Proof Search on Bilateralist Judgments over Non-deterministic Semantics

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Abstract. The bilateralist approach to logical consequence maintains that judgments of different qualities should be taken into account in determining what-follows-from-what. We argue that such an approach may be actualized by a two-dimensional notion of entailment induced by semantic structures that also accommodate non-deterministic and partial interpretations, and propose a proof-theoretical apparatus to reason over bilateralist judgments using symmetrical two-dimensional analytical Hilbert-style calculi. We also provide a proof-search algorithm for finite analytic calculi that runs in at most exponential time, in general, and in polynomial time when only rules having at most one formula in the succedent are present in the concerned calculus.

Keywords: Bilateralism · Two-dimensional consequence · Proof search

1 Introduction

The conventional approach to bilateralism in logic treats denial as a primitive judgment, on a par with assertion. One way of allowing these two kinds of judgments to coexist without necessarily allowing them to interfere with one another is by considering a two-dimensional notion of consequence, in which the validity of logical statements obtains in terms of preservation of acceptance along one dimension and of rejection along the other. From a semantical standpoint, as we will show, this idea may be actualized by the canonical notion of entailment induced by a $\mathsf{B}^{\mathsf{PN}}_{\Sigma}$-matrix, a partial non-deterministic logical matrix in which the latter judgments, or cognitive attitudes, are represented by separate collections of truth-values. This will, in particular, allow for distinct Tarskian (one-dimensional, generalized) consequence relations to coinhabit the same logical structure while keeping their interactions disciplined.

A common practice for incorporating bilateralism into a proof formalism consists in attaching to the underlying formulas a force indicator or signal, say + for assertion and − for denial [15,22]. For example, the inference $-(\mathrm{A} \to \mathrm{B}) \vdash +\mathrm{A}$ describes a rule in the bilateral axiomatization of classical logic given in [22], representing the impossibility of, at once, denying $\mathrm{A} \to \mathrm{B}$ while failing to assert A. In [9], a concurrent approach is offered that consists in working with

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a two-dimensional notion of consequence, allowing for the cognitive attitudes of acceptance and rejection to act over two separate logical dimensions and taking their interaction into consideration in determining the meaning of logical connectives and of the statements involving them. The aforementioned inference, for instance, would be expressed by the two-dimensional judgment $\frac{\varnothing}{\varnothing}|\frac{\mathrm{A}}{\mathrm{A}\to\mathrm{B}}$, which is intended to enforce that an agent is not expected to find reasons for rejecting A → B while failing to find reasons for accepting A. From a semantical standpoint, the latter notion of consequence may be induced by a two-dimensional logical matrix [7,9], whose associated two-dimensional canonical entailment relation very naturally embraces bilateralism and involves two possibly distinct collections of distinguished truth-values: the 'designated' values and the 'anti-designated' values, respectively equated with acceptance and rejection.

Non-deterministic logical matrices have been extensively investigated in recent years, and proved useful in the construction of effective semantics for many families of logics in a systematic and modular way [5,12,12,19]. As in [6], in the present paper the interpretations of the connectives in a matrix outputs (possibly empty) sets of values, instead of a single value. In our study, we explore an essential feature of (partial) non-deterministic semantics, namely *effectiveness*, to provide analytic axiomatizations for a very inclusive class of finite *monadic* two-dimensional matrices. The latter consist in matrices whose underlying linguistic resources are sufficiently expressive so as to uniquely characterize each of the underlying truth-values, in a similar vein as in [10,13]. In contrast to the multi-dimensional Gentzen-style calculi used in the literature to axiomatize many-valued logics in the context of bilateralism (and multilateralism) [16], we introduce much simpler two-dimensional symmetrical Hilbert-style calculi to the same effect and show how they give rise to derivations that do not conform to the received view that axiomatic proofs consist simply in 'sequences of formulas'. In our approach, indeed, extending to the bilateralist case the one-dimensional tree-derivation mechanism considered in [10,20,23], the inference rules, instead of manipulating metalinguistic objects, deal only with pairs of accepted/rejected formulas, and derivations are trees whose nodes come labelled with such pairs and result from expansions determined by the rules. As we will show, the *analyticity* of the axiomatizations that we extract from our two-dimensional (partial) non-deterministic matrices, using symmetrical rules that internalize 'case exhaustion', allows for bounded proof search, and the design of a simple recursive decision algorithm that runs in exponential time.

The paper is organized as follows: Sect. 2 introduces the basic concepts and terminology involved in two-dimensional notions of consequence and in symmetrical analytic Hilbert-style calculi. Section 3 presents the general axiomatization procedure for finite monadic matrices, illustrating it and highlighting its modularity via the correspondence between refining a matrix and adding rules to a sound symmetrical two-dimensional calculus. Then, Sect. 4 describes our proposed proof-search algorithm, proves its correctness and investigates its worst-case exponential asymptotic complexity. In the final remarks, we reflect upon the obtained results and indicate some directions for future developments. Detailed proofs of the main results may be found at https://tinyurl.com/21-GMM-Bilat.

2 Preliminaries

2.1 Languages

A *propositional signature* Σ is a family $\{\Sigma_k\}_{k\in\omega}$, where each Σ_k is a collection of k-*ary connectives*. Given a denumerable set $\mathcal{P} := \{p_i \mid i \in \omega\}$ of *propositional variables*, the *propositional language over* Σ *generated by* $\mathcal{P}$, $\mathbf{L}_\Sigma(\mathcal{P})$, is the absolutely free algebra over Σ freely generated by $\mathcal{P}$. The elements of $L_\Sigma(\mathcal{P})$, the carrier set of the latter algebra, are called *formulas* and will be indicated below by capital Roman letters. As usual, whenever there is no risk of confusion, we will omit braces and unions in collecting sets and formulas, and leave a blank space in place of $\varnothing$. For convenience, given $\Phi \subseteq L_\Sigma(\mathcal{P})$, the set of formulas not in Φ will be denoted by Φ^{c}. On any given language, we may define the functions subf and props, which output, respectively, the subformulas and the propositional variables occurring in a given formula, and define as well the function size, such that $\mathsf{size}(p) := 1$ for each $p \in \mathcal{P}$, and $\mathsf{size}(©(A_1, \ldots, A_k)) := 1 + \sum_{i=1}^{k} \mathsf{size}(A_i)$, for each $k \in \omega$ and $© \in \Sigma_k$. Moreover, as usual, endomorphisms on $\mathbf{L}_\Sigma(\mathcal{P})$ are called *substitutions*, and, given a formula $B \in L_\Sigma(\mathcal{P})$ with $\mathsf{props}(B) \subseteq \{p_{i_1}, \ldots, p_{i_k}\}$, for some $k \in \omega$, we write $B(A_1, \ldots, A_k)$ for the image of B under a substitution σ where $\sigma(p_{i_j}) = A_j$, for all $1 \leq j \leq k$, and where $\sigma(p) = p$ otherwise; for a set Φ of one-variable formulas, we let $\Phi(A) := \{B(A) \mid B \in \Phi\}$.

2.2 Two-Dimensional Consequence Relations

Hereupon, we shall call B-*statement* any 2×2-place tuple $\begin{pmatrix}\Phi_{\mathsf{И}} & \Phi_{\mathsf{⅄}} \\ \Phi_{\mathsf{Y}} & \Phi_{\mathsf{N}}\end{pmatrix}$ of sets of formulas in a given language. By definition, a collection of B-statements will be said to constitute a B-*consequence relation* $\vdots|\vdots$ provided that any of the following conditions constitutes a sufficient guarantee for the *consequence judgment* $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}$ to be established:

(O) $\Phi_{\mathsf{Y}} \cap \Phi_{\mathsf{⅄}} \neq \varnothing$ or $\Phi_{\mathsf{N}} \cap \Phi_{\mathsf{И}} \neq \varnothing$

(D) $\dfrac{\Psi_{\mathsf{И}}}{\Psi_{\mathsf{Y}}}|\dfrac{\Psi_{\mathsf{⅄}}}{\Psi_{\mathsf{N}}}$ and $\Psi_\alpha \subseteq \Phi_\alpha$ for every $\alpha \in \{\mathsf{Y}, \mathsf{N}, \mathsf{⅄}, \mathsf{И}\}$

(C) $\dfrac{\Omega^{\mathsf{c}}_{\mathsf{Ƨ}}}{\Omega_{\mathsf{S}}}|\dfrac{\Omega^{\mathsf{c}}_{\mathsf{S}}}{\Omega_{\mathsf{Ƨ}}}$ for all $\Phi_{\mathsf{Y}} \subseteq \Omega_{\mathsf{S}} \subseteq \Phi^{\mathsf{c}}_{\mathsf{⅄}}$ and $\Phi_{\mathsf{N}} \subseteq \Omega_{\mathsf{Ƨ}} \subseteq \Phi^{\mathsf{c}}_{\mathsf{И}}$

(S) $\dfrac{\Psi_{\mathsf{И}}}{\Psi_{\mathsf{Y}}}|\dfrac{\Psi_{\mathsf{⅄}}}{\Psi_{\mathsf{N}}}$ and $\Phi_\alpha = \sigma(\Psi_\alpha)$ for every $\alpha \in \{\mathsf{Y}, \mathsf{N}, \mathsf{⅄}, \mathsf{И}\}$, for a substitution σ

In the above conditions, $\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}}, \Phi_{\mathsf{⅄}}, \Phi_{\mathsf{И}}$ denote arbitrary sets of formulas, that may intuitively be read as representing, respectively, collections of *accepted*, *rejected*, *non-accepted* and *non-rejected* formulas. It is not hard to check that such definition, employing the properties of (O)verlap, (D)ilution, (C)ut and (S)ubstitution-invariance, is equivalent to the one found in [9], and it generalizes the well-known abstract Tarskian one-dimensional account of logical consequence. In addition, a B-consequence relation will be called *finitary* when a consequence judgment $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}$ always implies that:

(F) $\frac{\Phi^{\mathsf{f}}_{\mathsf{И}}}{\Phi^{\mathsf{f}}_{\mathsf{Y}}} \mid \frac{\Phi^{\mathsf{f}}_{\mathsf{⅄}}}{\Phi^{\mathsf{f}}_{\mathsf{N}}}$, for some finite $\Phi^{\mathsf{f}}_{\alpha} \subseteq \Phi_{\alpha}$, for every $\alpha \in \{\mathsf{Y}, \mathsf{N}, \mathsf{⅄}, \mathsf{И}\}$

We will denote by $\frac{\cdot}{\cdot} \nmid \frac{\cdot}{\cdot}$ the complement of $\frac{\cdot}{\cdot} \mid \frac{\cdot}{\cdot}$, sometimes called the *compatibility relation* associated to $\frac{\cdot}{\cdot} \mid \frac{\cdot}{\cdot}$ (cf. [8]). Furthermore, we should note that later on we will sometimes write $\tilde{\mathsf{Y}}$ for ⅄, write $\tilde{⅄}$ for Y, write $\tilde{\mathsf{N}}$ for И, and write $\tilde{\text{И}}$ for N.

