Adding an Implication to Logics of Perfect Paradefinite Algebras

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Abstract

Perfect paradefinite algebras are De Morgan algebras expanded with a perfection (or classicality) operation. They form a variety that is term-equivalent to the variety of involutive Stone algebras. Their associated multiple-conclusion (Set-Set) and single-conclusion (Set-Fmla) order-preserving logics are non-algebraizable self-extensional logics of formal inconsistency and undeterminedness determined by a six-valued matrix, studied in depth by Gomes et al. (2022) from both the algebraic and the proof-theoretical perspectives. We continue hereby that study by investigating directions for conservatively expanding these logics with an implication connective (essentially, one that admits the deduction-detachment theorem). We first consider logics given by very simple and manageable non-deterministic semantics whose implication (in isolation) is classical. These, nevertheless, fail to be self-extensional. We then consider the implication realized by the relative pseudo-complement over the six-valued perfect paradefinite algebra. Our strategy is to expand such algebra with this connective and study the (self-extensional) Set-Set and Set-Fmla order-preserving and $\top$-assertional logics of the variety induced by the new algebra. We provide axiomatizations for such new variety and for such logics, drawing parallels with the class of symmetric Heyting algebras and with Moisil's 'symmetric modal logic'. For the Set-Set logic, in particular, the axiomatization we obtain is analytic. We close by studying interpolation properties for these logics and concluding that the new variety has the Maehara amalgamation property.

Keywords paradefinite logics · logics of formal inconsistency and undeterminedness · implication · Heyting algebras · De Morgan algebras.

1 Introduction

The algebraic structures we now call *involutive Stone algebras* appear to have been first considered by Roberto Cignoli and Marta Sagastume along their investigation of finite-valued Łukasiewicz logics, in connection with Łukasiewicz-Moisil algebras (Cignoli and Sagastume [1981, 1983]). Formally, involutive Stone algebras are usually presented as expansions of De Morgan algebras (whose languages consists of a conjunction $\wedge$, a disjunction $\vee$, a negation $\sim$ and the lattice bounds $\bot, \top$) in one of the following two alternative term-equivalent ways (we shall soon see that a third one has been recently proposed):

1. by adding a unary 'possibility' operator (usually denoted by ∇ in the literature);
2. by adding an 'intuitionistic' (namely, a pseudo-complement) negation (denoted by $\neg$) satisfying the well-known Stone equation: $\neg x \vee \neg\neg x \approx \top$.

Though obviously arising as "algebras of logic", the class of involutive Stone algebras was not employed as an algebraic semantics for logical systems in the above-mentioned seminal works; rather, this has been pursued in a series of recent papers which focused in particular on the logic that preserves the lattice order of involutive Stone algebras (Cantú [2019], Cantú and Figallo [2018], Marcelino and Rivieccio [2022], Cantú and Figallo [2022]).

In the paper Gomes et al. [2022], which is the immediate predecessor of the present one, we built on the observation (made in Cantú and Figallo [2018]) that the logic of involutive Stone algebras may be viewed as a *Logic of Formal Inconsistency* in the sense of Marcos [2005a], Carnielli et al. [2007], and axiomatized it as such. This was possible due to the fact that involutive Stone algebras may be presented in a third term-equivalent way, namely:

3. by adding a unary 'perfection' operator (denoted by $\circ$) of the kind considered in Marcos [2005b].

Reformulated in this language ($\{\wedge, \vee, \sim, \circ, \bot, \top\}$), involutive Stone algebras have been renamed *perfect paradefinite algebras* (PP-algebras) in Gomes et al. [2022]. In that paper we axiomatized the SET-FMLA logic of order of PP-algebras, called $\mathcal{PP}_{\leq}$, and also showed that $\mathcal{PP}_{\leq}$ is semantically determined by a finite matrix based on the six-element algebra (there dubbed $\mathbf{PP_6}$) that generates the class of PP-algebras as a variety. Actually, we developed most of our results with respect to a SET-SET version of this logic, called $\mathcal{PP}^{\triangleright}_{\leq}$. The SET-SET axiomatization we presented for it is analytic, and thus suitable for automated reasoning. We then used all these results to aid in the study of $\mathcal{PP}_{\leq}$.

In the present paper, further pursuing this approach, we shall focus on the question of *how to add an implication connective to* SET-SET *and* SET-FMLA *logics associated to PP-algebras*. In an effort to proceed in a systematic fashion, we shall be guided by the following main principles:

- the resulting system must be conservative over the logics being extended;
- the new connective must indeed qualify as an implication according to some general standard.

In order to narrow down our search and to eventually converge upon a short list of candidates, we shall presently consider a third guiding principle as well, namely:

- the new logic must also qualify as a *logic of order* of a suitable class of algebras.

In the present setting, as we shall see, the above principle turns out to be equivalent to requiring the logic to be *self-extensional*, or congruential (see Definition 2.3).

The approach outlined above for the study of implicational extensions is reminiscent of — indeed, directly inspired by— Avron's proposal for extending Belnap's logic (Avron [2020]), in which two main requirements are entertained: first, that the new connective should be an implication relative to a given set of designated elements (condition (A1) in Subsection 2.4); second, that the resulting logic should be self-extensional (A2). We shall discuss these requirements at length in Section 4, where we show that in our case there is unfortunately *no implication connective that meets both of them* (Theorem 4.1). This is where we shall choose to retain the latter requirement (self-extensionality) instead of the former, which we find too restrictive, as we shall also discuss. As this choice still leaves plenty of room for a large collection of alternative implications, we shall follow two North Stars: Algebra and Tradition.

Algebra suggests that, for a wide family of logics, a well-behaved implication connective may be obtained by considering, whenever available, the residuum of the algebraic operation that realizes the logical conjunction (see e.g. Galatos et al. [2007]): this leads us to expand the logics of PP-algebras by a relative pseudo-complement implication, which we shall call a *Heyting implication*, for our definition mirrors the usual one for the implication on Heyting algebras.

