin D$, and we are done. The other direction is similar, but using item 4 of Lemma 3.10. Item 2, above, is a clear consequence of item 1, and items 3 and 4 follow directly from items 1 and 2, respectively. □

Theorem 3.12. $\mathcal{PP}_\leq \;= \vdash_{\langle \mathbf{PP_6}, \uparrow \mathbf{b}\rangle}$.

Proof. By Proposition 3.11 and the fact that $\vdash_{\mathcal{IS}_\leq}$ is characterised by the matrix $\langle \mathbf{IS_6}, \uparrow \mathbf{b}\rangle$, we have $\Phi \vdash_{\langle \mathbf{PP_6}, \uparrow \mathbf{b}\rangle} \psi$ iff $\Phi^\nabla \vdash_{\langle \mathbf{IS_6}, \uparrow \mathbf{b}\rangle} \psi^\nabla$ iff $\Phi^\nabla \vdash_{\mathcal{IS}_\leq} \psi^\nabla$ iff $\Phi \vdash_{\mathcal{PP}_\leq} \psi$. □

We may explore the term-equivalence just presented to prove other important facts about $\mathcal{PP}_\leq$. For the definitions of full self-extensionality, protoalgebraizability and algebraizability that appear in the next result, we refer the reader to [16, Definitions 5.25, 6.1 and 3.11, resp.].

Proposition 3.13. $\mathcal{PP}_\leq$ *is fully self-extensional and non-protoalgebraic (hence non-algebraizable).*

Proof. Follows from [20, Prop. 4.2], [16, Theorem 7.18, item 4], and the term-equivalence of $\mathbb{IS}$ with $\mathbb{PP}$ given by Theorem 3.6. □

Proposition 3.14. $\vdash^{\top}_{\mathbb{PP}} = \vdash^{\top}_{\mathbb{V}(\mathbf{PP}_3)} = \vdash_{\langle \mathbf{PP}_3, \{\hat{\mathbf{t}}\}\rangle}$.

Proof. It is clear that $\vdash^{\top}_{\mathbb{PP}} \subseteq \vdash^{\top}_{\mathbb{V}(\mathbf{PP}_3)} \subseteq \vdash_{\langle \mathbf{PP}_3, \{\hat{\mathbf{t}}\}\rangle}$. The result then follows because $\vdash^{\top}_{\mathbb{PP}} = \vdash_{\langle \mathbf{PP}_3, \{\hat{\mathbf{t}}\}\rangle}$, as $\Phi \vdash_{\langle \mathbf{PP}_3, \{\hat{\mathbf{t}}\}\rangle} \psi$ iff $\Phi^\nabla \vdash_{\langle \mathbf{PP}_3^\nabla, \{\hat{\mathbf{t}}\}\rangle} \psi^\nabla$ (by Proposition 3.10) iff $\Phi^\nabla \vdash_{\langle \mathbf{IS}_3, \{\hat{\mathbf{t}}\}\rangle} \psi^\nabla$ (because $\mathbf{IS}_3 = \mathbf{PP}_3^\nabla$) iff $\Phi^\nabla \vdash^{\top}_{\mathbb{IS}} \psi^\nabla$ (by [20, Prop. 4.5]) iff $\Phi \vdash^{\top}_{\mathbb{PP}} \psi$ (by Proposition 3.10). □

We now present a recipe for constructing a perfect paradefinite algebra by endowing a De Morgan algebra with a perfection operator. This is of particular interest for an investigation on **LFI**s and **LFU**s when the De Morgan algebra at hand happens not to be Boolean. We will see in the next section how to axiomatise logics induced by PP-algebras produced through this recipe, starting from a calculus for the logic induced by a De Morgan algebra given as input.

Definition 3.15. *Let* $\mathbf{A}$ *be a* Σ^{DM}*-algebra. Given* $\hat{\mathbf{f}}, \hat{\mathbf{t}} \notin A$*, we define the* Σ^{PP}*-algebra* $\mathbf{A}^\circ := \langle A \cup \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}, \cdot^{\mathbf{A}^\circ}\rangle$ *by letting*

$$a \wedge^{\mathbf{A}^\circ} b := \begin{cases} a \wedge^{\mathbf{A}} b & \text{if } a,b \in A \\ \hat{\mathbf{t}} & \text{if } a = b = \hat{\mathbf{t}} \\ \hat{\mathbf{f}} & \text{if } a = \hat{\mathbf{f}} \text{ or } b = \hat{\mathbf{f}} \\ c & \text{if } \{a,b\} = \{\hat{\mathbf{t}}, c\} \text{ with } c \in A \end{cases} \qquad a \vee^{\mathbf{A}^\circ} b := \begin{cases} a \vee^{\mathbf{A}} b & \text{if } a,b \in A \\ \hat{\mathbf{f}} & \text{if } a = b = \hat{\mathbf{f}} \\ \hat{\mathbf{t}} & \text{if } a = \hat{\mathbf{t}} \text{ or } b = \hat{\mathbf{t}} \\ c & \text{if } \{a,b\} = \{\hat{\mathbf{f}}, c\} \text{ with } c \in A \end{cases}$$

$$\sim^{\mathbf{A}^\circ} a := \begin{cases} \sim^{\mathbf{A}} a & \text{if } a \in A \\ \hat{\mathbf{f}} & \text{if } a = \hat{\mathbf{t}} \\ \hat{\mathbf{t}} & \text{if } a = \hat{\mathbf{f}} \end{cases} \qquad \circ^{\mathbf{A}^\circ} a := \begin{cases} \hat{\mathbf{t}} & \text{if } a = \hat{\mathbf{f}} \text{ or } a = \hat{\mathbf{t}} \\ \hat{\mathbf{f}} & \text{otherwise} \end{cases}$$

$$\bot^{\mathbf{A}^\circ} := \hat{\mathbf{f}} \qquad \top^{\mathbf{A}^\circ} := \hat{\mathbf{t}}$$

In addition, we define the Σ^{IS}-algebra $\mathbf{A}^\nabla := \langle A \cup \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}, \cdot^{\mathbf{A}^\nabla} \rangle$ interpreting the connectives in Σ^{DM} as above, while letting $\nabla^{\mathbf{A}^\nabla} a := \hat{\mathbf{f}}$ if $a = \hat{\mathbf{f}}$ and $\nabla^{\mathbf{A}^\nabla} a := \hat{\mathbf{t}}$ otherwise (cf. [20]*).*

Proposition 3.16. *If $\mathbf{A}$ is a De Morgan algebra, then $\mathbf{A}^\circ$ is a PP-algebra.*

Example 3.17. *Comparing Figure* 1a *with Figure* 1b*, we see that $\mathbf{IS}_6$, $\mathbf{IS}_5$ and $\mathbf{IS}_4$ coincide, respectively, with $\mathbf{DM}_4^\circ$, $\mathbf{K}_3^\circ$ and $\mathbf{B}_2^\circ$.*

Given a Σ^{DM}-matrix $\mathfrak{M} := \langle \mathbf{A}, D \rangle$, let $\mathfrak{M}^\circ := \langle \mathbf{A}^\circ, D \cup \{\hat{\mathbf{t}}\} \rangle$ be the Σ^{PP}-matrix with the underlying (by Proposition 3.16, perfect paradefinite) algebra $\mathbf{A}^\circ$ given by Definition 3.15. We denote by $\widehat{\mathfrak{M}^\circ}$ the Σ^{DM}-reduct of $\mathfrak{M}^\circ$. Given a class of Σ^{DM}-matrices $\mathcal{M}$, we let $\mathcal{M}^\circ := \{\mathfrak{M}^\circ : \mathfrak{M} \in \mathcal{M}\}$ and $\widehat{\mathcal{M}^\circ} := \{\widehat{\mathfrak{M}^\circ} : \mathfrak{M} \in \mathcal{M}\}$.