A B-consequence relation $\frac{\cdot}{\cdot} \mid \frac{\cdot}{\cdot}$ may be said to induce a 2-place relation $\cdot \rhd^{\mathsf{t}} \cdot$ over $\mathsf{Pow}(\mathrm{L}_{\Sigma}(\mathcal{P}))$ by setting $\Phi_{\mathsf{Y}} \rhd^{\mathsf{t}} \Phi_{⅄}$ iff $\frac{\varnothing}{\Phi_{\mathsf{Y}}} \mid \frac{\Phi_{⅄}}{\varnothing}$. This is easily seen to constitute a generalized (one-dimensional) consequence relation. Another such relation is induced by setting $\Phi_{\mathsf{N}} \rhd^{\mathsf{f}} \Phi_{\text{И}}$ iff $\frac{\Phi_{\text{И}}}{\varnothing} \mid \frac{\varnothing}{\Phi_{\mathsf{N}}}$. Connected to that, we will say that $\cdot \rhd^{\mathsf{t}} \cdot$ *inhabits the* $\mathtt{t}$*-aspect of* $\frac{\cdot}{\cdot} \mid \frac{\cdot}{\cdot}$, and that $\cdot \rhd^{\mathsf{f}} \cdot$ *inhabits the* $\mathtt{f}$*-aspect of* $\frac{\cdot}{\cdot} \mid \frac{\cdot}{\cdot}$. These are but two of many possible aspects of interest of a given B-consequence relation; in principle, very different Tarskian —and also non-Tarskian!— logics may coinhabit the same given two-dimensional consequence relation (see [9]).

Finally, a B-consequence $\frac{\cdot}{\cdot} \mid \frac{\cdot}{\cdot}$ is said to be *decidable* when there is some *decision procedure* that takes a B-statement $\left(\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} \vdots \frac{\Phi_{⅄}}{\Phi_{\mathsf{N}}}\right)$ with finite component sets as input, outputs `true` when $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} \mid \frac{\Phi_{⅄}}{\Phi_{\mathsf{N}}}$ is the case, and outputs `false` when $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} \nmid \frac{\Phi_{⅄}}{\Phi_{\mathsf{N}}}$.

2.3 Two-Dimensional Non-deterministic Matrices

A *partial non-deterministic* B*–matrix* $\mathbb{M}$ *over a signature* Σ, or simply $\mathsf{B}^{\mathsf{PN}}_{\Sigma}$*-matrix*, is a structure $\langle \mathcal{V}^{\mathbb{M}}, \mathsf{Y}^{\mathbb{M}}, \mathsf{N}^{\mathbb{M}}, \cdot^{\mathbb{M}} \rangle$ where the set $\mathcal{V}^{\mathbb{M}}$ is said to contain *truth-values*, the sets $\mathsf{Y}^{\mathbb{M}}, \mathsf{N}^{\mathbb{M}} \subseteq \mathcal{V}^{\mathbb{M}}$ are said to contain, respectively, the *designated* and the *anti-designated* truth-values, and, for each $\mathrm{k} \in \omega$ and $© \in \Sigma_{\mathrm{k}}$, the mapping $©^{\mathbb{M}} : (\mathcal{V}^{\mathbb{M}})^{\mathrm{k}} \to \mathsf{Pow}(\mathcal{V}^{\mathbb{M}})$ is the *interpretation* of © in $\mathbb{M}$. For convenience, we define $⅄^{\mathbb{M}} := \mathcal{V}^{\mathbb{M}} \backslash \mathsf{Y}^{\mathbb{M}}$ and $\text{И}^{\mathbb{M}} := \mathcal{V}^{\mathbb{M}} \backslash \mathsf{N}^{\mathbb{M}}$. A $\mathsf{B}^{\mathsf{PN}}_{\Sigma}$-matrix is said to be *total* when $\varnothing$ is not in the range of the interpretation of any connective of Σ, *deterministic* when the range of any interpretation contains only singletons, also called *deterministic images*, and *fully indeterministic* if it allows for the maximum degree of non-determinism, that is, if $©^{\mathbb{M}}(x_1, \ldots, x_{\mathrm{k}}) = \mathcal{V}^{\mathbb{M}}$ for each $\mathrm{k} \in \omega$ and $© \in \Sigma_{\mathrm{k}}$, and all $x_1, \ldots, x_{\mathrm{k}} \in \mathcal{V}^{\mathbb{M}}$.

In the following definitions, $\mathbb{M}$ will represent an arbitrary $\mathsf{B}^{\mathsf{PN}}_{\Sigma}$*-matrix*.

Given a set of truth-values $\mathcal{X} \subseteq \mathcal{V}^{\mathbb{M}}$, the *sub–*$\mathsf{B}^{\mathsf{PN}}_{\Sigma}$*-matrix* $\mathbb{M}_{\mathcal{X}}$ *induced by* $\mathcal{X}$ is the $\mathsf{B}^{\mathsf{PN}}_{\Sigma}$-matrix $\langle \mathcal{X}, \mathsf{Y}^{\mathbb{M}} \cap \mathcal{X}, \mathsf{N}^{\mathbb{M}} \cap \mathcal{X}, \cdot^{\mathbb{M}_{\mathcal{X}}} \rangle$ such that $©^{\mathbb{M}_{\mathcal{X}}}(x_1, \ldots, x_{\mathrm{k}}) := ©^{\mathbb{M}}(x_1, \ldots, x_{\mathrm{k}}) \cap \mathcal{X}$, for all $x_1, \ldots, x_{\mathrm{k}} \in \mathcal{X}$, $\mathrm{k} \in \omega$ and $© \in \Sigma_{\mathrm{k}}$. The set of all subsets of the values of each non-empty total sub–$\mathsf{B}^{\mathsf{PN}}_{\Sigma}$-matrix of $\mathbb{M}$ will be denoted by $\mathbb{T}_{\mathbb{M}}$, that is,

$$\mathbb{T}_{\mathbb{M}} := \bigcup_{\substack{\varnothing \neq \mathcal{X} \subseteq \mathcal{V}^{\mathbb{M}} \\ \mathbb{M}_{\mathcal{X}} \text{ total}}} \mathsf{Pow}(\mathcal{X}).$$

Check Example 3 for an illustration of the latter.

We shall call $\mathbb{M}$*-valuation* any mapping $v : \mathrm{L}_{\Sigma}(\mathcal{P}) \to \mathcal{V}^{\mathbb{M}}$ such that $v(©(\mathrm{A}_1, \ldots, \mathrm{A}_{\mathrm{k}})) \in ©^{\mathbb{M}}(v(\mathrm{A}_1), \ldots, v(\mathrm{A}_{\mathrm{k}}))$ for all $\mathrm{k} \in \omega$, $© \in \Sigma_{\mathrm{k}}$ and

$A_1, \ldots, A_k \in L_\Sigma(\mathcal{P})$. As proved in [6], given a set $\Phi \subseteq L_\Sigma(\mathcal{P})$ closed under subformulas, any mapping $f : \Phi \to \mathcal{V}^{\mathbb{M}}$ extends to an $\mathbb{M}$-valuation provided that $f(©(A_1, \ldots, A_k)) \in ©^{\mathbb{M}}(f(A_1), \ldots, f(A_k))$, for every $©(A_1, \ldots, A_k) \in \Phi$, and $f(\Phi) \in \mathbb{T}_{\mathbb{M}}$. Notice that if we disregard the latter condition we obtain the property of *effectiveness* for total non-deterministic matrices ([2]); as this very condition holds for all such matrices, by making it explicit in the previous definition we obtain a generalization of effectiveness that also applies to partial non-deterministic matrices. Any formula $A \in L_\Sigma(\mathcal{P})$ with $\mathsf{props}(A) = \{p_{i_1}, \ldots, p_{i_k}\}$ may be interpreted on $\mathbb{M}$ as a k-ary mapping $A^{\mathbb{M}}$ such that $A^{\mathbb{M}}(x_1, \ldots, x_k) := \{v(A) \mid v \text{ is an } \mathbb{M}\text{-valuation and } v(p_{i_1}) = x_1, \ldots, v(p_{i_k}) = x_k\}$.

The B-*entailment relation induced by* $\mathbb{M}$ is a 2×2-place relation $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}\,\mathbb{M}$ over $L_\Sigma(\mathcal{P})$ such that:

(B-ent) $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}\Big|\frac{\Phi_{\mathsf{λ}}}{\Phi_{\mathsf{N}}}\,\mathbb{M}$ iff there is no $\mathbb{M}$-valuation v such that $v(\Phi_\alpha) \subseteq \alpha^{\mathbb{M}}$ for every $\alpha \in \{\mathsf{Y}, \mathsf{N}, \mathsf{λ}, \mathsf{И}\}$

for every $\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}}, \Phi_{\mathsf{λ}}, \Phi_{\mathsf{И}} \subseteq L_\Sigma(\mathcal{P})$. Whenever $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}|\frac{\Phi_{\mathsf{λ}}}{\Phi_{\mathsf{N}}}\,\mathbb{M}$, we say that the B-statement $\left(\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}\vdots\frac{\Phi_{\mathsf{λ}}}{\Phi_{\mathsf{N}}}\right)$ *holds* in $\mathbb{M}$. It is straightforward to check that (see [7]):

Proposition 1. *The* B-*entailment relation induced by a* $\mathsf{B}^{\underline{\mathsf{PN}}}_{\Sigma}$-*matrix is a* B-*consequence relation.*

Example 1. Let $\mathcal{V}_4 := \{\mathbf{f}, \bot, \top, \mathbf{t}\}$, $\mathsf{Y}_4 := \{\top, \mathbf{t}\}$, $\mathsf{N}_4 := \{\top, \mathbf{f}\}$, and consider a signature Σ^{FDE} containing but two binary connectives, $\wedge$ and $\vee$, and one unary connective, $\neg$. Next, define the $\mathsf{B}^{\underline{\mathsf{PN}}}_{\Sigma^{\mathrm{FDE}}}$-matrix $\mathbb{I} := \langle \mathcal{V}_4, \mathsf{Y}_4, \mathsf{N}_4, \cdot^{\mathbb{I}} \rangle$ that interprets the latter connectives according to the following (non-deterministic) truth-tables (here and below, braces will be omitted from the images of the interpretations):

$\wedge^{\mathbb{I}}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
$\bot$	$\mathbf{f}$	$\mathbf{f}, \bot$	$\mathbf{f}$	$\mathbf{f}, \bot$
$\top$	$\mathbf{f}$	$\mathbf{f}$	$\top$	$\top$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}, \bot$	$\top$	$\mathbf{t}, \top$

$\vee^{\mathbb{I}}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}, \top$	$\mathbf{t}, \bot$	$\top$	$\mathbf{t}$
$\bot$	$\mathbf{t}, \bot$	$\mathbf{t}, \bot$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\top$	$\mathbf{t}$	$\top$	$\mathbf{t}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

	$\neg^{\mathbb{I}}$
$\mathbf{f}$	$\mathbf{t}$
$\bot$	$\bot$
$\top$	$\top$
$\mathbf{t}$	$\mathbf{f}$

The t-aspect of $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}\,\mathbb{I}$ is inhabited by the logic introduced in [3], which incorporates some principles on how a processor would be expected to deal with information about an arbitrary set of formulas.