Tradition, in the present setting, turns out to point in the same direction as algebra. In fact, it turns out that a substantial part of the theory of involutive Stone algebras was already developed, even prior to Cignoli and Sagastume's works, in Monteiro [1980], itself a collection of earlier material. The main subject of Monteiro's monograph is the class of *symmetric Heyting algebras*, providing an algebraic counterpart to Moisil's *symmetric modal logic*. These algebras are presented precisely in the traditional language of involutive Stone algebras (with the ∇ operator which Monteiro dubs 'possibility') enriched with a Heyting implication.

The formal relation between the class of PP-algebras enriched with an implication that we shall define and Monteiro's symmetric Heyting algebras is discussed in detail in Section 7. For the time being, let us conclude these introductory remarks by mentioning an alternative proposal concerning our main question — how to add an implication to logics of PP-algebras, one of them being $\mathcal{PP}_{\leq}$ — which can be retrieved from a recent paper by Coniglio and Rodrigues [2023]. Their purpose is to add a perfection operator $\circ$ to the well-known Belnap-Dunn four-valued logic, but the authors actually start from a logic which is itself a conservative expansion of the Belnap-Dunn logic with a *classic-like implication* (in the sense of Definition 2.11). The resulting logic – dubbed $\mathrm{LET}_{\mathrm{K}}^{+}$ – turns out to be determined by a six-valued matrix that (if we disregard the implication) coincides with the matrix $\langle \mathbf{PP_6}, \uparrow\mathbf{b}\rangle$ which determines the logic $\mathcal{PP}_{\leq}$ (see Gomes et al. [2022]). As noted in Coniglio and Rodrigues [2023, Subsec. 5.1], it follows that the implication-free fragment of $\mathrm{LET}_{\mathrm{K}}^{+}$ coincides with $\mathcal{PP}_{\leq}$. The implication connective of $\mathrm{LET}_{\mathrm{K}}^{+}$ consists thus in a candidate for expanding $\mathcal{PP}_{\leq}$ with an implication. An inspection of its truth-table (see Coniglio and Rodrigues [2023, Def. 3.9]) reveals (see also Section 4) that this implication satisfies (with respect to the designated set $\uparrow\mathbf{b}$) the first among Avron's requirements, $(\mathsf{A1})$, and therefore destroys self-extensionality (or, one may say, preserves the non-self-extensional character of paraconsistent Nelson logic). This leads to an interesting observation, namely, that $\mathrm{LET}_{\mathrm{K}}^{+}$ is determined by a refinement of the non-deterministic matrix introduced in Section 4, and may be obtained, thus, from the corresponding logic by adding suitable rules. In this sense, we may say our approach subsumes that of Coniglio and Rodrigues [2023], at least insofar as the task of conservatively expanding $\mathcal{PP}_{\leq}$ is concerned.

The paper is organized as follows. After the preliminary sections in which we fix the notation and basic definitions, we delve into the problem of expanding the logics of order of PP-algebras with an implication. To that effect, we first consider in Section 4 adding an implication satisfying at once $(\mathsf{A1})$ and $(\mathsf{A2})$. After proving the impossibility of this task, we focus on $(\mathsf{A1})$, which makes the implication qualify as classic-like. We explore possibilities of doing so using the semantical framework of non-deterministic logical matrices, providing analytic axiomatizations for the obtained SET-SET logics, from which we can readily obtain SET-FMLA axiomatizations for the corresponding SET-FMLA companions.

Next, having observed that the matrix $\langle \mathbf{PP_6}, \uparrow\mathbf{b}\rangle$, which determines $\mathcal{PP}_{\leq}^{\triangleright}$ and $\mathcal{PP}_{\leq}$, is based on an algebra ($\mathbf{PP_6}$) that is a finite distributive lattice, in Section 5 we enrich its language with an operation $\Rightarrow_{\mathsf{H}}$ which corresponds to the relative pseudo-complement implication determined by the lattice order. In fact, the six-element Heyting algebra thus obtained (denoted by $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$) turns out to be a symmetric Heyting algebra in Monteiro's sense. We then consider the family of all matrices $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, D\rangle$ such that D is a non-empty lattice filter of the lattice reduct of $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$. The (SET-SET and SET-FMLA) logics determined by this family are the order-preserving logics induced by the variety $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$, guaranteed to be self-extensional.

In Section 6, we proceed to axiomatize both the SET-SET and the SET-FMLA logics thus defined. First focusing on the SET-SET logic, we propose an axiomatization by a detailed analysis of the algebraic structure of the matrices and of their designated sets. More than sound and complete, the calculus turn out to be analytic. From such axiomatization we show how to extract an axiomatization for the corresponding SET-FMLA logic of order.

In Section 7, we look at the algebraic models that correspond to the SET-FMLA logics $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ and $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ (the $\top$-assertional logic associated to $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$) within the general theory of algebraization of logics. We begin by realizing that both logics are closely related to Moisil's 'symmetric modal logic', whose algebraic counterpart is the class of symmetric Heyting algebras (Definition 7.1). Indeed, $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$ is (term-equivalent to) a symmetric Heyting algebra, and the class of algebras providing an algebraic semantics for both $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ and $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ is precisely the subvariety $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$ of symmetric Heyting algebras generated by $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$ (Theorem 7.5); an equational presentation for $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$ is introduced in Definition 7.3, and shown to be sound and complete in Theorem 7.4. The algebraizability of Moisil's logic (Proposition 7.2) entails that $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ is also algebraizable; its equivalent algebraic semantics is precisely $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$, and its reduced matrix models are the matrices of the form $\langle \mathbf{A}, \{\top\}\rangle$ with $\mathbf{A} \in \mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$ (Theorem 7.5). On the other hand, $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ is not algebraizable (Proposition 5.8) but its algebraic counterpart is also $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$; the shape of the reduced matrix models of $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ is described in Proposition 7.13. We conclude the section by looking at the subvarieties of $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$. There are only three of them (corresponding precisely to the self-extensional extensions of $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$) which can be axiomatized by adding the axioms that translate the equations shown in Corollary 7.11.

In Section 8, we draw some conclusions regarding interpolation for the implicative extensions of interest, as well as amalgamation for the class of algebraic models of $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$. Further paths of investigation are highlighted in Section 9.