Lemma 3.18. *Let $\mathfrak{M}$ be a reduced model of $\mathcal{B}$. Then $\mathfrak{M} \cong \left(\widehat{\mathfrak{M}^\circ}\right)^*$.*

Proof. Let $\mathfrak{M} := \langle \mathbf{A}, D \rangle$ be a reduced model of $\mathcal{B}$. We know from [20, Lemma 4.6] that $\mathfrak{M} \cong \left(\widehat{\mathfrak{M}^\nabla}\right)^*$, and clearly $\widehat{\mathfrak{M}^\nabla}$ and $\widehat{\mathfrak{M}^\circ}$ are isomorphic matrices under the identity mapping on $A \cup \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$, and so are their reductions. □

Corollary 3.19. *Where $\mathfrak{M}$ is a reduced model of $\mathcal{B}$, we have $\rhd_{\mathfrak{M}} = \rhd_{\widehat{\mathfrak{M}^\circ}}$ and $\vdash_{\mathfrak{M}} = \vdash_{\widehat{\mathfrak{M}^\circ}}$.*

Corollary 3.20. *Where $\mathfrak{M}$ is a reduced model of $\mathcal{B}$, we have that $\rhd_{\mathfrak{M}^\circ}$ is a conservative expansion of $\rhd_{\mathfrak{M}}$ and $\vdash_{\mathfrak{M}^\circ}$ is a conservative expansion of $\vdash_{\mathfrak{M}}$.*

Once $\mathcal{PP}_\leq$ is characterised by the matrix $\langle \mathbf{PP}_6, \uparrow\mathbf{b} \rangle$, which is obtained from the matrix $\langle \mathbf{DM}_4, \uparrow\mathbf{b} \rangle$ by the construction introduced in Definition 3.15, we may then apply Corollary 3.20 and conclude that:

Corollary 3.21. *$\mathcal{PP}_\leq$ is a conservative expansion of $\mathcal{B}$.*

Reduced models of $\mathcal{B}$ are also reduced models of extensions of $\mathcal{B}$. The next result follows from Corollary 3.19 and gives a sufficient condition to conclude that two such reduced models define the same extension of $\mathcal{B}$.

Corollary 3.22. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be reduced models of $\mathcal{B}$. If $\vdash_{\mathfrak{M}_1^\circ} = \vdash_{\mathfrak{M}_2^\circ}$, then $\vdash_{\mathfrak{M}_1} = \vdash_{\mathfrak{M}_2}$.*

Let $\vdash$ be a super-Belnap logic. Define $\vdash^\circ$ as the logic induced by the family $\{\mathfrak{M}^\circ \mid \mathfrak{M} \in \mathsf{Mat}^*(\vdash)\}$.

Lemma 3.23. *Let $\vdash$ be a super-Belnap logic. Then $\vdash^\circ$ is a conservative expansion of $\vdash$.*

Proof. Let $\Phi \cup \{\psi\} \subseteq L_{\Sigma^{\mathsf{DM}}}(P)$ and suppose that $\Phi \nvdash \psi$. Then, there is a reduced model $\mathfrak{M} \in \mathsf{Mat}^*(\vdash)$ such that $\Phi \nvdash_{\mathfrak{M}} \psi$. By Corollary 3.20, we conclude that $\Phi \nvdash_{\mathfrak{M}^\circ} \psi$, thus $\Phi \nvdash^\circ \psi$. □

Corollary 3.24. *Let $\vdash_1$ and $\vdash_2$ be super-Belnap logics. Then $\vdash_1 \subseteq \vdash_2$ iff $\vdash_1^\circ \subseteq \vdash_2^\circ$.*

Proof. From the left to the right, assuming $\vdash_1 \subseteq \vdash_2$ gives that $\mathsf{Mat}^*(\vdash_2) \subseteq \mathsf{Mat}^*(\vdash_1)$, so $(\mathsf{Mat}^*(\vdash_2))^\circ \subseteq (\mathsf{Mat}^*(\vdash_1))^\circ$, which clearly entails that $\vdash_1^\circ \subseteq \vdash_2^\circ$. Conversely, suppose that $\vdash_1^\circ \subseteq \vdash_2^\circ$ and that $\Phi \vdash_1 \psi$. Hence $\Phi \vdash_1^\circ \psi$, and then $\Phi \vdash_2^\circ \psi$, which gives $\Phi \vdash_2 \psi$ by Corollary 3.23. □

Corollary 3.25. *The map given by $\vdash \mapsto \vdash^\circ$ is an embedding (that is, an injective homomorphism) of the lattice of super-Belnap logics into the lattice of extensions of $\mathcal{PP}_\leq$. The latter lattice has (at least) the cardinality of the continuum.*

Proof. By Corollary 3.24 and [26, Theorem 4.13]. □

The next result shows that paradefinite extensions of $\mathcal{B}$, when expanded with $\circ$ in the way we propose, result in logics which are **C**-systems and **D**-systems. This result applies, in particular, to the logic $\mathcal{PP}_\leq$.

Proposition 3.26. *Let $\mathcal{M}$ be a class of reduced models of $\mathcal{B}$ that induces a paradefinite logic. Then the* SET-SET *logic induced by $\mathcal{M}^\circ$ is a **C**-system and a **D**-system.*

Proof. That paradefiniteness is preserved when passing from $\mathcal{M}$ to $\mathcal{M}^\circ$ follows by Corollary 3.20. As it is well-known that the positive fragments of $\mathcal{CL}$ and $\mathcal{B}$ coincide, by taking $\circ$ as the consistency connective and $\sim\circ$ as the determinedness connective, we may straightforwardly use the values $\hat{\mathbf{f}}$ and $\hat{\mathbf{t}}$ to build suitable valuations for showing that the logic induced by $\mathcal{M}^\circ$ is a **C**-system and a **D**-system. □

Corollary 3.27. *$\mathcal{PP}_\leq$ is a **C**-system and a **D**-system.*

A unary connective © constitutes a *classical negation* in a SET-FMLA Σ-logic $\vdash$ when, for all $\varphi, \psi \in L_\Sigma(P)$, we have that (i): $\Phi, \varphi \vdash \psi$ and $\Phi, ©(\varphi) \vdash \psi$ imply $\Phi \vdash \psi$, and (ii): $\varphi, ©(\varphi) \vdash \psi$. In case $\vdash$ has a disjunction, we may equivalently replace (i) by (iii): $\varnothing \vdash \varphi \vee ©(\varphi)$ in this characterization. We next prove that any composite unary connective fails to satisfy both (i) and (iii) in $\mathcal{PP}_\leq$. Since $\mathcal{PP}_\leq$ has a disjunction, this entails that a classical negation is not definable in this logic.