Given two $\mathsf{B}^{\underline{\mathsf{PN}}}_{\Sigma}$-matrices $\mathbb{M}_1$ and $\mathbb{M}_2$, we say that $\mathbb{M}_2$ is a *refinement* of $\mathbb{M}_1$ when $\mathcal{V}^{\mathbb{M}_2} \subseteq \mathcal{V}^{\mathbb{M}_1}$ and $©^{\mathbb{M}_2}(x_1, \ldots, x_k) \subseteq ©^{\mathbb{M}_1}(x_1, \ldots, x_k)$ for each $k \in \omega$ and $© \in \Sigma_k$, and for every $x_1, \ldots, x_k \in \mathcal{V}^{\mathbb{M}_2}$. Also, we say that $\cdot^{\mathbb{M}_2}$ *agrees with* $\cdot^{\mathbb{M}_1}$ when both provide the same interpretations for the connectives of Σ. Evidently, every $\mathsf{B}^{\underline{\mathsf{PN}}}_{\Sigma}$-matrix is a refinement of the corresponding fully indeterministic $\mathsf{B}^{\underline{\mathsf{PN}}}_{\Sigma}$-matrix. In the examples that follow, we illustrate a couple of refinements of the

$\mathsf{B}^{\mathsf{PN}}_{\Sigma}$-matrix $\mathbb{I}$ presented in Example 1, giving rise to (two-dimensional versions of) other well-known logics.

Example 2. Let $\mathbb{E} := \langle \mathcal{V}_4, \mathsf{Y}_4, \mathsf{N}_4, \cdot^{\mathbb{E}} \rangle$ be the $\mathsf{B}^{\mathsf{PN}}_{\Sigma^{\mathrm{FDE}}}$–matrix consisting of a refinement of $\mathbb{I}$ with interpretations given by the following tables:

$\wedge^{\mathbb{E}}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
$\bot$	$\mathbf{f}$	$\bot$	$\mathbf{f}$	$\bot$
$\top$	$\mathbf{f}$	$\mathbf{f}$	$\top$	$\top$
$\mathbf{t}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$

$\vee^{\mathbb{E}}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\bot$	$\bot$	$\bot$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\top$	$\mathbf{t}$	$\top$	$\mathbf{t}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

	$\neg^{\mathbb{E}}$
$\mathbf{f}$	$\mathbf{t}$
$\bot$	$\bot$
$\top$	$\top$
$\mathbf{t}$	$\mathbf{f}$

One may readily see that these interpretations correspond to the ones of First Degree Entailment and that this $\mathsf{B}^{\mathsf{PN}}_{\Sigma^{\mathrm{FDE}}}$–matrix corresponds to the logic $\mathbf{E}^{\mathrm{B}}$ presented in [7].

Example 3. We may still refine $\mathbb{E}$ (and thus $\mathbb{I}$) a little more. Let $\mathbb{K} := \langle \mathcal{V}_4, \mathsf{Y}_4, \mathsf{N}_4, \cdot^{\mathbb{K}} \rangle$ be the $\mathsf{B}^{\mathsf{PN}}_{\Sigma^{\mathrm{FDE}}}$–matrix such that $\cdot^{\mathbb{K}}$ agrees with $\cdot^{\mathbb{E}}$ except that $\wedge^{\mathbb{K}}(\top, \bot) = \vee^{\mathbb{K}}(\top, \bot) = \wedge^{\mathbb{K}}(\bot, \top) = \vee^{\mathbb{K}}(\bot, \top) = \varnothing$. Note that $\mathbb{T}_{\mathbb{K}} = \{\mathcal{X} \subseteq \mathcal{V}_4 \mid \{\top, \bot\} \not\subseteq \mathcal{X}\}$. As shown in [10], Kleene's strong three-valued logic inhabits the t-aspect of $\div|\div\, \mathbb{K}$.

Example 4. Let $\mathcal{V}_5 := \{\mathrm{f}, \mathrm{F}, \mathrm{I}, \mathrm{T}, \mathrm{t}\}$, $\mathsf{Y}_5 := \{\mathrm{T}, \mathrm{I}, \mathrm{t}\}$, $\mathsf{N}_5 := \{\mathrm{T}, \mathrm{I}, \mathrm{f}\}$, and consider a signature Σ^{mCi} containing but three binary connectives, $\wedge$, $\vee$ and $\supset$, and two unary connectives, $\neg$ and $\circ$. Inspired by the 5-valued non-deterministic logical matrix presented in [1] for the logic of formal inconsistency called **mCi** [21], we define the $\mathsf{B}^{\mathsf{PN}}_{\Sigma^{\mathrm{mCi}}}$–matrix $\mathbb{P} := \langle \mathcal{V}_5, \mathsf{Y}_5, \mathsf{N}_5, \cdot^{\mathbb{P}} \rangle$ with the following interpretations:

$$\wedge^{\mathbb{P}}(x_1, x_2) := \begin{cases} \{\mathrm{f}\} & \text{if either } x_1 \notin \mathsf{Y}_5 \text{ or } x_2 \notin \mathsf{Y}_5 \\ \{\mathrm{t}, \mathrm{I}\} & \text{otherwise} \end{cases}$$

$$\vee^{\mathbb{P}}(x_1, x_2) := \begin{cases} \{\mathrm{t}, \mathrm{I}\} & \text{if either } x_1 \in \mathsf{Y}_5 \text{ or } x_2 \in \mathsf{Y}_5 \\ \{\mathrm{f}\} & \text{if } x_1, x_2 \notin \mathsf{Y}_5 \end{cases}$$

$$\supset^{\mathbb{P}}(x_1, x_2) := \begin{cases} \{\mathrm{t}, \mathrm{I}\} & \text{if either } x_1 \notin \mathsf{Y}_5 \text{ or } x_2 \in \mathsf{Y}_5 \\ \{\mathrm{f}\} & \text{if } x_1 \in \mathsf{Y}_5 \text{ and } x_2 \notin \mathsf{Y}_5 \end{cases}$$

	f	F	I	T	t
$\neg^{\mathbb{P}}$	t,I	T	t,I	F	f

	f	F	I	T	t
$\circ^{\mathbb{P}}$	T	T	F	T	T

We note that the logic **mCi** inhabits the t-aspect of $\div|\div\, \mathbb{P}$. It is worth pointing out that, up to now, no *finite* Hilbert-style calculus was known to axiomatize this logic; however, a finite two-dimensional symmetrical Hilbert-style calculus for **mCi** results smoothly from the procedure described in the next section.

Given $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{V}^{\mathbb{M}}$ and $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$, we say that $\mathcal{X}$ and $\mathcal{Y}$ are α-*separated*, denoted by $\mathcal{X} \#_\alpha \mathcal{Y}$, if $\mathcal{X} \subseteq \alpha^{\mathbb{M}}$ and $\mathcal{Y} \subseteq \mathcal{V}^{\mathbb{M}} \setminus \alpha^{\mathbb{M}}$, or vice-versa. Given two truth-values $x, y \in \mathcal{V}^{\mathbb{M}}$, a single-variable formula S is a *monadic separator for* x *and* y whenever $\mathrm{S}^{\mathbb{M}}(x) \#_\alpha \mathrm{S}^{\mathbb{M}}(y)$, for some $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$. The $\mathsf{B}^{\underline{\mathsf{PN}}}_{\Sigma}$-matrix $\mathbb{M}$ is said to be *monadic* when for each pair of distinct truth-values of $\mathbb{M}$ there is a monadic separator for these values.[1] We say that a set of single-variable formulas $\mathcal{D}^x$ *isolates* x whenever, for every $y \neq x$, there exists a monadic separator $\mathrm{S} \in \mathcal{D}^x$ for x and y. A *discriminator for* $\mathbb{M}$, then, is a family $\mathcal{D} := \{(\mathcal{D}^x_{\mathsf{Y}}, \mathcal{D}^x_{\mathsf{⅄}}, \mathcal{D}^x_{\mathsf{N}}, \mathcal{D}^x_{\mathsf{И}})\}_{x \in \mathcal{V}^{\mathbb{M}}}$ such that $\mathcal{D}^x := \bigcup_\alpha \mathcal{D}^x_\alpha$ isolates x and $\mathrm{S}^{\mathbb{M}}(x) \subseteq \alpha^{\mathbb{M}}$ whenever $\mathrm{S} \in \mathcal{D}^x_\alpha$. We denote the set $\bigcup_{x \in \mathcal{V}^{\mathbb{M}}} \mathcal{D}^x$ by $\mathcal{D}^{\bowtie}$ and say that $\mathcal{D}$ *is based on* $\mathcal{D}^{\bowtie}$.

Example 5. The tables below describe, respectively, a discriminator based on $\{\mathrm{p}\}$ for any $\mathsf{B}^{\underline{\mathsf{PN}}}_{\Sigma}$-matrix of the form $\langle \mathcal{V}_4, \mathsf{Y}_4, \mathsf{N}_4, \cdot \rangle$ (see Examples 1, 2 and 3) and a discriminator for $\mathbb{P}$ based on $\{\mathrm{p}, \neg\mathrm{p}\}$ (of Example 4):

x	$\mathcal{D}^x_{\mathsf{Y}}$	$\mathcal{D}^x_{\mathsf{⅄}}$	$\mathcal{D}^x_{\mathsf{N}}$	$\mathcal{D}^x_{\mathsf{И}}$
$\mathbf{f}$	$\varnothing$	p	p	$\varnothing$
$\bot$	$\varnothing$	p	$\varnothing$	p
$\top$	p	$\varnothing$	p	$\varnothing$
$\mathbf{t}$	p	$\varnothing$	$\varnothing$	p

x	$\mathcal{D}^x_{\mathsf{Y}}$	$\mathcal{D}^x_{\mathsf{⅄}}$	$\mathcal{D}^x_{\mathsf{N}}$	$\mathcal{D}^x_{\mathsf{И}}$
f	$\varnothing$	p	p	$\varnothing$
F	$\varnothing$	p	$\varnothing$	p
I	p, ¬p	$\varnothing$	p	$\varnothing$
T	p	¬p	p	$\varnothing$
t	p	$\varnothing$	$\varnothing$	p

The following result —which will be instrumental, in particular, within the soundness proof of the axiomatizations that we will develop later on— shows that a discriminator is capable of uniquely characterizing each truth-value of the corresponding $\mathsf{B}^{\underline{\mathsf{PN}}}_{\Sigma}$-matrix:

Lemma 1. *If* $\mathbb{M}$ *is a monadic* $\mathsf{B}^{\underline{\mathsf{PN}}}_{\Sigma}$*-matrix and* $\mathcal{D}$ *is a discriminator for* $\mathbb{M}$*, then, for all* $\mathrm{A} \in \mathrm{L}_\Sigma(\mathcal{P})$*,* $x \in \mathcal{V}^{\mathbb{M}}$ *and* $\mathbb{M}$*-valuation* v*,*

$$v(\mathrm{A}) = x \quad \textit{iff} \quad v(\mathcal{D}^x_\alpha(\mathrm{A})) \subseteq \alpha^{\mathbb{M}} \textit{ and } v(\mathcal{D}^x_{\tilde{\alpha}}(\mathrm{A})) \subseteq \tilde{\alpha}^{\mathbb{M}} \textit{ for every } \alpha \in \{\mathsf{Y}, \mathsf{N}\}.$$

Proof. Analogous to the proof of Lemma 1 in [10].