2 Preliminaries

In this section, we introduce the main concepts related to algebras, logics and axiomatizations, with particular attention to perfect paradefinite algebras and their logics. We also fix what we shall understand as an implication and the selection criteria we will employ in investigating extensions of a logic by the addition of an implication.

2.1 Algebras, languages and logics

A *propositional signature* is a family $\Sigma := \{\Sigma_k\}_{k\in\omega}$, where each Σ_k is a collection of k-ary *connectives*. A Σ-*multialgebra* is a structure $\mathbf{A} := \langle A, \cdot^{\mathbf{A}}\rangle$, where A is a non-empty set called the *carrier* of $\mathbf{A}$ and, for each $© \in \Sigma_k$, the *multioperation* $©^{\mathbf{A}} : A^k \to \wp(A)$ is the *interpretation* of © in $\mathbf{A}$. We say that $\mathbf{A}$ is a Σ-*algebra* (or *deterministic* Σ-*multialgebra*) when $©^{\mathbf{A}}(a_1,\ldots,a_k)$ is a singleton for every $a_1,\ldots,a_k \in A$, $© \in \Sigma_k$ and $k \in \omega$. If $©^{\mathbf{A}}(a_1,\ldots,a_k) \neq \varnothing$ for every $a_1,\ldots,a_k \in A$, $© \in \Sigma_k$ and $k \in \omega$, we say that $\mathbf{A}$ is *total*. Note that the notion of Σ-algebra matches the usual notion of algebra in Universal Algebra (cf. Sankappanavar [1987]) (which we will employ throughout the paper without notice). Given $\Sigma' \subseteq \Sigma$ (that is, $\Sigma'_k \subseteq \Sigma_k$ for all $k \in \omega$), the Σ'-*reduct* of a Σ-multialgebra $\mathbf{A}$ is the Σ'-multialgebra over the same carrier of $\mathbf{A}$ that agrees with $\mathbf{A}$ on the interpretation of the connectives in Σ'.

A Σ-*homomorphism* between Σ-multialgebras $\mathbf{A}$ and $\mathbf{B}$ is a mapping $h : A \to B$ such that $h(©^{\mathbf{A}}(a_1,\ldots,a_k)) \in ©^{\mathbf{B}}(h(a_1),\ldots,h(a_k))$ for all $a_1,\ldots,a_k \in A$, $© \in \Sigma_k$ and $k \in \omega$. Note that for Σ-algebras this definition coincides with the usual notion of homomorphism. The collection of Σ-homomorphisms between two Σ-multialgebras $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathsf{Hom}(\mathbf{A},\mathbf{B})$.

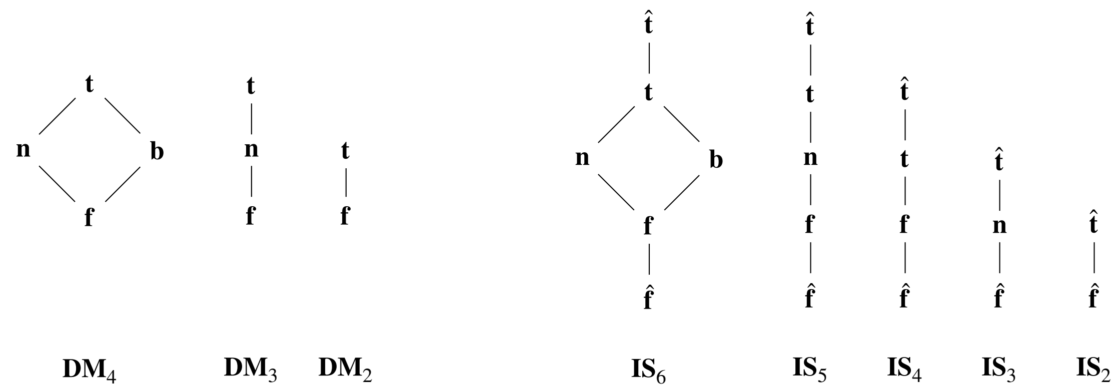
Given a denumerable set $P \supseteq \{p,q,r,s,x,y\}$ of *propositional variables*, the absolutely free Σ-algebra freely generated by P, or simply the *language* over Σ (generated by P), is denoted by $\mathbf{L}_\Sigma(P)$, and its members are called Σ-*formulas*. The collection of all propositional variables occurring in a formula $\varphi \in L_\Sigma(P)$ is denoted by $\mathsf{props}(\varphi)$, and we let $\mathsf{props}(\Phi) := \bigcup_{\varphi\in\Phi}\mathsf{props}(\varphi)$, for all $\Phi \subseteq L_\Sigma(P)$. The elements of $\mathsf{Hom}(\mathbf{L}_\Sigma(P),\mathbf{A})$ will sometimes be referred to as *valuations on* $\mathbf{A}$. When $\mathbf{A}$ is $\mathbf{L}_\Sigma(P)$, valuations are endomorphisms on $\mathbf{L}_\Sigma(P)$ and are usually dubbed *substitutions*. The set of all substitutions is denoted by $\mathsf{End}(\mathbf{L}_\Sigma(P))$. Given $h, h' \in \mathsf{Hom}(\mathbf{L}_\Sigma(P),\mathbf{A})$, we shall say that h' *agrees with* h *on* $\Phi \subseteq L_\Sigma(P)$ provided that $h'(\varphi) = h(\varphi)$ for all $\varphi \in \Phi$. In case $p_1,\ldots,p_k$ are the only propositional variables ocurring in $\varphi \in L_\Sigma(P)$, we say that φ is k-*ary* and denote by $\varphi^{\mathbf{A}}$ the k-ary multioperation on A such that $\varphi^{\mathbf{A}}(a_1,\ldots,a_k) := \{h(\varphi) \mid h \in \mathsf{Hom}(\mathbf{L}_\Sigma(P),\mathbf{A}) \text{ with } h(p_i) = a_i \text{ for each } 1 \leq i \leq k\}$, for all $a_1,\ldots,a_k \in A$. Also, if $\psi_1,\ldots,\psi_k \in L_\Sigma(P)$, we let $\varphi(\psi_1,\ldots,\psi_k)$ denote the formula $\varphi^{\mathbf{L}_\Sigma(P)}(\psi_1,\ldots,\psi_k)$. For a set Θ of formulas $\varphi(p_1,\ldots,p_k)$, we let $\Theta(\psi_1,\ldots,\psi_k) := \{\varphi(\psi_1,\ldots,\psi_k) \mid \varphi \in \Theta\}$.