Proposition 3.28. *There is no unary formula $\varphi \in L_{\Sigma^{\mathsf{PP}}}(P)$ such that $p, \varphi(p) \vdash_{\mathcal{PP}_\leq} q$ and $\varnothing \vdash_{\mathcal{PP}_\leq} p \vee \varphi(p)$.*

Proof. Let $\varphi \in L_{\Sigma^{\mathsf{PP}}}(P)$ be a unary formula and suppose that $p, \varphi(p) \vdash_{\mathcal{PP}_\leq} q$ and $\varnothing \vdash_{\mathcal{PP}_\leq} p \vee \varphi(p)$. Then, since $\mathcal{PP}_\leq$ is an order-preserving logic, we have, for all $h \in \mathsf{Hom}(\mathbf{L}_{\Sigma^{\mathsf{PP}}}(P), \mathbf{PP_6})$, that $h(p) \vee^{\mathbf{PP_6}} \varphi^{\mathbf{PP_6}}(h(p)) = \hat{\mathbf{t}}$ (the greatest element of $\mathbf{PP_6}$) and $h(p) \wedge^{\mathbf{PP_6}} \varphi^{\mathbf{PP_6}}(h(p)) = \hat{\mathbf{f}}$ (the least element of $\mathbf{PP_6}$) , which is to say that $\varphi^{\mathbf{PP_6}}(a)$ is a Boolean complement of a, for every element a of $\mathbf{PP_6}$. This is absurd, since, by the definition of $\wedge^{\mathbf{PP_6}}$ and $\vee^{\mathbf{PP_6}}$, only $\hat{\mathbf{t}}$ and $\hat{\mathbf{f}}$ have Boolean complements in $\mathbf{PP_6}$. □

As argued in [23], the ability to recover negation-consistent (resp. negation-determined) reasoning is the most fundamental feature of **LFI**s (resp. **LFU**s). This feature may be expressed in terms of a convenient *Derivability Adjustment Theorem* (DAT) with respect to Classical Logic, which states, in the present case, that classical reasoning may be fully recovered as long as premises restoring the lost 'perfection' and establishing the 'classicality' of a certain set of formulas are available. The result presented below is a DAT that applies to any super-Belnap logic extended with the perfection operator $\circ$ considered in this paper. As a corollary, we will, in particular, have a DAT for the logic $\mathcal{PP}_\leq$.

Theorem 3.29. *Let $\mathcal{M}$ be a class of reduced models of $\mathcal{B}$. Then, for all $\Phi, \Psi \subseteq L_{\Sigma^{\mathsf{DM}}}(P)$, we have*

$$\Phi \rhd_{\mathcal{CL}} \Psi \ \textit{iff} \ \Phi, \circ p_1, \ldots, \circ p_n \rhd_{\mathcal{M}^\circ} \Psi,$$

with $\{p_1, \ldots, p_n\} = \mathsf{props}(\Phi \cup \Psi)$.

Proof. Let $\mathcal{M}$ be a class of reduced models of $\mathcal{B}$. Notice that $\langle \mathbf{PP}_2, \{\hat{\mathbf{t}}\}\rangle$ is a submatrix of $\mathfrak{M}^\circ$ for all $\mathfrak{M}^\circ \in \mathcal{M}^\circ$.

From the left to the right, contrapositively, suppose that $\Phi, \circ p_1, \ldots, \circ p_n \blacktriangleright_{\mathcal{M}^\circ} \Psi$. Then, there are $\mathfrak{M}^\circ = \langle \mathbf{A}^\circ, D \cup \{\hat{\mathbf{t}}\}\rangle \in \mathcal{M}^\circ$ and $h \in \mathsf{Hom}(\mathbf{L}_{\Sigma^{\mathsf{PP}}}(P), \mathbf{A}^\circ)$ such that (a) $h(\Phi \cup \{\circ p_1, \ldots, \circ p_n\}) \subseteq D \cup \{\hat{\mathbf{t}}\}$ and (b) $h(\Psi) \subseteq \overline{D} \cup \{\hat{\mathbf{f}}\}$. The interpretation of $\circ$ given in Definition 3.15 and (a) entail that $h(p_i) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ for all $1 \leq i \leq n$. As $\mathbf{PP}_2$ is a subalgebra of $\mathbf{A}^\circ$, we may define an $h' : \{p_1, \ldots, p_n\} \to \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ by setting $h'(p_i) := h(p_i)$; this extends to the full language and, in view of Definition 3.15, agrees with h on the set $\Phi \cup \Psi$. Thus, by (a), $h'(\Phi) \subseteq \{\hat{\mathbf{t}}\}$ (as $\hat{\mathbf{f}} \notin D$), while $h'(\Psi) \subseteq \{\hat{\mathbf{f}}\}$ by (b), meaning that $\Phi \blacktriangleright^{\top}_{\mathbf{PP}_2} \Psi$. Hence, by Proposition 3.9, we have $\Phi \blacktriangleright_{\mathcal{CL}} \Psi$.

From the right to the left, again contrapositively, assume that $\Phi \blacktriangleright_{\mathcal{CL}} \Psi$. Thus, by Proposition 3.9, we have $\Phi \blacktriangleright^{\top}_{\mathbf{PP}_2} \Psi$. Then there is $h \in \mathsf{Hom}(\mathbf{L}_{\Sigma^{\mathsf{PP}}}(P), \mathbf{PP}_2)$ such that $h(\Phi) \subseteq \{\hat{\mathbf{t}}\}$ and $h(\Psi) \subseteq \{\hat{\mathbf{f}}\}$. Notice that, if $\mathfrak{M}^\circ = \langle \mathbf{A}^\circ, D \cup \{\hat{\mathbf{t}}\}\rangle \in \mathcal{M}^\circ$, then we may define $h' : P \to A \cup \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ with $h'(p) = h(p)$, for all $p \in P$. As $\mathbf{PP}_2$ is a subalgebra of $\mathbf{A}^\circ$, h' extends to the full language and agrees with h. Moreover, as $h'(p_i) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$, we have, by Definition 3.15, $h'(\circ p_i) = \hat{\mathbf{t}}$, for all $1 \leq i \leq n$. Hence $h'(\Phi \cup \{\circ p_1, \ldots, \circ p_n\}) \subseteq \{\hat{\mathbf{t}}\}$, while $h'(\Psi) \subseteq \{\hat{\mathbf{f}}\}$. Therefore, $\Phi, \circ p_1, \ldots, \circ p_n \blacktriangleright_{\mathfrak{M}^\circ} \Psi$ for each $\mathfrak{M}^\circ \in \mathcal{M}^\circ$, and, in particular, we obtain $\Phi, \circ p_1, \ldots, \circ p_n \blacktriangleright_{\mathcal{M}^\circ} \Psi$. □

Corollary 3.30. *For all $\Phi \cup \{\psi\} \subseteq L_{\Sigma^{\mathsf{DM}}}(P)$, we have*

$$\Phi \vdash_{\mathcal{CL}} \psi \;\; \textit{iff} \;\; \Phi, \circ p_1, \ldots, \circ p_n \vdash_{\mathcal{PP}_{\leq}} \psi,$$

with $\{p_1, \ldots, p_n\} = \mathsf{props}(\Phi \cup \{\psi\})$.