2.4 Calculi for Two-Dimensional Statements

We may consider the B-statements themselves as the formal objects whose provability by a given (Hilbert-style) deductive proof system we will be interested upon. The B-statements with finite component sets will be hereupon called B-*sequents*. A *(Set*^{2}*–Set*^{2}*) rule schema* $\mathfrak{r} := \dfrac{\Phi_{\mathsf{Y}} \,;\, \Phi_{\mathsf{N}}}{\Phi_{\mathsf{⅄}} \,;\, \Phi_{\mathsf{И}}}$ is a B-statement $\left(\begin{smallmatrix}\Phi_{\mathsf{И}} & \Phi_{\mathsf{⅄}} \\ \Phi_{\mathsf{Y}} & \Phi_{\mathsf{N}}\end{smallmatrix}\right)$ that, when having its component sets subjected to a substitution σ, produce a *(rule) instance (with schema* $\mathfrak{r}$*)*, denoted simply by $\mathfrak{r}^\sigma$; for each rule

[1] Whether monadicity of a $\mathsf{B}^{\underline{\mathsf{PN}}}_{\Sigma}$-matrix is decidable is still an open problem.

instance $\mathfrak{r}^{\sigma}$, the pair $(\sigma(\Phi_{\mathsf{Y}}), \sigma(\Phi_{\mathsf{N}}))$ is said to be the *antecedent* and the pair $(\sigma(\Phi_{\mathsf{⅄}}), \sigma(\Phi_{\mathsf{И}}))$ is said to be the *succedent* of $\mathfrak{r}^{\sigma}$. For later reference, we also set $\mathsf{branch}(\mathfrak{r}^{\sigma}) := |\sigma(\Phi_{\mathsf{⅄}}) \cup \sigma(\Phi_{\mathsf{И}})|$ and $\mathsf{size}(\mathfrak{r}^{\sigma}) := \sum_{\alpha} \mathsf{size}(\sigma(\Phi_{\alpha}))$, which extends to sets of rule instances in the natural way. Notice that our notation for rule schemas differs from that of B-statements with respect to the positioning of the sets of formulas. The purpose is to facilitate the development of proofs in tree form growing downwards from the premises to the conclusion as described in the sequel. B-statements, in turn, follow the notation for consequence judgements, which is motivated by the bilattice representation of the four logical values underlying a B-consequence relation [9], in addition to the desire of better expressing the possible interactions between the two dimensions.

A *(Set^2–Set^2) calculus* $\mathcal{C}$ is a collection of rule schemas. We shall sometimes refer to the set of all rule instances of a schema $\mathfrak{r}$ of $\mathcal{C}$ as an *inference rule (with schema $\mathfrak{r}$) of* $\mathcal{C}$. An inference rule with schema $\mathfrak{r} := \dfrac{\Phi_{\mathsf{Y}} \ ; \ \Phi_{\mathsf{N}}}{\Phi_{\mathsf{⅄}} \ ; \ \Phi_{\mathsf{И}}}$ is called *finitary* whenever Φ_{α} is finite for every $\alpha \in \{\mathsf{Y}, \mathsf{N}, \mathsf{⅄}, \mathsf{И}\}$. A calculus is finitary when each of its inference rules is finitary.

In order to explain what it means for a B-statement $\mathfrak{s} := \left(\begin{smallmatrix}\Phi_{\mathsf{И}} & \Phi_{\mathsf{⅄}} \\ \Phi_{\mathsf{Y}} & \Phi_{\mathsf{N}}\end{smallmatrix}\right)$ to be provable—in other words, for its succedent $(\Phi_{\mathsf{⅄}}, \Phi_{\mathsf{И}})$ to follow from its antecedent $(\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}})$—using the inference rules of a calculus, we will first introduce the notion of a derivation structured in tree form. A *directed rooted tree* t is a poset $\langle \mathsf{nds}(t), \preceq^{t} \rangle$ such that, for every *node* $n \in \mathsf{nds}(t)$, the set $\mathsf{acts}^{t}(n) := \{n' \mid n' \prec^{t} n\}$ of the *ancestors* of n is well-ordered under $\prec^{t}$, and there is a single minimal element $\mathsf{rt}(t)$, called the *root* of t. We denote by $\mathsf{dcts}^{t}(n) := \{n' \mid n \prec^{t} n'\}$ the set of *descendants* of t, by $\mathsf{chn}^{t}(n)$ the minimal elements of $\mathsf{dcts}^{t}(n)$ (the *children* of n in t), and by $\mathsf{lvs}(t)$ the set of maximal elements of $\preceq^{t}$, the *leaves* of t. A rooted tree t is said to be *bounded* when every branch of t has a leaf. Moreover, we will call *labelled* a rooted tree t that comes equipped with a mapping $\ell^{t} : \mathsf{nds}(t) \to \mathsf{Pow}(\mathrm{L}_{\Sigma}(\mathcal{P}))^{2} \cup \{\star\}$, each node n of t being *labelled with* $\ell^{t}(n)$. A node labelled with $\star$ is said to be *discontinued*. In what follows, labelled bounded rooted trees will be referred to simply as *trees*. A tree with a single node labelled with $\ell \in \mathsf{Pow}(\mathrm{L}_{\Sigma}(\mathcal{P}))^{2} \cup \{\star\}$ will be denoted by $\mathsf{sntree}(\ell)$.

Given a node n labelled with (Φ, Ψ) and given a formula A, we shall use $n_{\mathsf{S}}^{\mathrm{A}}$ to refer to a node labelled with $(\Phi \cup \{\mathrm{A}\}, \Psi)$ and use $n_{\mathsf{Ƨ}}^{\mathrm{A}}$ to refer to a node labelled with $(\Phi, \Psi \cup \{\mathrm{A}\})$. We say that a tree t is a $\mathcal{C}$-*derivation* provided that for each non-leaf node n of t labelled with $(\Psi_{\mathsf{Y}}, \Psi_{\mathsf{N}})$ there is an instance of an inference rule of $\mathcal{C}$, say $\mathfrak{r}^{\sigma} = \dfrac{\sigma(\Phi_{\mathsf{Y}}) \ ; \ \sigma(\Phi_{\mathsf{N}})}{\sigma(\Phi_{\mathsf{⅄}}) \ ; \ \sigma(\Phi_{\mathsf{И}})}$, that *expands* n or, equivalently, that is *applicable* to the label of n, meaning that $\sigma(\Phi_{\alpha}) \subseteq \Psi_{\alpha}$, for every $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$, and

- if $\Phi_{\mathsf{⅄}} \cup \Phi_{\mathsf{И}} = \varnothing$, then $\mathsf{chn}^{t}(n) = \{n_{\star}\}$ and $\ell^{t}(n_{\star}) = \star$
- otherwise, $\mathsf{chn}^{t}(n) = \{n_{\mathsf{S}}^{\mathrm{A}} \mid \mathrm{A} \in \sigma(\Phi_{\mathsf{⅄}})\} \cup \{n_{\mathsf{Ƨ}}^{\mathrm{A}} \mid \mathrm{A} \in \sigma(\Phi_{\mathsf{И}})\}$

We should observe that, with our present notation, traditional Hilbert-style derivations (when only inference rules with a single formula in the succedent are applied) turn out to be linear trees; for all practical purposes, at any given node we may count with all the information from previous nodes in the branch, and, accordingly, a rule application with a single succedent just adds a new bit of information to that very branch.

Given a B-statement $\mathscr{s} := \left(\begin{smallmatrix}\Phi_{\mathsf{И}} & \vdots & \Phi_{\mathsf{⅄}}\\ \cdots & & \cdots \\ \Phi_{\mathsf{Y}} & \vdots & \Phi_{\mathsf{N}}\end{smallmatrix}\right)$ and a calculus $\mathcal{C}$, a $\mathcal{C}$-derivation t with $\ell^t(\mathsf{rt}(t)) = (\Psi_{\mathsf{Y}}, \Psi_{\mathsf{N}})$ is a $\mathcal{C}$-*proof* of $\mathscr{s}$ provided that $\Psi_\alpha \subseteq \Phi_\alpha$ for every $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$ and, for all $n \in \mathsf{lvs}(t)$ with $\ell^t(n) = (\Psi_{\mathsf{⅄}}, \Psi_{\mathsf{И}})$, we have $\Psi_\alpha \cap \Phi_\alpha \neq \varnothing$ for some $\alpha \in \{\mathsf{⅄}, \mathsf{И}\}$. We also say that a node is $(\Phi_{\mathsf{⅄}}, \Phi_{\mathsf{И}})$-*closed* when the latter condition holds for such node and we say that t is $(\Phi_{\mathsf{⅄}}, \Phi_{\mathsf{И}})$-*closed* when all of its leaf nodes are $(\Phi_{\mathsf{⅄}}, \Phi_{\mathsf{И}})$-closed. When a $\mathcal{C}$-proof exists for the B-statement $\mathscr{s}$, we say that $\mathscr{s}$ is $\mathcal{C}$-*provable*. The reader is referred to Example 10 in order to see some proofs of the form we have just described. A calculus $\mathcal{C}$ induces a $2{\times}2$-place relation $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}\,\mathcal{C}$ over $\mathsf{Pow}(\mathrm{L}_\Sigma(\mathcal{P}))$ such that $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}\,\mathcal{C}$ whenever $\left(\begin{smallmatrix}\Phi_{\mathsf{И}} & \vdots & \Phi_{\mathsf{⅄}}\\ \cdots & & \cdots \\ \Phi_{\mathsf{Y}} & \vdots & \Phi_{\mathsf{N}}\end{smallmatrix}\right)$ is $\mathcal{C}$-provable. As we point out in Proposition 2 below, this provides another realization (compare with Proposition 1) of a B-consequence relation.