A Σ-*equation* is a pair (φ,ψ) of Σ-formulas that we denote by $\varphi \approx \psi$, and a Σ-multialgebra $\mathbf{A}$ is said to *satisfy* $\varphi \approx \psi$ if $h(\varphi) = h(\psi)$ for every $h \in \mathsf{Hom}(\mathbf{L}_\Sigma(P),\mathbf{A})$. For any given collection of Σ-equations, the class of all Σ-algebras that satisfy such equations is called a Σ-*variety*. An equation is said to be *valid* in a given variety if it is satisfied by each algebra in the variety. The variety generated by a class K of Σ-algebras, denoted by $\mathbb{V}(\mathsf{K})$, is the closure of K under homomorphic images, subalgebras and direct products. We denote the latter operations, respectively, by $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$. We write $\mathsf{Cng}\,\mathbf{A}$ to refer to the collection of all congruence relations on $\mathbf{A}$, which is known to form a complete lattice under inclusion.

In what follows, we assume the reader is familiar with basic notations and terminology of lattice theory (Davey and Priestley [2002]). We denote by Σ^{bL} the signature containing but two binary connectives, $\land$ and $\lor$, and two nullary connectives $\top$ and $\bot$, and by Σ^{DM} the extension of the latter signature by the addition of a unary connective $\sim$. Moreover, we let Σ^{IS} (resp. Σ^{PP}) be the signature obtained from Σ^{DM} by adding the unary connective ∇ (resp. $\circ$). We provide below the definitions and some examples of De Morgan algebras and of involutive Stone algebras, in order to illustrate some of the notions introduced above.

Definition 2.1. *Given a Σ^{DM}-algebra whose Σ^{bL}-reduct is a bounded distributive lattice, we say that it constitutes a* De Morgan algebra *if it satisfies the equations:* **(DM1)** ${\sim}{\sim}x \approx x$ **(DM2)** ${\sim}(x \land y) \approx {\sim}x \lor {\sim}y$

Example 1. *Let $\mathcal{V}_4 := \{\mathbf{t},\mathbf{b},\mathbf{n},\mathbf{f}\}$ and let $\mathbf{DM}_4 := \langle \mathcal{V}_4, \cdot^{\mathbf{DM}_4}\rangle$ be the Σ^{DM}-algebra known as the Dunn-Belnap lattice, whose interpretations for the lattice connectives are those induced by the Hasse diagram in Figure* 1a*, and the interpretation for $\sim$ is such that ${\sim}^{\mathbf{DM}_4}\mathbf{f} := \mathbf{t}$, ${\sim}^{\mathbf{DM}_4}\mathbf{t} := \mathbf{f}$ and ${\sim}^{\mathbf{DM}_4}a := a$, for $a \in \{\mathbf{n},\mathbf{b}\}$; as expected, for the nullary connectives, we have $\top^{\mathbf{DM}_4} := \mathbf{t}$ and $\bot^{\mathbf{DM}_4} := \mathbf{f}$. In Figure* 1a*, besides depicting the lattice structure of $\mathbf{DM}_4$, we also show its subalgebras $\mathbf{DM}_3$ and $\mathbf{DM}_2$, which coincide with the three-element Kleene algebra and the two-element Boolean algebra. These three algebras are the only subdirectly irreducible De Morgan algebras* [Balbes and Dwinger, 1975, Sec. XI.2, Thm. 6].



(a) The subdirectly irreducible De Morgan algebras.

(b) The subdirectly irreducible IS-algebras.

Figure 1

Definition 2.2. *Given a Σ^{IS}-algebra whose Σ^{DM}-reduct is a De Morgan algebra, we say that it constitutes an* involutive Stone algebra *(*IS-algebra*) if it satisfies the equations:* **(IS1)** $\nabla\bot \approx \bot$ **(IS2)** $x \wedge \nabla x \approx x$ **(IS3)** $\nabla(x \wedge y) \approx \nabla x \wedge \nabla y$ **(IS4)** $\sim\nabla x \wedge \nabla x \approx \bot$

Example 2. *Let $\mathcal{V}_6 := \mathcal{V}_4 \cup \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ and let $\mathbf{IS}_6 := \langle \mathcal{V}_6, \cdot^{\mathbf{IS}_6} \rangle$ be the Σ^{IS}-algebra whose lattice structure is depicted in Figure* 1b *and interprets $\sim$ and ∇ as per the following:*

$$\sim^{\mathbf{IS}_6} a := \begin{cases} \sim^{\mathbf{DM}_4} a & a \in \mathcal{V}_4 \\ \hat{\mathbf{f}} & a = \hat{\mathbf{t}} \\ \hat{\mathbf{t}} & a = \hat{\mathbf{f}} \end{cases} \qquad \nabla^{\mathbf{IS}_6} a := \begin{cases} \hat{\mathbf{t}} & a \in \mathcal{V}_6 \setminus \{\hat{\mathbf{f}}\} \\ \hat{\mathbf{f}} & a = \hat{\mathbf{f}} \end{cases}$$

The subalgebras of $\mathbf{IS}_6$ exhibited in Figure 1b *constitute the only subdirectly irreducible IS-algebras (*Cignoli and Sagastume [1983]*).*

The notion of *residuation* in Σ-algebras will be essential for us here, as it is tightly connected to the intuitionistic implication. Let K be a class of Σ-algebras, with $\Sigma \supseteq \Sigma^{\mathsf{bL}}$, whose $\{\wedge, \top\}$-reducts are meet-semilattices with a greatest element assigned to $\top$. Given $\mathbf{A} \in \mathsf{K}$, we say that © is the *residuum* of $\wedge$ in $\mathbf{A}$ provided that $a \wedge^{\mathbf{A}} b \leq c$ if, and only if, $a \leq b \copyright^{\mathbf{A}} c$ for all $a, b, c \in A$. It is well-known that in classical logic and in intuitionistic logic the implication plays the role of residuum of conjunction. We will also consider the notion of *pseudocomplement* of $a \in A$: this will be defined as the greatest element $\neg a \in A$ such that $a \wedge^{\mathbf{A}} \neg a = \bot^{\mathbf{A}}$. When every element of $\mathbf{A}$ has a pseudocomplement (unique by definition), $\neg^{\mathbf{A}}$ is called the *pseudocomplement operation* of $\mathbf{A}$.