Proof. Follows from the facts that $\mathbf{DM}_4$ is reduced [15] and $\mathcal{PP}_{\leq}$ is determined by the single matrix $\mathbf{PP}_6$, which coincides with $\mathbf{DM}_4^\circ$. □

4 Axiomatising Logics of De Morgan Algebras Enriched with Perfection

In the first part of this section, we provide a general recipe for producing a symmetrical Hilbert-style calculus for the SET-SET logic determined by any class $\mathcal{M}$ of Σ^{DM}-matrices expanded with the perfection operator $\circ$ according to the mechanism set up in the previous section. Our approach is based on adding some rules governing $\circ$ to a given axiomatization of $\mathcal{M}$, resulting in what we call a *relative axiomatization* of $\mathcal{M}^\circ$ by the added rules with respect to the SET-SET logic induced by $\mathcal{M}$. In the sequel, we will show, for a particular class of matrices, how to turn the given SET-SET relative axiomatizations into SET-FMLA axiomatizations, using the fact proved in [27, Theorem 5.37] that a symmetrical calculus R can be transformed into a SET-FMLA calculus provided that $\vdash_{\mathsf{R}}$ has a disjunction. If R axiomatises a class $\mathcal{M}$ of Σ^{DM}-matrices, a sufficient condition for the latter property to hold is that all members of $\mathcal{M}$ have prime filters as sets of designated values. For this reason, our result will be focused on providing a SET-FMLA Hilbert-style axiomatization for logics induced by classes of Σ^{DM}-matrices whose designated values form prime filters.

4.1 Analyticity-preserving symmetrical calculi

In what follows, if $\rhd_1$ and $\rhd_2$ are SET-SET logics over Σ, we set $\rhd_1 \simeq \rhd_2$ iff $\rhd_1 \cup \{(L_\Sigma(P), \varnothing)\} = \rhd_2 \cup \{(L_\Sigma(P), \varnothing)\}$. It is clear that two logics satisfying this condition induce the same SET-FMLA logic.[2] We

[2]This has been observed by R. Carnap, already in the 1940s [10]. It might seem that extending a logic this way would imply that a semantics characterising the extended logic would have to provide 'models for contradictory formulas'. However, such a

will employ this weaker relation instead of the equality relation to make the results in this section more general and simpler to prove. The first result below provides a generic recipe for axiomatising the SET-SET logic determined by the class $\mathcal{M}^\circ$, assuming we have a calculus R that axiomatises the SET-SET logic determined by $\widehat{\mathcal{M}^\circ}$.

Theorem 4.1. *Let $\mathcal{M}$ be a class of Σ^{DM}-matrices. If $\rhd_{\widehat{\mathcal{M}^\circ}} \simeq \rhd_{\mathsf{R}}$, then $\rhd_{\mathcal{M}^\circ} = \rhd_{\mathsf{R}\cup\mathsf{R}_\circ}$, where $\mathsf{R}_\circ$ consists of the following inference rules:*

$$\frac{}{\circ\bot}\,\mathsf{r}_1 \qquad \frac{}{\circ\top}\,\mathsf{r}_2 \qquad \frac{}{\circ\circ p}\,\mathsf{r}_3 \qquad \frac{\circ p}{\circ{\sim}p}\,\mathsf{r}_4 \qquad \frac{\circ{\sim}p}{\circ p}\,\mathsf{r}_5 \qquad \frac{\circ p}{p,{\sim}p}\,\mathsf{r}_6 \qquad \frac{\circ p, p, {\sim}p}{}\,\mathsf{r}_7$$

$$\frac{\circ p}{\circ(p\wedge q), p}\,\mathsf{r}_8 \qquad \frac{\circ q}{\circ(p\wedge q), q}\,\mathsf{r}_9 \qquad \frac{\circ(p\wedge q), q}{\circ p}\,\mathsf{r}_{10} \qquad \frac{\circ(p\wedge q), p}{\circ q}\,\mathsf{r}_{11} \qquad \frac{\circ p, \circ q}{\circ(p\wedge q)}\,\mathsf{r}_{12} \qquad \frac{\circ(p\wedge q)}{\circ p, \circ q}\,\mathsf{r}_{13}$$

$$\frac{\circ p, \circ q}{\circ(p\vee q)}\,\mathsf{r}_{14} \qquad \frac{\circ(p\vee q)}{\circ p, \circ q}\,\mathsf{r}_{15} \qquad \frac{\circ p, p}{\circ(p\vee q)}\,\mathsf{r}_{16} \qquad \frac{\circ q, q}{\circ(p\vee q)}\,\mathsf{r}_{17} \qquad \frac{\circ(p\vee q)}{\circ p, q}\,\mathsf{r}_{18} \qquad \frac{\circ(p\vee q)}{\circ q, p}\,\mathsf{r}_{19}$$

Proof. Checking the soundness of those rules is routine; we provide only a couple of examples. Let v be an $\mathfrak{M}^\circ$-valuation. The rule r_3 is sound in $\mathfrak{M}^\circ$, given that, $v(\circ\varphi) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$, so we have that $v(\circ\circ\varphi) = \hat{\mathbf{t}}$. On what concerns rule r_8, we have that, if $v(\circ\varphi) = \hat{\mathbf{t}}$, then either (i) $v(\varphi) = \hat{\mathbf{f}}$ or (ii) $v(\varphi) = \hat{\mathbf{t}}$. Soundness is obvious in case (ii). In case (i), $v(\varphi\wedge\psi) = \hat{\mathbf{f}}$, so $v(\circ(\varphi\wedge\psi)) = \hat{\mathbf{t}}$.

For completeness, assume $\Phi \blacktriangleright_{\mathsf{R}_\circ} \Psi$. Then, by cut for sets, there is a partition $\langle T, F\rangle$ of $L_{\Sigma^{\mathsf{PP}}}(P)$ such that $\Phi \subseteq T$ and $\Psi \subseteq F$ and $T \blacktriangleright_{\mathsf{R}_\circ} F$. Note that (by r_3, r_6 and r_7) for each φ, we have either $\circ\varphi \in T$ or ${\sim}\circ\varphi \in T$, but never both. In particular, F is never empty. Also, by r_6 and r_7, if we have $\circ\varphi \in T$, we have either $\varphi \in T$ or ${\sim}\varphi \in T$, but never both. Hence, each φ must belong to exactly one of three cases: (a) ${\sim}\circ\varphi \in T$, (b) $\circ\varphi, \varphi \in T$ or (c) $\circ\varphi, {\sim}\varphi \in T$.