Proposition 2. *Given a calculus* $\mathcal{C}$*, the* $2{\times}2$*-place relation* $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}\,\mathcal{C}$ *is the smallest* B*-consequence containing the rules of* $\mathcal{C}$*.*

Given a collection R of rule instances, we say that a B-statement $\mathscr{s}$ is R-provable whenever there is a proof of $\mathscr{s}$ using only rule instances in R. We may define a $2{\times}2$-place relation $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}\,\mathrm{R}$ by setting $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}\,\mathrm{R}$ to hold iff $\left(\begin{smallmatrix}\Phi_{\mathsf{И}} & \vdots & \Phi_{\mathsf{⅄}}\\ \cdots & & \cdots \\ \Phi_{\mathsf{Y}} & \vdots & \Phi_{\mathsf{N}}\end{smallmatrix}\right)$ is R-provable. Although not necessarily substitution-invariant, one may readily check that this relation respects properties (O), (D) and (C).

Given a $\mathsf{B}^{\mathsf{PN}}_{\Sigma}$-matrix $\mathbb{M}$, we say that a calculus $\mathcal{C}$ is *sound* with respect to $\mathbb{M}$ whenever $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}\,\mathcal{C} \subseteq \frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}\,\mathbb{M}$ and say that it is *complete* with respect to $\mathbb{M}$ when the converse inclusion holds. Being sound and complete means that $\mathcal{C}$ *axiomatizes* $\mathbb{M}$.

Example 6. Any fully indeterministic $\mathsf{B}^{\mathsf{PN}}_{\Sigma}$-matrix is axiomatized by the empty set of rules.

Example 7. We present below a calculus that axiomatizes the $\mathsf{B}^{\mathsf{PN}}_{\Sigma}$-matrix $\mathbb{I}$ introduced in Example 1, resulting from the simplification of the calculus produced via the recipe described in Definition 1, given further ahead.

$$\frac{p\ ;}{p\vee q\ ;}\,\vee^4_1 \qquad \frac{q\ ;}{p\vee q\ ;}\,\vee^4_2 \qquad \frac{;\ p,q}{;\ p\vee q}\,\vee^4_3 \qquad \frac{;\ p\vee q}{;\ \ q}\,\vee^4_4 \qquad \frac{;\ p\vee q}{;\ \ p}\,\vee^4_5$$

$$\frac{p\wedge q\ ;}{p\ \ ;}\,\wedge^4_1 \qquad \frac{p\wedge q\ ;}{q\ \ ;}\,\wedge^4_2 \qquad \frac{p,q\ ;}{p\wedge q\ ;}\,\wedge^4_3 \qquad \frac{;\ \ q}{;\ p\wedge q}\,\wedge^4_4 \qquad \frac{;\ \ p}{;\ p\wedge q}\,\wedge^4_5$$

$$\frac{;\ \neg p}{p\ ;}\,\neg^4_1 \qquad \frac{;\ p}{\neg p\ ;}\,\neg^4_2 \qquad \frac{\neg p\ ;}{;\ p}\,\neg^4_3 \qquad \frac{p\ ;}{;\ \neg p}\,\neg^4_4$$

The next example illustrates how adding rules to an axiomatization of a $\mathsf{B}^{\mathsf{PN}}_{\Sigma}$-matrix $\mathbb{M}$ imposes refinements on $\mathbb{M}$ in order to guarantee soundness of these very rules. Such mechanism is essential to the axiomatization procedure presented in the next section.

Example 8. We obtain an axiomatization for $\mathbb{E}$ by adding rules $\frac{p\vee q\ ;}{p,q\ ;}\vee^4_6$ and $\frac{;\ p\wedge q}{;\ p,q}\wedge^4_6$ to the calculus of Example 7. If, in addition, we include the rule $\frac{q\ ;\ q}{p\ ;\ p}\mathrm{T}^4$ we axiomatize $\mathbb{K}$ (see Example 3).

Let us explain the intuition behind this mechanism considering the case of rule $\wedge^4_6$; the other rules will follow the same principle. What rule $\wedge^4_6$ enforces is that any refinement of $\mathbb{I}$ with respect to which this rule is sound must disallow valuations that assign values in $\{\bot,\mathbf{t}\}$ to formulas A and B while assigning a value in $\{\top,\mathbf{f}\}$ to $A\wedge B$, for otherwise such valuation would constitute a countermodel for that very rule. This is reflected in $\wedge^{\mathbb{E}}$ (Example 2) by the absence of the values from the set $\{\top,\mathbf{f}\}$ in the entries corresponding to the truth-value assignments in which both inputs belong to $\{\bot,\mathbf{t}\}$.

Example 9. By the same mechanism used in the previous example, in adding the rules $\frac{;}{p\ ;\ p}\bot\mathrm{E}$ and $\frac{p\ ;\ p}{;}\top\mathrm{E}$ to the axiomatization of $\mathbb{E}$, we force empty outputs on any truth-table entry whose input involves either $\bot$ or $\top$. It follows that Classical Logic inhabits the t-aspect of the resulting $\mathsf{B}^{\mathsf{PN}}_{\Sigma}$-matrix, hereby called $\mathbb{C}$.

Example 10. In Fig. 1, we offer proofs of $\left(\begin{array}{c:c} & \neg p\vee\neg q \\ \hdashline \neg(p\wedge q) & \end{array}\right)$, $\left(\begin{array}{c:c} s & s \\ \hdashline r\wedge p & p\vee q \end{array}\right)$ and $\left(\begin{array}{c:c} & \\ \hdashline p,\neg p & \end{array}\right)$, respectively, in the calculi for $\mathbb{E}$, $\mathbb{K}$ and $\mathbb{C}$ presented in the previous examples.

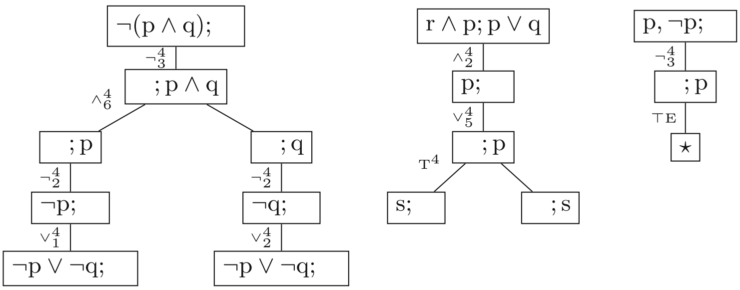

Fig. 1. Examples of derivations in tree form. For the sake of a cleaner presentation, we omit the formulas that are inherited when expanding a node.

We conclude this section by introducing the notion of (generalized) analyticity of a calculus, an important feature for proof-search procedures that is built in the axiomatizations delivered by the recipe of the next section. Given a B-statement $\mathfrak{s} := \left(\begin{array}{c:c} \Phi_{\mathsf{И}} & \Phi_{\mathsf{⅄}} \\ \hdashline \Phi_{\mathsf{Y}} & \Phi_{\mathsf{N}} \end{array}\right)$, let $\mathcal{S}(\mathfrak{s}) := \bigcup_{\alpha\in\{\mathsf{Y},\mathsf{N},\mathsf{⅄},\mathsf{И}\}}\mathsf{subf}(\Phi_\alpha)$ be the collection of *subformulas of* $\mathfrak{s}$, and $\mathcal{S}^{\Psi}(\mathfrak{s}) := \mathcal{S}(\mathfrak{s})\cup\{\sigma(A) \mid A\in\Psi, \sigma : \mathcal{P}\to\mathcal{S}(\mathfrak{s})\}$ be the *generalized subformulas of* $\mathfrak{s}$ *(with respect to* Ψ*)*. Define the 2×2-place relation $\frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}{}^{\mathcal{S}^{\Psi}}_{\mathcal{C}}$ over $\mathsf{Pow}(L_{\Sigma}(\mathcal{P}))$ by setting $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}\big|\frac{\Phi_{\mathsf{⅄}}}{\Phi_{\mathsf{N}}}{}^{\mathcal{S}^{\Psi}}_{\mathcal{C}}$ iff there is a $\mathcal{C}$-proof t of $\mathfrak{s} := \left(\begin{array}{c:c} \Phi_{\mathsf{И}} & \Phi_{\mathsf{⅄}} \\ \hdashline \Phi_{\mathsf{Y}} & \Phi_{\mathsf{N}} \end{array}\right)$

such that $\ell^t(\mathsf{nds}(t)) \subseteq \mathsf{Pow}(\mathcal{S}^\Psi(\jmath))^2 \cup \{\star\}$. Such a proof is said to be Ψ-*analytic*. We say that $\mathcal{C}$ is Ψ-*analytic* in case $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}} | \frac{\Phi_{\mathsf{\lambda}}}{\Phi_{\mathsf{N}}} \, \mathcal{C}$ implies $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}} | \frac{\Phi_{\mathsf{\lambda}}}{\Phi_{\mathsf{N}}} \, {}^{\mathcal{S}^\Psi}_{\mathcal{C}}$. We will denote by $\mathcal{C}[\jmath]$ the set of all rule instances of $\mathcal{C}$ resulting from substitutions that only use formulas in $\mathcal{S}^\Psi(\jmath)$.

3 Axiomatizing Monadic $\mathsf{B}^{\mathsf{PN}}_{\Sigma}$-matrices

We now describe four collections of rule schemas by which any sufficiently expressive $\mathsf{B}^{\mathsf{PN}}_{\Sigma}$-matrix $\mathbb{M}$ is constrained. Together, these schemas constitute a presentation of a calculus that will be denoted by $\mathcal{C}^{\mathcal{D}}$, where $\mathcal{D}$ is a discriminator for $\mathbb{M}$. The first collection, $\mathcal{C}^{\mathcal{D}}_{\exists}$, is intended to exclude all combinations of separators that do not correspond to truth-values. The second, $\mathcal{C}^{\mathcal{D}}_{\mathsf{D}}$, sets the combinations of separators that characterize acceptance apart from those that characterize non-acceptance, and sets the combinations of separators that characterize rejection apart from those that characterize non-rejection. The third one, $\mathcal{C}^{\mathcal{D}}_{\Sigma}$, fully describes, through appropriate refinements, the interpretation of the connectives of Σ in $\mathbb{M}$. At last, the rules in $\mathcal{C}^{\mathcal{D}}_{\mathbb{T}}$ guarantee that values belong to total sub–$\mathsf{B}^{\mathsf{PN}}_{\Sigma}$-matrices of $\mathbb{M}$.

In what follows, given $\mathcal{X} \subseteq \mathcal{V}^{\mathbb{M}}$, we shall use $\left(\dot{\mathcal{D}}^{\mathcal{X}}_{\mathsf{Y}}, \dot{\mathcal{D}}^{\mathcal{X}}_{\mathsf{N}}\right)$ to denote a pair of sets in which $\dot{\mathcal{D}}^{\mathcal{X}}_{\alpha}$, with $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$, is obtained by choosing an element of $\mathcal{D}^{x}_{\alpha}$ for each $x \in \mathcal{X}$. Notice that, when $\mathcal{X} = \varnothing$, the only possibility is the pair $(\varnothing, \varnothing)$; moreover, when $\mathcal{D}^{x}_{\mathsf{Y}} \cup \mathcal{D}^{x}_{\mathsf{N}} = \varnothing$ for some $x \in \mathcal{X}$, no such pair exists. The pair $\left(\dot{\mathcal{D}}^{\mathcal{X}}_{\mathsf{\lambda}}, \dot{\mathcal{D}}^{\mathcal{X}}_{\mathsf{И}}\right)$ shall be used analogously.