We move now to the logical preliminaries. A SET-FMLA *logic (over Σ)* is a consequence relation $\vdash$ on $L_\Sigma(P)$ and a SET-SET *logic (over Σ)* is a generalized consequence relation $\rhd$ on $L_\Sigma(P)$ (Humberstone [2011]). The SET-FMLA *companion* of a given SET-SET logic $\rhd$ is the SET-FMLA logic $\vdash_\rhd$ such that $\Phi \vdash_\rhd \psi$ if, and only if, $\Phi \rhd \{\psi\}$. We will write $\Phi \lhd\rhd \Psi$ when $\Phi \rhd \Psi$ and $\Psi \rhd \Phi$; in that case, we say these sets are *logically equivalent* (these definitions apply analogously for SET-FMLA logics). The complement of a given SET-SET logic $\rhd$ will be denoted by $\blacktriangleright$.

Let $\rhd$ and $\rhd'$ be SET-SET logics over signatures Σ and Σ', respectively, with $\Sigma \subseteq \Sigma'$. We say that $\rhd'$ *expands* $\rhd$ when $\rhd' \supseteq \rhd$. When $\Sigma = \Sigma'$, we say more simply that $\rhd'$ *extends* $\rhd$. It is worth recalling that the collection of all extensions of a given logic forms a complete lattice under inclusion. Moreover, we say that $\rhd'$ is a *conservative expansion* of $\rhd$ when $\rhd'$ expands $\rhd$ and, for all $\Phi, \Psi \subseteq L_\Sigma(P)$, we have $\Phi \rhd' \Psi$ if, and only if, $\Phi \rhd \Psi$. The *finitary companion* of a SET-SET logic $\rhd$ is the SET-SET logic $\rhd_{\mathsf{fin}}$ such that $\Phi \rhd_{\mathsf{fin}} \Psi$ if, and only if, there are finite $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$ such that $\Phi' \rhd \Psi'$. These concepts may be adapted to SET-FMLA logics in the obvious ways.

We call special attention to the notion of self-extensionality, as it will play an important role in the implicative extensions we consider in this paper:

Definition 2.3. *A logic $\rhd$ over the signature Σ is called* self-extensional *(or* congruential*) if $\varphi_i \lhd\rhd \psi_i$ implies $\copyright(\varphi_1, \ldots, \varphi_k) \lhd\rhd \copyright(\psi_1, \ldots, \psi_k)$, for all $\copyright \in \Sigma_k$ and $\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_k \in L_\Sigma(P)$. For* SET-FMLA *logics, just replace '$\rhd$' by '$\vdash$'.*

A *partial non-deterministic Σ-matrix* (or, more simply, Σ-*PNmatrix*) $\mathfrak{M}$ is a structure $\langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is a Σ-multialgebra and the members of $D \subseteq A$ are called *designated values*. We will write $\overline{D}$ to refer to $A \backslash D$. When $\mathbf{A}$ is a Σ-algebra,

$\mathfrak{M}$ is called a Σ-*matrix*, and coincides with the usual definition of logical matrix in the literature. A *refinement of* $\mathfrak{M}$ is a Σ-PNmatrix obtained from $\mathbf{A}$ by deleting values from some entries of the interpretations of the connectives in $\mathbf{A}$ (the resulting interpretations are also said to be *refinements* of the ones in $\mathbf{A}$). The mappings in $\mathsf{Hom}(\mathbf{L}_\Sigma(P), \mathbf{A})$ are called $\mathfrak{M}$-*valuations*. Every Σ-PNmatrix determines a SET-SET logic $\rhd_{\mathfrak{M}}$ such that $\Phi \rhd_{\mathfrak{M}} \Psi$ iff $h(\Phi) \cap \overline{D} \neq \emptyset$ or $h(\Psi) \cap D \neq \emptyset$ for all $\mathfrak{M}$-valuations h, as well as a SET-FMLA logic $\vdash_{\mathfrak{M}}$ with $\Phi \vdash_{\mathfrak{M}} \psi$ iff $h(\Phi) \cap \overline{D} \neq \emptyset$ or $h(\Psi) \in D$ for all $\mathfrak{M}$-valuations h (notice that $\vdash_{\mathfrak{M}}$ is the SET-FMLA companion of $\rhd_{\mathfrak{M}}$). Given a SET-SET logic $\rhd$ (resp. a SET-FMLA logic $\vdash$), if $\rhd \subseteq \rhd_{\mathfrak{M}}$ (resp. $\vdash \subseteq \vdash_{\mathfrak{M}}$), we shall say that $\mathfrak{M}$ *is a matrix model of* $\rhd$ (resp. $\vdash$), and if the converse also holds we shall say that $\mathfrak{M}$ *determines* $\rhd$ (resp. $\vdash$). The SET-SET (resp. SET-FMLA) logic *determined* by a class $\mathcal{M}$ of Σ-matrices is given by $\bigcap\{\rhd_{\mathfrak{M}} \mid \mathfrak{M} \in \mathcal{M}\}$ (resp. $\bigcap\{\vdash_{\mathfrak{M}} \mid \mathfrak{M} \in \mathcal{M}\}$).