Since $\mathsf{R} \subseteq \mathsf{R}\cup\mathsf{R}_\circ$, we also have $T \blacktriangleright_{\mathsf{R}} F$. From the fact that $\rhd_{\mathsf{R}} \simeq \rhd_{\widehat{\mathfrak{M}}}$ and $F \neq \varnothing$ we know that $T \blacktriangleright_{\widehat{\mathfrak{M}}} F$. We may therefore pick some $\widehat{\mathfrak{M}}$-valuation v, for some $\mathfrak{M} \in \mathcal{M}$, such that $v(T) \subseteq D$ and $v(F) \cap D = \varnothing$. Consider now the mapping $v' : L_{\Sigma^{\mathsf{PP}}}(P) \to \mathfrak{M}^\circ$ defined by:

$$v'(\varphi) := \begin{cases} v(\varphi_i) & \text{if } \varphi = \varphi_1 \wedge \varphi_2 \text{ and } {\sim}\circ\varphi, \circ\varphi_{3-i} \in T \text{ for } i \in \{1,2\} \\ v(\varphi_i) & \text{if } \varphi = \varphi_1 \vee \varphi_2 \text{ and } {\sim}\circ\varphi, \circ\varphi_i \in T \text{ for } i \in \{1,2\} \\ v(\varphi) & \text{if } {\sim}\circ\varphi \in T \\ \hat{\mathbf{t}} & \text{if } \circ\varphi, \varphi \in T \\ \hat{\mathbf{f}} & \text{if } \circ\varphi, {\sim}\varphi \in T \end{cases}$$

We will check that v' is an $\mathfrak{M}^\circ$-valuation:

1. $v'(\circ\varphi) = \circ v'(\varphi)$: If (i) ${\sim}\circ\varphi \in T$ then, by r_3, $\circ\circ\varphi \in T$ (so $v'(\circ\varphi) = \hat{\mathbf{f}}$). Thus $v'(\circ\varphi) = \hat{\mathbf{f}} = \circ(v'(\varphi))$. If (ii) $\circ\varphi, \varphi \in T$, then, by r_3, $\circ\circ\varphi \in T$ (so $v'(\circ\varphi) = \hat{\mathbf{t}}$). So $v'(\circ\varphi) = \hat{\mathbf{t}} = \circ(v'(\varphi))$. Case (iii) is analogous to (ii).

2. $v'({\sim}\varphi) = {\sim}v'(\varphi)$: If (i) ${\sim}\circ{\sim}\varphi \in T$, then, by r_3 and r_7, $\circ{\sim}\varphi \notin T$. Then, by r_4, $\circ\varphi \notin T$. Thus, by r_3 and r_6, ${\sim}\circ\varphi \in T$ (so $v'(\varphi) = v(\varphi)$). So $v'({\sim}\varphi) = v({\sim}\varphi) = {\sim}_{\mathfrak{M}}(v(\varphi)) = {\sim}(v'(\varphi))$. If (ii) $\circ{\sim}\varphi, {\sim}\varphi \in T$, by r_5, $\circ\varphi \in T$ (so $v'(\varphi) = \hat{\mathbf{f}}$)). Then $v'({\sim}\varphi) = \hat{\mathbf{t}} = {\sim}(v'(\varphi))$. Case (iii) is analogous to (ii).

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model, in this case, would be trivial, for it would make all formulas equally true. As argued in [24], this is not the kind of models that a paraconsistent logician is interested upon. This explains, by the way, why our definition of paraconsistency, presented towards the end of Section 2, has been formulated in terms of $p, {\sim}p \blacktriangleright q$ rather than $p, {\sim}p \blacktriangleright \varnothing$.

3. $v'(\varphi \wedge \psi) = v'(\varphi) \wedge v'(\psi)$: If (i) $\sim\circ(\varphi \wedge \psi) \in T$, then, by r_3 and r_7, we have that $\circ(\varphi \wedge \psi) \notin T$. By r_{12}, we have that (a) $\circ\varphi, \circ\psi \notin T$, (b) $\circ\varphi \in T$ and $\circ\psi \notin T$ or (c) $\circ\varphi \notin T$ and $\circ\psi \in T$. So:
   (a) By r_3 and r_6, $\sim\circ\varphi, \sim\circ\psi \in T$ (so $v'(\varphi) = v(\varphi)$ and $v'(\psi) = v(\psi)$). So $v'(\varphi \wedge \psi) = v(\varphi \wedge \psi) = v(\varphi) \wedge_{\mathfrak{M}} v(\psi) = v'(\varphi) \wedge v'(\psi)$.
   (b) By r_3 and r_6, $\sim\circ\psi \in T$ (so $v'(\psi) = v(\psi)$). By r_8, $\varphi \in T$ (so $v'(\varphi) = \hat{\mathbf{t}}$). Therefore $v'(\varphi \wedge \psi) = v(\psi) = v'(\psi) = \hat{\mathbf{t}} \wedge v'(\psi) = v'(\varphi) \wedge v'(\psi)$.
   (c) This case is analogous to the previous one, but now using r_9.

   If (ii) $\circ(\varphi \wedge \psi), \varphi \wedge \psi \in T$, then $\varphi, \psi \in T$. By r_{10} and r_{11}, $\circ\varphi, \circ\psi \in T$. (so $v'(\varphi) = v'(\psi) = \hat{\mathbf{t}}$) hence $v'(\varphi \wedge \psi) = \hat{\mathbf{t}} = v'(\varphi) \wedge v'(\psi)$. If (iii) $\circ(\varphi \wedge \psi), \sim(\varphi \wedge \psi) \in T$, then either $\sim\varphi \in T$ or $\sim\psi \in T$. By r_{13}, we have that (a) $\circ\varphi, \circ\psi \in T$, (b) $\circ\varphi \in T$ and $\circ\psi \notin T$ or (c) $\circ\varphi \notin T$ and $\circ\psi \in T$. So:
   (a) Here, we have that $v'(\varphi) = \hat{\mathbf{f}}$ or $v'(\psi) = \hat{\mathbf{f}}$. So $v'(\varphi \wedge \psi) = \hat{\mathbf{f}} = v'(\varphi) \wedge v'(\psi)$.
   (b) By r_{11} and r_6 $\sim\varphi \in T$ (so $v'(\varphi) = \hat{\mathbf{f}}$). So $v'(\varphi \wedge \psi) = \hat{\mathbf{f}} = v'(\varphi) \wedge v'(\psi)$.
   (c) This case is analogous to the previous one, using r_{10}.
4. $v'(\varphi \vee \psi) = v'(\varphi) \vee v'(\psi)$: analogous to the case of $\wedge$.
5. $v'(\bot) = \hat{\mathbf{f}}$ and $v'(\top) = \hat{\mathbf{t}}$: directly from rules r_1 and r_2. □

Given $\Xi \subseteq L_{\Sigma^{\mathsf{PP}}}(P)$, let $\Xi^\circ := \Xi \cup \{\sim p, \circ p, \sim\circ p\}$. The theorem below shows that the recipe presented above preserves analyticity.