Definition 1. *Let* $\mathbb{M}$ *be a* $\mathsf{B}^{\mathsf{PN}}_{\Sigma}$*-matrix, and let* $\mathcal{D}$ *be a discriminator for* $\mathbb{M}$*. The calculus* $\mathcal{C}^{\mathcal{D}}$ *is presented by way of the following rule schemas:*

($\mathcal{C}^{\mathcal{D}}_{\exists}$) *for each* $\mathcal{X}_1 \subseteq \mathcal{V}^{\mathbb{M}}$ *and each possible choices of* $\left(\dot{\mathcal{D}}^{\mathcal{X}_0}_{\mathsf{Y}}, \dot{\mathcal{D}}^{\mathcal{X}_0}_{\mathsf{N}}\right)$ *and of* $\left(\dot{\mathcal{D}}^{\mathcal{X}_1}_{\mathsf{\lambda}}, \dot{\mathcal{D}}^{\mathcal{X}_1}_{\mathsf{И}}\right)$*, with* $\mathcal{X}_0 := \mathcal{V}^{\mathbb{M}} \setminus \mathcal{X}_1$*,*

$$\frac{\dot{\mathcal{D}}^{\mathcal{X}_1}_{\mathsf{\lambda}} \; ; \; \dot{\mathcal{D}}^{\mathcal{X}_1}_{\mathsf{И}}}{\dot{\mathcal{D}}^{\mathcal{X}_0}_{\mathsf{Y}} \; ; \; \dot{\mathcal{D}}^{\mathcal{X}_0}_{\mathsf{N}}}$$

($\mathcal{C}^{\mathcal{D}}_{\mathsf{D}}$) *for an arbitrary propositional variable* $\mathrm{p} \in \mathcal{P}$*, and for each* $x \in \mathcal{V}^{\mathbb{M}}$*,*

$$\frac{\mathcal{D}^{x}_{\mathsf{Y}}(\mathrm{p}), p_{\mathsf{\lambda}}(x) \; ; \; \mathcal{D}^{x}_{\mathsf{N}}(\mathrm{p})}{\mathcal{D}^{x}_{\mathsf{\lambda}}(\mathrm{p}), p_{\mathsf{Y}}(x) \; ; \; \mathcal{D}^{x}_{\mathsf{И}}(\mathrm{p})} \qquad \frac{\mathcal{D}^{x}_{\mathsf{Y}}(\mathrm{p}) \; ; \; \mathcal{D}^{x}_{\mathsf{N}}(\mathrm{p}), p_{\mathsf{И}}(x)}{\mathcal{D}^{x}_{\mathsf{\lambda}}(\mathrm{p}) \; ; \; \mathcal{D}^{x}_{\mathsf{И}}(\mathrm{p}), p_{\mathsf{N}}(x)}$$

where, for $\alpha \in \{\mathsf{Y}, \mathsf{N}, \mathsf{\lambda}, \mathsf{И}\}$*,* $p_{\alpha} : \mathcal{V}^{\mathbb{M}} \to \mathsf{Pow}(\{\mathrm{p}\})$ *is such that* $p_{\alpha}(x) = \{\mathrm{p}\}$ *iff* $x \in \alpha^{\mathbb{M}}$*.*

($\mathcal{C}^{\mathcal{D}}_{\Sigma}$) *for each* k*-ary connective* ©*, each sequence* $X := (x_1, \ldots, x_k)$ *of truth-values of* $\mathbb{M}$*, each* $y \notin ©^{\mathbb{M}}X$*, and for a sequence* $(p_1, \ldots, p_k)$ *of distinct propositional variables,*

$$\frac{\Theta^{©,X,y}_{\mathsf{Y}} \;;\; \Theta^{©,X,y}_{\mathsf{N}}}{\Theta^{©,X,y}_{\text{⅄}} \;;\; \Theta^{©,X,y}_{\text{И}}}$$

where each $\Theta^{©,X,y}_{\alpha} := \bigcup_{1 \leq i \leq k} \mathcal{D}^{x_i}_{\alpha}(p_i) \cup \mathcal{D}^{y}_{\alpha}(©(p_1, \ldots, p_k))$.

($\mathcal{C}^{\mathcal{D}}_{\mathbb{T}}$) *for each* $\mathcal{X} \notin \mathbb{T}_{\mathbb{M}}$ *and an arbitrary family* $\{p_x\}_{x \in \mathcal{X}}$ *of distinct propositional variables,*

$$\frac{\bigcup_{x \in \mathcal{X}} \mathcal{D}^{x}_{\mathsf{Y}}(p_x) \;;\; \bigcup_{x \in \mathcal{X}} \mathcal{D}^{x}_{\mathsf{N}}(p_x)}{\bigcup_{x \in \mathcal{X}} \mathcal{D}^{x}_{\text{⅄}}(p_x) \;;\; \bigcup_{x \in \mathcal{X}} \mathcal{D}^{x}_{\text{И}}(p_x)}.$$

Theorem 1. *If* $\mathcal{D}$ *is a discriminator for a* $\mathsf{B}^{\mathsf{PN}}_{\Sigma}$*-matrix* $\mathbb{M}$*, then the calculus* $\mathcal{C}^{\mathcal{D}}$ *is sound with respect to* $\mathbb{M}$*.*

Proof. We can show by contradiction that no $\mathbb{M}$-valuation can be a counter-model for the schemas in each of the groups of schemas of $\mathcal{C}^{\mathcal{D}}$. We detail the case of ($\mathcal{C}^{\mathcal{D}}_{\exists}$). Consider a schema $\mathfrak{s} := \frac{\dot{\mathcal{D}}^{\mathcal{X}_1}_{\text{⅄}} \;;\; \dot{\mathcal{D}}^{\mathcal{X}_1}_{\text{И}}}{\dot{\mathcal{D}}^{\mathcal{X}_0}_{\mathsf{Y}} \;;\; \dot{\mathcal{D}}^{\mathcal{X}_0}_{\mathsf{N}}}$, for some $\mathcal{X}_1 \subseteq \mathcal{V}^{\mathbb{M}}$ and some choice of $\left(\dot{\mathcal{D}}^{\mathcal{X}_0}_{\mathsf{Y}}, \dot{\mathcal{D}}^{\mathcal{X}_0}_{\mathsf{N}}\right)$ and $\left(\dot{\mathcal{D}}^{\mathcal{X}_1}_{\text{⅄}}, \dot{\mathcal{D}}^{\mathcal{X}_1}_{\text{И}}\right)$. Suppose that $\mathfrak{s}$ does not hold in $\mathbb{M}$, with the valuation v witnessing this fact. We will prove that, given a propositional variable p, $v(p) \neq x$, for all $x \in \mathcal{V}^{\mathbb{M}}$, an absurd. For that purpose, let $x \in \mathcal{V}^{\mathbb{M}}$. In case $x \in \mathcal{X}_1$, there must be a separator S in $\mathcal{D}^{x}_{\tilde{\alpha}}$, for some $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$, such that $v(S(p)) \in \alpha^{\mathbb{M}}$. By Lemma 1, this implies that $v(p) \neq x$. The reasoning is similar in case $x \in \mathcal{X}_0$.

In what follows, denote by $\mathcal{S}^{\mathcal{D}}$ the mapping $\mathcal{S}^{\mathcal{D}^{\bowtie}}$, which indicates what formulas may appear in a $\mathcal{D}^{\bowtie}$-analytic proof. In order to prove completeness and $\mathcal{D}^{\bowtie}$-analyticity of $\mathcal{C}^{\mathcal{D}}$ with respect to $\mathbb{M}$, we shall make use of Lemma 2 presented below, which contains four items, each one referring to a group of schemas of $\mathcal{C}^{\mathcal{D}}$. Intuitively, given a B-statement $\mathfrak{s}$ and assuming that there is no $\mathcal{D}^{\bowtie}$-analytic proof of it in $\mathcal{C}^{\mathcal{D}}$, items 1 and 2 give us the resources to define a mapping $f : \mathsf{subf}(\mathfrak{s}) \to \mathcal{V}^{\mathbb{M}}$ that, by items 3 and 4, can be extended to a countermodel for $\mathfrak{s}$ in $\mathbb{M}$.

Lemma 2. *For all* B*-statements* $\mathfrak{s}$ *of the form* $\left(\begin{array}{c:c} \Omega^{c}_{\text{Ƨ}} & \Omega^{c}_{\mathsf{S}} \\ \hdashline \Omega_{\mathsf{S}} & \Omega_{\text{Ƨ}} \end{array}\right)$:

1. *if* $\frac{\Omega^{c}_{\text{Ƨ}}}{\Omega_{\mathsf{S}}} \ast \frac{\Omega^{c}_{\mathsf{S}}}{\Omega_{\text{Ƨ}}} {}^{\mathcal{S}^{\mathcal{D}}}_{\mathcal{C}^{\mathcal{D}}_{\exists}}$, *then for all* $A \in \mathsf{subf}(\mathfrak{s})$ *there is an* $x \in \mathcal{V}^{\mathbb{M}}$ *such that* $\mathcal{D}^{x}_{\alpha}(A) \subseteq \Omega_{\beta}$ *and* $\mathcal{D}^{x}_{\tilde{\alpha}}(A) \subseteq \Omega^{c}_{\beta}$*, for* $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \text{Ƨ})\}$*;*
2. *if* $\frac{\Omega^{c}_{\text{Ƨ}}}{\Omega_{\mathsf{S}}} \ast \frac{\Omega^{c}_{\mathsf{S}}}{\Omega_{\text{Ƨ}}} {}^{\mathcal{S}^{\mathcal{D}}}_{\mathcal{C}^{\mathcal{D}}_{\mathsf{D}}}$, *then for every* $A \in \mathsf{subf}(\mathfrak{s})$ *and* $x \in \mathcal{V}^{\mathbb{M}}$ *such that* $\mathcal{D}^{x}_{\alpha}(A) \subseteq \Omega_{\beta}$ *and* $\mathcal{D}^{x}_{\tilde{\alpha}}(A) \subseteq \Omega^{c}_{\beta}$*, we have* $x \in \alpha^{\mathbb{M}}$ *iff* $A \in \Omega_{\beta}$*, for* $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \text{Ƨ})\}$*;*