Example 3. *The* Σ^{DM}*-matrix* $\langle \mathbf{DM}_4, \uparrow\mathbf{b}\rangle$ *determines the logic known as the 4-valued Dunn-Belnap logic (*Belnap [1977]*), or First-Degree Entailment (FDE), which we hereby denote by* $\mathcal{B}$*. Extensions of* $\mathcal{B}$ *are known as* super-Belnap logics *(*Rivieccio [2012]*).*

Example 4. *Classical Logic, henceforth denoted by* $\mathcal{CL}$*, is determined by the* Σ^{DM}*-matrix* $\langle \mathbf{DM}_2, \{\mathbf{t}\}\rangle$ *(see Figure 1a).*

Given a Σ-matrix $\mathfrak{M} = \langle \mathbf{A}, D\rangle$, a congruence $\theta \in \mathsf{Cng}\,\mathbf{A}$ is said to be *compatible with* $\mathfrak{M}$ when $b \in D$ whenever both $a \in D$ and $a\theta b$, for all $a, b \in A$. We denote by $\Omega^{\mathfrak{M}}$ the *Leibniz congruence* associated to $\mathfrak{M}$, namely the greatest congruence of $\mathbf{A}$ compatible with $\mathfrak{M}$. The matrix $\mathfrak{M}^* = \langle \mathbf{A}/\Omega^{\mathfrak{M}}, D/\Omega^{\mathfrak{M}}\rangle$ is the *reduced version* of $\mathfrak{M}$. It is well known that $\rhd_{\mathfrak{M}} = \rhd_{\mathfrak{M}^*}$ (and thus $\vdash_{\mathfrak{M}} = \vdash_{\mathfrak{M}^*}$) and, since every logic is determined by a class of matrix models, we have that every logic coincides with the logic determined by its reduced matrix models. As a shortcut, we call a matrix *reduced* when it coincides with its own reduced version (or, equivalently, when its Leibniz congruence is the identity relation on A).

When a Σ-algebra $\mathbf{A}$ has a lattice structure with underlying order $\leq$, for any $a \in A$ we write $\uparrow a$ to refer to the set $\{b \in A \mid a \leq b\}$. For instance, over $\mathbf{IS_6}$ we may consider the set $\uparrow\mathbf{b} = \{\mathbf{b}, \mathbf{t}, \hat{\mathbf{t}}\}$ (see Figure 1b). A *lattice filter* of a meet-semilattice $\mathbf{A}$ with a top element $\top^{\mathbf{A}}$ is a subset $D \subseteq A$ with $\top^{\mathbf{A}} \in D$ and closed under $\wedge^{\mathbf{A}}$; moreover, D is a *proper* lattice filter of $\mathbf{A}$ when $D \neq A$. A *principal filter* of $\mathbf{A}$ is a lattice filter of the form $\uparrow a$, for some $a \in A$. Note that, if $\mathbf{A}$ is finite, every lattice filter is principal. If $\mathbf{A}$ also has a join-semilattice structure, a *prime filter* of $\mathbf{A}$ is a proper lattice filter D of $\mathbf{A}$ such that $a \vee b \in D$ iff $a \in D$ or $b \in D$, for all $a, b \in A$.

2.2 Logics associated to classes of (ordered) algebras

Throughout this section, consider a propositional signature $\Sigma \supseteq \Sigma^{\mathsf{bL}}$. In addition, for a set of Σ-formulas $\Phi := \{\varphi_1, \ldots, \varphi_n\}$, let $\bigwedge \Phi := \varphi_1 \wedge \ldots \wedge \varphi_n$ and $\bigvee \Phi := \varphi_1 \vee \ldots \vee \varphi_n$, while, by convention, let $\bigwedge \emptyset := \top$ and $\bigvee \emptyset := \bot$. Moreover, let the *inequality* $\varphi \leq \psi$ abbreviate the equation $\varphi \approx \varphi \wedge \psi$.

Definition 2.4. *Let* K *be a class of* Σ*-algebras such that each* $\mathbf{A} \in \mathsf{K}$ *has a bounded distributive lattice reduct with greatest element* $\top^{\mathbf{A}}$ *and a least element* $\bot^{\mathbf{A}}$*. The* order-preserving logic associated to K*, denoted by* $\rhd^{\leq}_{\mathsf{K}}$*, is such that* $\Phi \rhd^{\leq}_{\mathsf{K}} \Psi$ *if, and only if, there are finite* $\Phi' \subseteq \Phi$ *and* $\Psi' \subseteq \Psi$ *such that* $\bigwedge \Phi' \leq \bigvee \Psi'$ *is valid in* K*.*

For this logic, the following characterization in terms of prime filters applies:

Proposition 2.5. $\rhd^{\leq}_{\mathsf{K}}$ *is the finitary companion of the logic determined by* $\mathcal{M}^{\uparrow\vee} := \{\langle \mathbf{A}, D\rangle \mid \mathbf{A} \in \mathsf{K}$ *and* D *is a prime filter of* $\mathbf{A}\}$*.*

Proof. From the left to the right, assume that $\Phi \rhd^{\leq}_{\mathsf{K}} \Psi$. Then (a): the above condition holds for finite $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$. Let $\langle \mathbf{A}, D\rangle \in \mathcal{M}^{\uparrow\vee}$ and $v \in \mathsf{Hom}(\mathbf{L}_\Sigma(P), \mathbf{A})$ be a valuation. Assume that $v[\Phi'] \subseteq D$. Thus $v(\bigwedge \Phi') \in D$, as D is a lattice filter. By (a), we know that $v(\bigwedge \Phi') \leq v(\bigvee \Psi')$, so $v(\bigvee \Psi') \in D$. Since D is prime, we must have $v(\psi) \in D$ for some $\psi \in \Psi'$ and we are done.