Theorem 4.2. *Let $\mathcal{M}$ be a class of Σ^{DM}-matrices. If R is a Ξ-analytic axiomatization of $\rhd_{\widehat{\mathcal{M}^\circ}}$, then $\mathsf{R} \cup \mathsf{R}_\circ$ is a Ξ°-analytic axiomatization of $\rhd_{\mathcal{M}^\circ}$.*

Proof. Let $\Upsilon := \mathsf{sub}(\Phi \cup \Psi)$ and $\Lambda := \Upsilon_{\Xi^\circ}$. Assume that $\Phi \blacktriangleright^{\Lambda}_{\mathsf{R} \cup \mathsf{R}_\circ} \Psi$. Then, by cut for sets, there is a partition $\langle T, F \rangle$ of Λ such that $\Phi \subseteq T$ and $\Psi \subseteq F$ and $T \blacktriangleright^{\Lambda}_{\mathsf{R} \cup \mathsf{R}_\circ} F$. Since $\mathsf{R} \subseteq \mathsf{R} \cup \mathsf{R}_\circ$, we also have $T \blacktriangleright^{\Lambda}_{\mathsf{R}} F$. From the fact that R axiomatises $\rhd_{\widehat{\mathcal{M}^\circ}}$ and $F \neq \varnothing$ we know that $T \blacktriangleright_{\widehat{\mathcal{M}^\circ}} F$. We may therefore pick $v \in \mathsf{Hom}_{\Sigma^{\mathsf{DM}}}(\mathbf{L}_{\Sigma^{\mathsf{PP}}}(P), \widehat{\mathfrak{M}^\circ})$, for some $\mathfrak{M} \in \mathcal{M}$, such that $v(T) \subseteq D$ and $v(F) \cap D = \varnothing$. Since, for each $\varphi \in \Upsilon$, we have $\sim\varphi, \circ\varphi, \sim\circ\varphi \in \Lambda$, we may use the same construction given in Theorem 4.1 to define a certain mapping $v' : \Upsilon \to \mathfrak{M}^\circ$. That v' respects all the connectives follows from the fact that in the proof of Theorem 4.1 we only used instances of the rules employing formulas present in Λ. This, together with the fact that Υ is closed under subformulas, implies that v' is a partial $\mathfrak{M}^\circ$-valuation. Hence, v' may be extended to a total $\mathfrak{M}^\circ$-valuation, witnessing the fact that $\Phi \blacktriangleright_{\mathcal{M}^\circ} \Psi$, thus concluding the proof. □

From Theorem 3.15 and Theorem 4.2, it follows that:

Corollary 4.3. *Let $\mathcal{S} := \{p, \sim p\}$. The calculus presented in Example* 2.8 *together with the rules of* $\mathsf{R}_\circ$ *is an $\mathcal{S}^\circ$-analytic axiomatization of $\mathcal{PP}_{\leq}$.*

As explained in [21], analytic calculi as those we have been discussing are associated to a proof-search algorithm and an algorithm for searching for countermodels, and consequently to a decision procedure for its corresponding SET-SET logic. Briefly, if we want to know whether $\Phi \rhd_{\mathsf{R}} \Psi$, where R is a Ξ-analytic symmetrical calculus, obtaining a proof when the answer is positive and a countermodel otherwise, we may attempt to build a derivation in the following way: start from a single node labelled with Φ and search for a rule instance of R not used in the same branch with formulas in the set Υ^{Ξ} whose premises are in Φ. If there is one, expand that node by creating a child node labelled with $\Phi \cup \{\varphi\}$ for each formula φ in the succedent of the chosen rule instance and repeat this step for each new node. In case it fails in finding a rule instance for applying to some node, we may conclude that no proof exists, and from each non-Ψ-closed branch we may extract a countermodel. In case every branch eventually gets Ψ-closed, the resulting tree is a proof of the desired statement. The next example illustrates how this works.

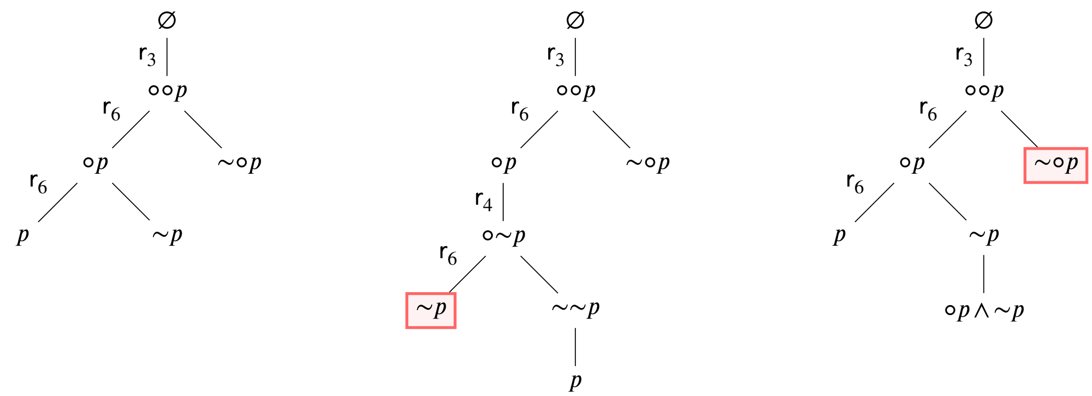

Figure 3:
Outputs of the proof-search and of the countermodel-search algorithm induced by our analytic symmetrical calculus, witnessing that $\varnothing \rhd_{\mathsf{R}\cup\mathsf{R}^\circ} p, \sim p, \sim\circ p$; that $\varnothing \blacktriangleright_{\mathsf{R}_{\mathcal{B}}\cup\mathsf{R}^\circ_{\mathcal{B}}} p, \sim\circ p$ and that $\varnothing \blacktriangleright_{\mathsf{R}_{\mathcal{B}}\cup\mathsf{R}^\circ_{\mathcal{B}}} p, \circ p \wedge \sim p$.