3. if $\frac{\Omega_{\text{Ƨ}}^{\mathsf{c}}}{\Omega_{\mathsf{S}}} \not\ast \frac{\Omega_{\mathsf{S}}^{\mathsf{c}}}{\Omega_{\text{Ƨ}}} \,{}^{\mathcal{S}^{\mathcal{D}}}_{\mathcal{C}^{\mathcal{D}}_{\Sigma}}$*, then for every* $© \in \Sigma_{\mathrm{k}}$*,* $\mathrm{A} := ©(\mathrm{A}_1, \ldots, \mathrm{A}_{\mathrm{k}}) \in \mathsf{subf}(\jmath)$ *and* $x_1, \ldots, x_{\mathrm{k}} \in \mathcal{V}^{\mathbb{M}}$ *with* $\mathcal{D}^{x_{\mathrm{i}}}_{\alpha}(\mathrm{A}_{\mathrm{i}}) \subseteq \Omega_{\beta}$ *and* $\mathcal{D}^{x_{\mathrm{i}}}_{\tilde{\alpha}}(\mathrm{A}_{\mathrm{i}}) \subseteq \Omega^{\mathsf{c}}_{\beta}$*, for each* $1 \leq \mathrm{i} \leq \mathrm{k}$ *and* $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \text{Ƨ})\}$*, we have that* $\mathcal{D}^{y}_{\alpha}(\mathrm{A}) \subseteq \Omega_{\beta}$ *and* $\mathcal{D}^{y}_{\tilde{\alpha}}(\mathrm{A}) \subseteq \Omega^{\mathsf{c}}_{\beta}$ *for each* $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \text{Ƨ})\}$ *implies* $y \in ©^{\mathbb{M}}(x_1, \ldots, x_{\mathrm{k}})$*;*

4. if $\frac{\Omega_{\text{Ƨ}}^{\mathsf{c}}}{\Omega_{\mathsf{S}}} \not\ast \frac{\Omega_{\mathsf{S}}^{\mathsf{c}}}{\Omega_{\text{Ƨ}}} \,{}^{\mathcal{S}^{\mathcal{D}}}_{\mathcal{C}^{\mathcal{D}}_{\mathbb{T}}}$*, then* $\{x \in \mathcal{V}^{\mathbb{M}} \mid \mathcal{D}^{x}_{\alpha}(\mathrm{A}) \subseteq \Omega_{\beta}$ *and* $\mathcal{D}^{x}_{\tilde{\alpha}}(\mathrm{A}) \subseteq \Omega^{\mathsf{c}}_{\beta}$*,*
for each $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \text{Ƨ})\}$ *and* $\mathrm{A} \in \mathsf{subf}(\jmath)\} \in \mathbb{T}_{\mathbb{M}}$*.*

Proof. The strategy to prove each item is the same: by contraposition, use the data from the assumptions to compose an instance of a rule schema of the corresponding group of rule schemas. We detail below the proof for the third item. Suppose that there is a connective $© \in \Sigma_{\mathrm{k}}$, a formula $\mathrm{A} := ©(\mathrm{A}_1, \ldots, \mathrm{A}_{\mathrm{k}}) \in \mathsf{subf}(\jmath)$, a sequence $(x_1, \ldots, x_{\mathrm{k}})$ of truth-values with $\mathcal{D}^{x_{\mathrm{i}}}_{\alpha}(\mathrm{A}_{\mathrm{i}}) \subseteq \Omega_{\beta}$ and $\mathcal{D}^{x_{\mathrm{i}}}_{\tilde{\alpha}}(\mathrm{A}_{\mathrm{i}}) \subseteq \Omega^{\mathsf{c}}_{\beta}$ for each $1 \leq \mathrm{i} \leq \mathrm{k}$ and $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \text{Ƨ})\}$, and some $y \notin ©^{\mathbb{M}}(x_1, \ldots, x_{\mathrm{k}})$ such that $\mathcal{D}^{y}_{\alpha}(\mathrm{A}) \subseteq \Omega_{\beta}$ and $\mathcal{D}^{y}_{\tilde{\alpha}}(\mathrm{A}) \subseteq \Omega^{\mathsf{c}}_{\beta}$ for each $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \text{Ƨ})\}$. Then $\bigcup_{1 \leq \mathrm{i} \leq \mathrm{k}} \mathcal{D}^{x_{\mathrm{i}}}_{\alpha}(\mathrm{A}_{\mathrm{i}}) \cup \mathcal{D}^{y}_{\alpha}(\mathrm{A}) \subseteq \Omega_{\beta} \cap \mathcal{S}^{\mathcal{D}}(\jmath)$ and $\bigcup_{1 \leq \mathrm{i} \leq \mathrm{k}} \mathcal{D}^{x_{\mathrm{i}}}_{\tilde{\alpha}}(\mathrm{A}_{\mathrm{i}}) \cup \mathcal{D}^{y}_{\tilde{\alpha}}(\mathrm{A}) \subseteq \Omega^{\mathsf{c}}_{\beta} \cap \mathcal{S}^{\mathcal{D}}(\jmath)$ for each $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \text{Ƨ})\}$, and thus we have $\frac{\Omega_{\text{Ƨ}}^{\mathsf{c}}}{\Omega_{\mathsf{S}}} \mid \frac{\Omega_{\mathsf{S}}^{\mathsf{c}}}{\Omega_{\text{Ƨ}}} \,{}^{\mathcal{S}^{\mathcal{D}}}_{\mathcal{C}^{\mathcal{D}}_{\Sigma}}$.

Theorem 2. *If* $\mathcal{D}$ *is a discriminator for a* $\mathsf{B}^{\underline{PN}}_{\Sigma}$*-matrix* $\mathbb{M}$*, then the calculus* $\mathcal{C}^{\mathcal{D}}$ *is complete with respect to* $\mathbb{M}$*. Furthermore, this calculus is* $\mathcal{D}^{\bowtie}$*–analytic.*

Proof. Let $\jmath := \left(\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} \vdots \frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}}\right)$ be a B-statement and suppose that (a) $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} \not\ast \frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}} \,{}^{\mathcal{S}^{\mathcal{D}}}_{\mathcal{C}^{\mathcal{D}}}$. Our goal is to build an $\mathbb{M}$-valuation witnessing $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} \not\ast \frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}} \,\mathbb{M}$. From (a), by (C), we have that (b) there are $\Phi_{\mathsf{Y}} \subseteq \Omega_{\mathsf{S}} \subseteq \Phi^{\mathsf{c}}_{\text{⅄}}$ and $\Phi_{\mathsf{N}} \subseteq \Omega_{\text{Ƨ}} \subseteq \Phi^{\mathsf{c}}_{\text{И}}$ such that $\frac{\Omega_{\text{Ƨ}}^{\mathsf{c}}}{\Omega_{\mathsf{S}}} \not\ast \frac{\Omega_{\mathsf{S}}^{\mathsf{c}}}{\Omega_{\text{Ƨ}}} \,{}^{\mathcal{S}^{\mathcal{D}}}_{\mathcal{C}^{\mathcal{D}}}$. Consider then a mapping $\mathrm{f} : \mathsf{subf}(\jmath) \to \mathcal{V}^{\mathbb{M}}$ with (c) $\mathrm{f}(\mathrm{A}) \in \alpha^{\mathbb{M}}$ iff $\mathrm{A} \in \Omega_{\beta}$, for $(\alpha, \beta) \in \{(\mathsf{Y}, \mathsf{S}), (\mathsf{N}, \text{Ƨ})\}$, whose existence is guaranteed by items (1) and (2) of Lemma 2. Notice that items (3) and (4) of this same lemma imply, respectively, that $\mathrm{f}(©(\mathrm{A}_1, \ldots, \mathrm{A}_{\mathrm{k}})) \in ©^{\mathbb{M}}(\mathrm{f}(\mathrm{A}_1), \ldots, \mathrm{f}(\mathrm{A}_{\mathrm{k}}))$ for every $©(\mathrm{A}_1, \ldots, \mathrm{A}_{\mathrm{k}}) \in \mathcal{S}^{\mathcal{D}}(\jmath)$, and $\mathrm{f}(\mathsf{subf}(\jmath)) \in \mathbb{T}_{\mathbb{M}}$. Hence, f may be extended to an $\mathbb{M}$-valuation v and, from (b) and (c), we have $v(\Phi_{\alpha}) \subseteq \alpha^{\mathbb{M}}$ for each $\alpha \in \{\mathsf{Y}, \mathsf{N}, \text{⅄}, \text{И}\}$, so $\frac{\Phi_{\text{И}}}{\Phi_{\mathsf{Y}}} \not\ast \frac{\Phi_{\text{⅄}}}{\Phi_{\mathsf{N}}} \,\mathbb{M}$.

The calculi presented so far (Examples 7 and 8) were produced by means of the axiomatization procedure just described, followed by some simplifications consisting of removing instances of conditions (O) and (D), and using condition (C) on pairs of schemas having the forms $\frac{\Phi_{\mathsf{Y}}, \mathrm{A} \;;\; \Phi_{\mathsf{N}}}{\Phi_{\text{⅄}} \;;\; \Phi_{\text{И}}}$ and $\frac{\Phi_{\mathsf{Y}} \;;\; \Phi_{\mathsf{N}}}{\Phi_{\text{⅄}}, \mathrm{A} \;;\; \Phi_{\text{И}}}$, or the forms $\frac{\Phi_{\mathsf{Y}} \;;\; \Phi_{\mathsf{N}}}{\Phi_{\text{⅄}} \;;\; \Phi_{\text{И}}, \mathrm{A}}$ and $\frac{\Phi_{\mathsf{Y}} \;;\; \Phi_{\mathsf{N}}, \mathrm{A}}{\Phi_{\text{⅄}} \;;\; \Phi_{\text{И}}}$, yielding in either case the schema $\frac{\Phi_{\mathsf{Y}} \;;\; \Phi_{\mathsf{N}}}{\Phi_{\text{⅄}} \;;\; \Phi_{\text{И}}}$. By Theorem 2 and the fact that these simplifications preserve analyticity, it follows that such calculi are analytic. It is also worth mentioning that this same procedure may be applied to the matrix $\mathbb{P}$ in view of its monadicity (see a discriminator for it in Example 5), which means that we also obtain a *finite* Hilbert-style symmetrical axiomatization for **mCi**.

4 Proof Search in Two Dimensions

Throughout this section, let $\mathscr{s} := \left(\frac{\Phi_{И}}{\Phi_{\mathsf{Y}}} \vdots \frac{\Phi_{⅄}}{\Phi_{\mathsf{N}}}\right)$ be an arbitrary B-sequent, $\mathcal{C}$ be a finite and finitary calculus, and Ψ be a finite set of formulas. Notice that, whenever $\mathcal{C}$ is Ψ-analytic, it is enough to consider the rule instances in $\mathcal{C}[\mathscr{s}]$ in order to provide a proof of $\mathscr{s}$ in $\mathcal{C}$. Searching for such a proof is clearly a particular case of finding a proof of $\mathscr{s}$ using only candidates in a finite set R of finitary rule instances. A proof-search algorithm for this more general setting is presented in Algorithm 1 by means of a function called Expand. The algorithm searches for a proof by expanding nodes that are not closed or discontinued using only instances in R that were not used yet in the branch of the node under expansion. As we shall see in the sequel, the order in which applicable instances are selected does not affect the result, although for sure smarter choice heuristics may well improve the performance of the algorithm in particular cases.