From the right to the left, assume that (b): $\Phi \rhd^{\mathsf{f}}_{\mathcal{M}^{\uparrow\vee}} \Psi$. This consequence relation is finitary, so $\Phi' \rhd_{\mathcal{M}^{\uparrow\vee}} \Psi'$ for finite $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$. Let $\mathbf{A} \in \mathsf{K}$ and $v \in \mathsf{Hom}(\mathbf{L}_\Sigma(P), \mathbf{A})$. Let $a := v(\bigwedge \Phi')$ (note that $a = \top^{\mathbf{A}}$ if $\Phi' = \emptyset$). If $a = \bot^{\mathbf{A}}$, the result is straightforward. Otherwise, $\uparrow a$ is a proper filter. Suppose, by reductio, that $a \nleq v(\bigvee \Psi')$, that is, $v(\bigvee \Psi') \notin \uparrow a$. Since $\mathbf{A}$ is distributive, consider, by the Prime Filter Theorem, the extension of $\uparrow a$ to a prime filter $D \supseteq \uparrow a$ such that $v(\bigvee \Psi') \notin D$. Since $a \leq v(\varphi)$ for each $\varphi \in \Phi'$, we have $v(\varphi) \in \uparrow a \subseteq D$, for all $\varphi \in \Phi'$ and, by (b), we have $v(\psi) \in D$ for some $\psi \in \Psi'$. Since $v(\psi) \leq v(\bigvee \Psi')$, we must have $v(\bigvee \Psi') \in D$. As this leads to a contradiction, we conclude that $a \leq v(\bigvee \Psi')$, as desired. $\square$

For a variety generated by a single finite distributive lattice, we have this simpler characterization in terms of a finite family of finite matrices:

Proposition 2.6. *Let $\mathsf{K} = \mathbb{V}(\{\mathbf{B}\})$, for $\mathbf{B}$ a finite distributive lattice. Then $\rhd^{\leq}_{\mathsf{K}}$ is determined by $\mathcal{M}^{\uparrow\vee}_{\mathsf{fin}} := \{\langle \mathbf{B}, D\rangle \mid D$ is a prime filter of $\mathbf{B}\}$.*

Proof. As $\mathcal{M}^{\uparrow\vee}_{\mathsf{fin}} \subseteq \mathcal{M}^{\uparrow\vee}$, we have $\rhd^{\leq}_{\mathsf{K}} \subseteq \rhd_{\mathcal{M}^{\uparrow\vee}_{\mathsf{fin}}}$. Conversely, suppose that (b): $\Phi \blacktriangleright^{\leq}_{\mathsf{K}} \Psi$. Consider finite $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$. We want to show that $\Phi' \blacktriangleright_{\mathcal{M}^{\uparrow\vee}_{\mathsf{fin}}} \Psi'$. By (b), we know that $\bigwedge \Phi' \leq \bigvee \Psi'$ fails in K, thus it fails in $\mathbf{B}$, say under a valuation v. Let $b := v(\bigwedge \Phi')$. So, $b \nleq v(\bigvee \Psi')$. Note that $\uparrow b$ must be proper. As $v(\bigvee \Psi') \notin \uparrow b$, consider the extension of $\uparrow b$ to a prime filter D, by the Prime Filter Theorem, such that $v(\bigvee \Psi') \notin D$. Since $\mathbf{B}$ is finite, we have that D is principal. Clearly, we must have $v(\psi) \notin D$ for every $\psi \in \Psi'$, as $v(\psi) \leq v(\bigvee \Psi')$ and D is upwards closed, and $\varphi \in D$ for every Φ'. Therefore, $\Phi' \blacktriangleright_{\mathcal{M}^{\uparrow\vee}_{\mathsf{fin}}} \Psi'$, as desired. ☐

Definition 2.7. *Let K be a class of Σ-algebras as in Definition 2.4. The* SET-FMLA order-preserving logic induced by K*, which we denote by $\vdash^{\leq}_{\mathsf{K}}$, is such that $\Phi \vdash^{\leq}_{\mathsf{K}} \psi$ if, and only if, there are $\varphi_1, \ldots, \varphi_n \subseteq \Phi$ ($n \geq 1$) for which $\bigwedge_i \varphi_i \leq \psi$ is valid in K.*

Note that $\vdash^{\leq}_{\mathsf{K}}$ is the SET-FMLA companion of $\rhd^{\leq}_{\mathsf{K}}$.

The above logics are particularly interesting for us in view of the following:

Proposition 2.8. *$\rhd^{\leq}_{\mathsf{K}}$ and $\vdash^{\leq}_{\mathsf{K}}$ are self-extensional.*

Proof. Directly from Jansana [2006, Sec. 3] and from the fact that $\vdash^{\leq}_{\mathsf{K}}$ is the SET-FMLA companion of $\rhd^{\leq}_{\mathsf{K}}$. ☐

We close by defining the $\top$-assertional logics (also known as 1-assertional logics) associated to a class of bounded lattices.

Definition 2.9. *Let K be a class of algebras with a bounded lattice reduct. The* $\top$-assertional logics *$\rhd^{\top}_{\mathsf{K}}$ and $\vdash^{\top}_{\mathsf{K}}$ correspond respectively to the* SET-SET *and* SET-FMLA *logics determined by the class of matrices $\{\langle \mathbf{A}, \{\top^{\mathbf{A}}\}\rangle \mid \mathbf{A} \in \mathsf{K}\}$.*

Notice that $\vdash^{\top}_{\mathsf{K}}$ is the SET-FMLA companion of $\rhd^{\top}_{\mathsf{K}}$.

2.3 Hilbert-style axiomatizations

Based on Shoesmith and Smiley [1978] and Caleiro and Marcelino [2019], we define a *symmetrical* (*Hilbert-style*) *calculus* R (or SET-SET *calculus*, for short) as a collection of pairs $(\Phi, \Psi) \in \wp L_\Sigma(P) \times \wp L_\Sigma(P)$, denoted by $\frac{\Phi}{\Psi}$ and called (*symmetrical* or SET-SET) *inference rules*, where Φ is the *antecedent* and Ψ is the *succedent* of the said rule. We will adopt the convention of omitting curly braces when writing sets of formulas and leaving a blank space instead of writing $\varnothing$ when presenting inference rules and statements involving (generalized) consequence relations. We proceed to define what constitutes a proof in such calculi.

A *bounded rooted tree* t is a poset $\langle \mathsf{nds}(t), \leq^t\rangle$ with a single minimal element $\mathsf{rt}(t)$, the *root* of t, such that, for each *node* $n \in \mathsf{nds}(t)$, the set $\{n' \in \mathsf{nds}(t) \mid n' \leq^t n\}$ of *ancestors* of n (or the *branch* up to n) is well-ordered under $\leq^t$, and every branch of t has a maximal element (a *leaf* of t). We may assign a label $l^t(n) \in \wp L_\Sigma(P) \cup \{*\}$ to each node n of t, in which case t is said to be *labelled*. Given $\Psi \subseteq L_\Sigma(P)$, a leaf n is Ψ-*closed* in t when $l^t(n) = *$ or $l^t(n) \cap \Psi \neq \varnothing$. The tree t itself is Ψ-*closed* when all of its leaves are Ψ-closed. The immediate successors of a node n with respect to $\leq^t$ are called the *children* of n in t.