Example 4.4. *The first tree in Figure 3 proves that $\varnothing \rhd_{\mathsf{R}\cup\mathsf{R}^\circ} p, \sim p, \sim\circ p$ in any Ξ°-analytic calculus $\mathsf{R}\cup\mathsf{R}^\circ$ obtained from Theorem 4.1, and may be easily built by the algorithm described above. If we consider the calculus $\mathsf{R}_{\mathcal{B}}$ given in Example 2.8, the second tree in the same figure shows an output of the described algorithm when we search for a countermodel witnessing $\varnothing \blacktriangleright_{\mathsf{R}_{\mathcal{B}}\cup\mathsf{R}^\circ_{\mathcal{B}}} p, \sim\circ p$. In this tree, the leftmost branch is a non-Ψ-closed branch for which no rule instance based only on subformulas of $\Phi\cup\Psi$ and not used yet in the same branch is available. This implies that, for $\Theta = \{\circ\circ p, \circ p, \circ\sim p, \sim p\}$, which are the formulas in the leaf of this non-Ψ-closed branch, we have $\Theta \blacktriangleright_{\mathsf{R}_{\mathcal{B}}\cup\mathsf{R}^\circ_{\mathcal{B}}} \Upsilon^{\Xi^\circ}\backslash\Theta$. As the semantical counterpart of this calculus is the matrix $\langle \mathbf{PP}_6, \uparrow\mathbf{b}\rangle$, a valuation v such that $v(\Theta) \subseteq \uparrow\mathbf{b}$ necessarily sets $v(p) = \hat{\mathbf{f}}$, since $\{\sim p, \circ p\} \subseteq \Theta$. A similar situation occurs in the third tree, which constitutes evidence for $\varnothing \blacktriangleright_{\mathsf{R}_{\mathcal{B}}\cup\mathsf{R}^\circ_{\mathcal{B}}} p, \circ p \wedge \sim p$, meaning that the pseudo-complement given by $\neg x := \circ x \wedge \sim x$ is non-implosive and, thus, not a classical negation in $\mathcal{PP}_{\leq}$. (This is not surprising, in view of Proposition 3.28, but it is worth contrasting this with what happens in many other **LFI**s [22], in which the latter definition of $\neg$ does correspond to a classical negation.)*

4.2 Set-Fmla Hilbert-style calculi for logics of De Morgan algebras with prime filters

We may extend the recipe given in Theorem 4.1, which delivers a symmetrical Hilbert-style calculus, to provide a Set-Fmla Hilbert-style calculus for the class $\mathcal{M}^\circ$ when $\mathcal{M}$ itself is axiomatised by a Set-Fmla Hilbert-style calculus. Before showing how, we will define a collection of such conventional Hilbert-style inference rules associated to a given collection of symmetrical rules. In what follows, when $\Phi = \{\varphi_1, \ldots, \varphi_n\} \subseteq L_\Sigma(P)$ $(n \geq 1)$, let $\bigvee\Phi := (\ldots(\varphi_1 \vee \varphi_2) \vee \ldots) \vee \varphi_n$. Also, let $\Phi \vee \psi := \{\varphi \vee \psi \mid \varphi \in \Phi\}$.

Definition 4.5. *Let R be a symmetrical calculus. Define the set $\mathsf{R}^\vee := \left\{ \frac{p}{p\vee q}, \frac{p\vee q}{q\vee p}, \frac{p\vee(q\vee r)}{(p\vee q)\vee r} \right\} \cup \left\{ \mathsf{r}^\vee \mid \mathsf{r} \in \mathsf{R} \right\}$ where $\mathsf{r}^\vee$ is $\frac{\varnothing}{\varphi}$ if $\mathsf{r} = \frac{\varnothing}{\varphi}$, $\frac{\Phi\vee p_0}{(\bigvee\Psi)\vee p_0}$ if $\mathsf{r} = \frac{\Phi}{\Psi}$, and $\frac{\Phi\vee p_0}{p_0}$ if $\mathsf{r} = \frac{\Phi}{\varnothing}$, where p_0 is a propositional variable not occurring in the rules that belong to R.*

The next result states that, when R is the calculus given by Theorem 4.1, the calculus $\mathsf{R}^\vee$ is the Set-Fmla Hilbert-style calculus we are looking for.

Theorem 4.6. *Let $\mathcal{M}$ be a class of Σ^{DM}-matrices whose designated sets are prime filters, and let R be a* Set-Fmla *Hilbert-style calculus. If $\vdash_{\mathsf{R}} = \vdash_{\mathcal{M}} = \vdash_{\widehat{\mathcal{M}^\circ}}$, then $\vdash_{(\mathsf{R}\cup\mathsf{R}_\circ)^\vee} = \vdash_{\mathcal{M}^\circ}$.*

Proof. If $\vdash_{\mathsf{R}} = \vdash_{\widehat{\mathcal{M}^\circ}}$ then $\rhd_{\mathsf{R}} \simeq \rhd_{\widehat{\mathcal{M}^\circ}}$, so, by Theorem 4.1, we have that $\rhd_{\mathcal{M}^\circ} = \rhd_{\mathsf{R} \cup \mathsf{R}_\circ}$, and thus $\vdash_{\mathcal{M}^\circ} = \vdash_{\mathsf{R} \cup \mathsf{R}_\circ}$. Given that $\mathcal{M}$ is a class of Σ^{DM}-matrices whose designated sets are prime filters and $\vdash_{\mathsf{R}} = \vdash_{\mathcal{M}}$, we have $p \vdash_{\mathsf{R}} p \vee q$, $q \vdash_{\mathsf{R}} p \vee q$ and $p \vee q \vdash_{\mathsf{R}} p, q$. Since $\widehat{\mathcal{M}^\circ}$ preserves the latter inferences, then $p \vdash_{(\mathsf{R} \cup \mathsf{R}_\circ)} p \vee q$, $q \vdash_{(\mathsf{R} \cup \mathsf{R}_\circ)} p \vee q$ and $p \vee q \vdash_{(\mathsf{R} \cup \mathsf{R}_\circ)} p, q$. The latter statements guarantee that $\vdash_{\mathsf{R}}$ has a disjunction, so by [27, Theorem 5.37] we have that $\vdash_{\mathsf{R} \cup \mathsf{R}_\circ} = \vdash_{(\mathsf{R} \cup \mathsf{R}_\circ)^\vee}$. Therefore, $\vdash_{\mathcal{M}^\circ} = \vdash_{(\mathsf{R} \cup \mathsf{R}_\circ)^\vee}$. □

Example 4.7. *Consider a* SET-FMLA *Hilbert calculus that axiomatises $\vdash_{\mathcal{B}}$. Since $\mathcal{B} = \vdash_{\langle \mathbf{DM}_4, \uparrow \mathbf{b} \rangle} = \vdash_{\widehat{\langle \mathbf{DM}_4, \uparrow \mathbf{b} \rangle^\circ}}$ (cf. [20]), we obtain a conventional Hilbert-style axiomatization for $\mathcal{PP}_{\leq} = \vdash_{\langle \mathbf{DM}_4, \uparrow \mathbf{b} \rangle^\circ} = \vdash_{\langle \mathbf{PP}_6, \uparrow \mathbf{b} \rangle}$ by adding to that calculus the $\mathsf{R}^\vee_\circ$ rules. We illustrate, below, with some of the resulting rules:*

$$\frac{\circ p \vee r}{(p \vee {\sim} p) \vee r}\,\mathsf{r}^\vee_6 \qquad \frac{\circ p \vee r, p \vee r, {\sim} p \vee r}{r}\,\mathsf{r}^\vee_7 \qquad \frac{\circ p \vee r}{(\circ(p \wedge q) \vee p) \vee r}\,\mathsf{r}^\vee_8 \qquad \frac{\circ p \vee r, p \vee r}{\circ(p \vee q) \vee r}\,\mathsf{r}^\vee_{16}$$

In what follows we consider a few extensions of $\mathcal{PP}_{\leq}$, illustrating how our methods may be used to axiomatise them. The following result, which is an immediate consequence of Theorem 4.6, shows that Example 4.7 smoothly generalises to all super-Belnap logics.