Algorithm 1: Proof search over a finite set of finitary rule instances

function Expand$(\mathrm{F} := (\Psi_{\mathsf{Y}}, \Psi_{\mathsf{N}}),\ \mathrm{C} := (\Phi_{И}, \Phi_{⅄}),\ \mathrm{R})$**:**
  Input: antecedents in F, succedents in C and a finite set R of finitary rule instances
  $t \leftarrow \mathsf{sntree}(\mathrm{F})$
  if $\Psi_\alpha \cap \Phi_{\tilde{\alpha}} \neq \varnothing$ *for some* $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$ **then return** t
  foreach *rule instance* $\mathfrak{r}^\sigma := \frac{\Theta_{\mathsf{Y}}\ ;\ \Theta_{\mathsf{N}}}{\Theta_{⅄}\ ;\ \Theta_{И}} \in \mathrm{R}$ **do**
    if $\Theta_{\tilde{\alpha}} \cap \Psi_\alpha = \varnothing$ *and* $\Theta_\alpha \subseteq \Psi_\alpha$ *for each* $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$ **then**
      if $\Theta_{⅄} \cup \Theta_{И} = \varnothing$ **then return** t *with a single child* $\mathsf{sntree}(\star)$
      foreach $\alpha \in \{\mathsf{Y}, \mathsf{N}\}$ *and* $\mathrm{A} \in \Theta_{\tilde{\alpha}}$ **do**
        $t' \leftarrow$ Expand$((\Psi_{\mathsf{Y}} \cup \mathrm{P}_{\mathsf{Y}}(\mathrm{A}), \Psi_{\mathsf{N}} \cup \mathrm{P}_{\mathsf{N}}(\mathrm{A})),\ \mathrm{C},\ \mathrm{R}\backslash\{\mathfrak{r}^\sigma\})$, where $\mathrm{P}_\alpha(\mathrm{A})$ is $\varnothing$ if $\mathrm{A} \notin \Theta_\alpha$ and $\{\mathrm{A}\}$ otherwise
        add $\mathsf{rt}(t')$ as a child of $\mathsf{rt}(t)$ in t
        if t' *is not* C*-closed* **then return** t
      if t *is* C*-closed* **then return** t
  return t

The following lemma (verifiable by induction on the size of R) proves the termination of Expand and its correctness. The subsequent result establishes the applicability of this algorithm for proof search over Ψ-analytic calculi.

Lemma 3. *Let* R *be a finite set of finitary rule instances. Then the procedure* Expand$((\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}}), (\Phi_{⅄}, \Phi_{И}), \mathrm{R})$ *always terminates, returning a tree that is* $(\Phi_{⅄}, \Phi_{И})$*-closed iff* $\frac{\Phi_{И}}{\Phi_{\mathsf{Y}}} | \frac{\Phi_{⅄}}{\Phi_{\mathsf{N}}}\ \mathrm{R}$.

Lemma 4. *If* $\mathcal{C}$ *is* Ψ*-analytic, then* Expand *is a proof-search algorithm for* $\mathcal{C}$ *and a decision procedure for* $\frac{\cdot}{\cdot}|\frac{\cdot}{\cdot}\,\mathcal{C}$*.*

Proof. We know that $\mathcal{C}[\mathscr{s}]$ provides enough material for a derivation of $\mathscr{s}$ to be produced, since $\mathcal{C}$ is Ψ-analytic. Clearly, such set is finite and contains only finitary rule instances, hence the present result is a direct consequence of Lemma 3.

The next results concern the complexity of Algorithm 1. In what follows, let R be a finite set of finitary rule instances, $\mathrm{b} := \max_{\mathfrak{r}^\sigma \in \mathrm{R}} \mathsf{branch}(\mathfrak{r}^\sigma)$, $\mathrm{s} := \mathsf{size}(\{\mathscr{s}\} \cup \mathrm{R})$ and $\mathrm{n} := |\mathrm{R}|$. We shall use $\mathsf{p}(\mathrm{m})$ to refer to "a polynomial in m".

Lemma 5. *The worst-case running time of* Expand$((\Phi_{\mathsf{Y}}, \Phi_{\mathsf{N}}), (\Phi_{\mathsf{\lambda}}, \Phi_{\mathsf{И}}), \mathrm{R})$ *is* $\mathrm{O}(\mathrm{b}^\mathrm{n} + \mathrm{n} \cdot \mathsf{p}(\mathrm{s}))$*.*

Proof. Let T(n, s) be the worst-case running-time of Expand. Note that it occurs under three conditions: first, $\frac{\Phi_{\mathsf{И}}}{\Phi_{\mathsf{Y}}}|\frac{\Phi_{\mathsf{\lambda}}}{\Phi_{\mathsf{N}}}\,\mathrm{R}$; second, the set R needs to be entirely inspected until an applicable rule instance is found; and third, such an instance does not have an empty set of succedents. Notice that $\mathrm{T}(0, \mathrm{s}) = \mathrm{c}_1 + \mathsf{p}(\mathrm{s})$ and, based on the assignments above and after some algebraic manipulations, we have, for $\mathrm{n} \geq 1$, $\mathrm{T}(\mathrm{n}, \mathrm{s}) \leq \mathrm{b} \cdot \mathrm{T}(\mathrm{n}-1, \mathrm{s} + \mathsf{p}(\mathrm{s})) + 2\mathrm{n} \cdot \mathsf{p}(\mathrm{s})$. It is then straightforward to check by induction on n that $\mathrm{T}(\mathrm{n}, \mathrm{s}) \in \mathrm{O}(\mathrm{b}^\mathrm{n} + \mathrm{n} \cdot \mathsf{p}(\mathrm{s}))$.

Theorem 3. *If* $\mathcal{C}$ *is* Ψ*-analytic,* Expand *is a proof-search algorithm for* $\mathcal{C}$ *that runs in exponential time in general, and in polynomial time if* $\mathcal{C}$ *contains only rules with at most one formula in the succedent.*

Proof. Clearly, the set of all instances of rules of $\mathcal{C}$ using only formulas in $\mathcal{S}^\Psi(\mathscr{s})$ is finite and contains only finitary rule instances, and its size is polynomial in $\mathsf{size}(\mathscr{s})$. The announced result then follows directly from Lemma 5.

The previous result makes the axiomatization procedure presented in Sect. 3 even more attractive, since it delivers a $\mathcal{D}^{\bowtie}$–analytic calculus for $\mathbb{M}$, where $\mathcal{D}^{\bowtie}$ is a finite set of formulas acting as separators. It follows then that Expand is a proof-search algorithm for such axiomatization running in at most exponential time. More than that, Expand outputs a tree with at least one open branch when the B-sequent $\mathscr{s}$ of interest is not provable. From such branch, one may obtain a partition of $\mathcal{S}^{\mathcal{D}}(\mathscr{s})$ and, by Proposition 2, define a mapping on $\mathsf{subf}(\mathscr{s})$ that extends to an $\mathbb{M}$-valuation. It follows that the discussed algorithm may easily be adapted so as to deliver a countermodel when $\mathscr{s}$ is unprovable. For experimenting with the axiomatization procedure and searching for proofs over the generated calculus, one can make use of the implementation that may be found at https://github.com/greati/logicantsy. We should also emphasize that, by Theorem 3 and the axiomatization procedure given in Sect. 3, we have:

Corollary 1. *Any finite monadic* $\mathsf{B}_\Sigma^{\mathsf{PN}}$*-matrix* $\mathbb{M}$ *whose induced axiomatization contains only rules with at most one succedent is decidable in polynomial time.*

By the above result, then, the B-entailment relation $\div|\div\,\mathbb{I}$ (from Example 1) is decidable in polynomial time. Consequently, the same also holds for its t-aspect, which is inhabited by the 4-valued logic introduced in [3].

In addition, it is worth stressing that, although no better in the limiting cases, the axiomatization provided in Sect. 3 together with the algorithm presented in this section translate the problem of deciding a B-entailment relation into a purely symbolic procedure that may perform better than searching for $\mathbb{M}$-valuations in some cases.

We close with another complexity result concerning the decidability of $\div|\div\,\mathcal{C}$, complementing the one given by the discussed algorithm; it follows by an argument similar to the one presented for the one-dimensional case in [18].

Theorem 4. *If* $\mathcal{C}$ *is* Ψ*-analytic, then the problem of deciding* $\div|\div\,\mathcal{C}$ *is in* coNP.

5 Conclusion

In this paper, we approached bilateralism by exploring a two-dimensional notion of consequence, considering the cognitive attitudes of acceptance and rejection instead of the conventional speech acts of assertion and of denial. Our intervention has been two-fold: on the semantical front we have employed two-dimensional (partial) non-deterministic logical matrices, and on proof-theoretical grounds we have employed two-dimensional symmetrical proof formalisms which generalize traditional Hilbert-style calculi and their associated unilinear notion of derivation. As a result, and generalizing [10], we have provided an axiomatization procedure that delivers analytic calculi for a very expressive class of finite monadic matrices. On what concerns proof development, in spite of well-known evidence about the p-equivalence between Hilbert-style calculi and Gentzen-style calculi ([14]), die-hard popular belief concerning their 'deep inequivalence' seems hard to wash away. To counter that belief with facts, we developed for our calculi a general proof-search algorithm that was secured to run in exponential time.

We highlight that our two-dimensional proof-formalism differs in important respects from the many-placed sequent calculi used in [4] to axiomatize (one-dimensional total) non-deterministic matrices (requiring no sufficient expressiveness) and in [16] for approaching multilateralism. First, a many-placed sequent calculus is not Hilbert-style: rules manipulate complex objects whose structures involve contexts and considerably deviate from the shape of the consequence relation being captured; our calculi, on the other hand, are contained in their corresponding B-consequences. Second, when axiomatizing a matrix, the structure of many-placed sequents grows according to the number of values (n places for n truth-values); our rule schemas, in turn, remain with four places, and reflect the complexity of the underlying semantics in the complexity of the formulas being manipulated. Moreover, the study of many-placed sequents currently contemplates only one-dimensional consequence relations; extending them to the two-dimensional case is a line of research worth exploring.

As further future work, we envisage generalizing the two-dimensional notion of consequence relation by allowing logics over different languages ([17]) —for instance, conflating different logics or different fragments of some given logic of interest— to coinhabit the same logical structure, each one along its own dimension, while controlling their interaction at the object-language level, taking advantage of the framework and the results in [18]. This opens the doors for a line of investigation on whether or to what extent the individual characteristics of these ingredient logics, such as their decidability status, may be preserved. With respect to our proof search algorithm, an important research path to be explored would involve the design of heuristics for smarter choices of rule instances used to expand nodes during the search, as this may improve the performance of the algorithm on certain classes of logics. At last, we also expect to extend the present research so as to cover multidimensional notions of consequence, in order to provide increasingly general technical and philosophical grounds for the study of logical pluralism.

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