Proposition 4.8. *Let $\mathcal{M}$ be a class of models of $\mathcal{B}$ whose designated sets are prime filters. If $\vdash_{\mathcal{M}}$ is axiomatised relative to $\mathcal{B}$ by a set R of* SET-FMLA *rules, then $\vdash_{\mathcal{M}^\circ}$ is also axiomatised by R relative to $\mathcal{B}^\circ$.*

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be classes of models of $\mathcal{B}$, such that $\vdash_{\mathcal{M}_1} = \vdash_{\mathcal{M}_2}$. Then $\vdash_{\mathcal{M}_1}$ and $\vdash_{\mathcal{M}_2}$ are axiomatised by the same set R of singleton-succedent rules. Hence, $\vdash_{\mathcal{M}_1^\circ} = \vdash_{\mathcal{M}_2^\circ}$ is axiomatised by the set $\mathsf{R}^\vee_\circ$ defined above. This entails, in particular, that, if a super-Belnap logic $\vdash$ is finitary, then $\vdash^\circ$ (described in Lemma 3.23) is also finitary. Since the lattice of super-Belnap logics contains continuum many finitary logics [25, Corollary 8.17], we obtain the following sharpening of Corollary 3.25:

Proposition 4.9. *There are continuum many finitary extensions of $\mathcal{PP}_{\leq}$.*

The super-Belnap logics (see [1] for further details) considered below for the sake of illustration are the Asenjo-Priest Logic of Paradox $\mathcal{LP}$, the two logics $\mathcal{K}_{\leq}$ and $\mathcal{K}_1$ named after S. C. Kleene, and Classical Logic $\mathcal{CL}$. The next result establishes that each of these logics can be axiomatised, relative to $\mathcal{B}$, by a combination of the rules given below.

$$\frac{p \wedge ({\sim} p \vee q)}{q}\,(\mathrm{DS}) \qquad \frac{(p \wedge {\sim} p) \vee q}{q}\,(K_1) \qquad \frac{(p \wedge {\sim} p) \vee r}{q \vee {\sim} q \vee r}\,(K_{\leq}) \qquad \frac{}{p \vee {\sim} p}\,(\mathrm{EM})$$

Proposition 4.10. ([1, Theorem 3.4])

(i) $\mathcal{LP} = \mathsf{Log}\langle \mathbf{K}_3, \uparrow \mathbf{n} \rangle = \mathcal{B} + (EM)$

(ii) $\mathcal{K}_1 = \mathsf{Log}\langle \mathbf{K}_3, \{\mathbf{t}\} \rangle = \mathcal{B} + (K_1)$

(iii) $\mathcal{K}_{\leq} = \mathsf{Log}\{\langle \mathbf{K}_3, \uparrow \mathbf{n} \rangle, \langle \mathbf{K}_3, \{\mathbf{t}\} \rangle\} = \mathcal{B} + (K_{\leq})$

(iv) $\mathcal{CL} = \mathsf{Log}\langle \mathbf{B}_2, \{\mathbf{t}\} \rangle = \mathcal{B} + (DS) + (EM)$

Theorem 4.11. *For logics above $\mathcal{PP}_{\leq}$ we have the following relative axiomatizations:*

(i) $\mathsf{Log}\langle \mathbf{PP}_5, \uparrow \mathbf{n} \rangle = \mathcal{LP}^\circ = \mathcal{PP}_{\leq} + (EM)$

(ii) $\mathsf{Log}\langle \mathbf{PP}_5, \uparrow \mathbf{t} \rangle = \mathcal{K}_1^\circ = \mathcal{PP}_{\leq} + (\mathcal{K}_1)$

(iii) $\mathsf{Log}\{\langle \mathbf{PP}_5, \uparrow \mathbf{n} \rangle, \langle \mathbf{PP}_5, \uparrow \mathbf{t} \rangle\} = \mathcal{K}_{\leq}^\circ = \mathcal{PP}_{\leq} + (K_{\leq})$

(iv) $\mathsf{Log}\langle \mathbf{PP}_4, \uparrow \mathbf{t} \rangle = \mathcal{CL}^\circ = \mathcal{PP}_{\leq} + (DS) + (EM)$

Proof. This follows directly from Proposition 4.8 and Proposition 4.10, taking into account that $\langle \mathbf{B}_2, \{\mathbf{t}\} \rangle^\circ = \langle \mathbf{PP}_4, \uparrow \mathbf{t} \rangle$, and for $x \in \{\mathbf{t}, \mathbf{n}\}$, we have $\langle \mathbf{K}_3, \uparrow x \rangle^\circ = \langle \mathbf{PP}_5, \uparrow x \rangle$. □

5 Final remarks

We have seen how to endow with a perfection connective logics characterised by matrices having a De Morgan algebraic reduct, offering two possible directions: either by appropriately expanding the corresponding matrices or by adding new rules of inference to an existing Hilbert-style axiomatization. In particular, by so enriching Dunn-Belnap's 4-valued logic we obtain the 6-valued order-preserving logic $\mathcal{PP}_{\leq}$, associated with the variety of expanded algebras, which we called 'perfect paradefinite algebras' (PP-algebras) and proved to be term-equivalent with the variety of involutive Stone algebras (IS-algebras). It is worth mentioning that the one-one correspondence between the latter varieties can be used to introduce back-and-forth functors between an algebraic category associated to the variety of IS-algebras and an algebraic category associated to the variety of PP-algebras defined in the expected way.

By providing a Derivability Adjustment Theorem for $\mathcal{PP}_{\leq}$ and its extensions, we have also shown that Boolean reasoning is fully recovered using De Morgan negation and the perfection operator. Notice, indeed, that adding the equation $\circ x \approx \top$ to a perfect paradefinite algebra, intuitively stating the well-behavedness of each of its elements, results, up to a "linguistic adjustment", in a Boolean algebra.

The equational basis of the variety of PP-algebras studied here was conceived having in mind the expected term-equivalence with the variety of IS-algebras. A natural path for future work is to drop such constraint and study De Morgan algebras (and associated logics) enriched with perfection operators satisfying weaker equations. Within such more general algebraic structures and the logics based thereon, and taking also into account the need to compare $\mathcal{PP}_{\leq}$ with other **C**-systems and **D**-systems in the literature, one might consider having two distinct negations (not necessarily respecting all De Morgan rules) instead of just a single negation that is at once paraconsistent and paracomplete, and possibly also two recovery connectives, as in [13, 18]. Yet another path of investigation would be to consider endowing $\mathcal{PP}_{\leq}$ with an implication connective. A promising starting point for that would be to consider the normal and self-extensional extension of Dunn-Belnap's logic presented in [2], which, when expanded with the perfection connective considered in this paper, would lead to an expansion of $\mathcal{PP}_{\leq}$ that is still self-extensional yet, now, protoalgebraizable. Similar directions have recently been trodden in [14], but considering involutive distributive residuated lattices instead of De Morgan algebras.

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