Let R be a symmetrical calculus. An R-*derivation* is a labelled bounded rooted tree such that for every non-leaf node n of t there exists a rule of inference $\mathsf{r} = \frac{\Pi}{\Theta} \in \mathsf{R}$ and a substitution σ such that $\sigma(\Pi) \subseteq l^t(n)$, and the set of children of n is either (i) $\{n^\varphi \mid \varphi \in \sigma(\Theta)\}$, in case $\Theta \neq \varnothing$, where n^φ is a node labelled with $l^t(n) \cup \{\varphi\}$, or (ii) a singleton $\{n^*\}$ with $l^t(n) = *$, in case $\Theta = \varnothing$. We say that $\Phi \rhd_{\mathsf{R}} \Psi$ whenever there is a Ψ-closed derivation t such that $\Phi \supseteq \mathsf{rt}(t)$; such a tree consists in a *proof* that Ψ follows from Φ in R. As a matter of simplification when drawing such trees, we usually avoid copying the formulas inherited from the parent nodes (see Example 5 below). The relation $\rhd_{\mathsf{R}}$ so defined is a SET-SET logic and, when $\rhd_{\mathsf{R}} = \rhd_{\mathfrak{M}}$, we say that R *axiomatizes* $\rhd_{\mathfrak{M}}$. A rule $\frac{\Phi}{\Psi}$ is *sound* in $\mathfrak{M}$ when $\Phi \rhd_{\mathfrak{M}} \Psi$. It should be pointed out that this deductive formalism generalises the conventional (SET-FMLA) Hilbert-style calculi: the latter corresponds to symmetrical calculi whose rules have, each, a finite antecedent and a singleton as succedent.

Given $\Lambda \subseteq L_\Sigma(P)$, we write $\Phi \rhd^{\Lambda}_{\mathsf{R}} \Psi$ whenever there is a proof of Ψ from Φ using only formulas in Λ. Given $\Theta, \Xi \subseteq L_\Sigma(P)$, we define the set Υ^{Ξ} of Ξ-*generalized subformulas of* Θ as the set $\mathsf{sub}(\Theta) \cup \{\sigma(\varphi) \mid \varphi \in \Xi$ and $\sigma : P \to \mathsf{sub}(\Theta)\}$. We say that R is Ξ-*analytic* when, for all $\Phi, \Psi \subseteq L_\Sigma(P)$, we always have $\Phi \rhd^{\Upsilon^{\Xi}}_{\mathsf{R}} \Psi$ whenever we have $\Phi \rhd_{\mathsf{R}} \Psi$; intuitively, it means that proofs in R that Ψ follows from Φ, whenever they exist, do not need to use more material than subformulas of $\Phi \cup \Psi$ or substitution instances of the formulas in Ξ built from those same subformulas.

A general method is introduced in Caleiro and Marcelino [2019], Marcelino and Caleiro [2021] for obtaining analytic calculi (in the sense of analyticity introduced in the previous paragraph) for logics given by a Σ-PNmatrix $\langle \mathbf{A}, D\rangle$ whenever a certain expressiveness requirement (called 'monadicity' in Shoesmith and Smiley [1978]) is met: for every $a, b \in A$, there is a single-variable formula S such that $\mathrm{S}^{\mathbf{A}}(a) \in D$ and $\mathrm{S}^{\mathbf{A}}(b) \notin D$ or vice-versa. We call such formula a *separator (for a and b).*

The following example illustrates a symmetrical calculus for $\mathcal{B}$ generated by this method, as well as some proofs in this calculus.

Example 5. *The matrix* $\langle \mathbf{DM}_4, {\uparrow}\mathbf{b}\rangle$ *fulfills the above expressiveness requirement, with the following set of separators:* $\mathcal{S} := \{p, {\sim}p\}$. *We may therefore apply the method introduced in* Marcelino and Caleiro [2021] *to obtain for* $\mathcal{B}$ *the following* $\mathcal{S}$*-analytic axiomatization we call* $\mathsf{R}_{\mathcal{B}}$*:*

$$\frac{}{\top}\,\mathsf{r}_1 \quad \frac{{\sim}\top}{}\,\mathsf{r}_2 \quad \frac{}{{\sim}\bot}\,\mathsf{r}_3 \quad \frac{\bot}{}\,\mathsf{r}_4 \quad \frac{p}{{\sim}{\sim}p}\,\mathsf{r}_5 \quad \frac{{\sim}{\sim}p}{p}\,\mathsf{r}_6$$

$$\frac{p \wedge q}{p}\,\mathsf{r}_7 \quad \frac{p \wedge q}{q}\,\mathsf{r}_8 \quad \frac{p, q}{p \wedge q}\,\mathsf{r}_9 \quad \frac{{\sim}p}{{\sim}(p \wedge q)}\,\mathsf{r}_{10} \quad \frac{{\sim}q}{{\sim}(p \wedge q)}\,\mathsf{r}_{11} \quad \frac{{\sim}(p \wedge q)}{{\sim}p, {\sim}q}\,\mathsf{r}_{12}$$

$$\frac{p}{p \vee q}\,\mathsf{r}_{13} \quad \frac{q}{p \vee q}\,\mathsf{r}_{14} \quad \frac{p \vee q}{p, q}\,\mathsf{r}_{15} \quad \frac{{\sim}p, {\sim}q}{{\sim}(p \vee q)}\,\mathsf{r}_{16} \quad \frac{{\sim}(p \vee q)}{{\sim}p}\,\mathsf{r}_{17} \quad \frac{{\sim}(p \vee q)}{{\sim}q}\,\mathsf{r}_{18}$$

Figure 2 *illustrates some proofs in* $\mathsf{R}_{\mathcal{B}}$.

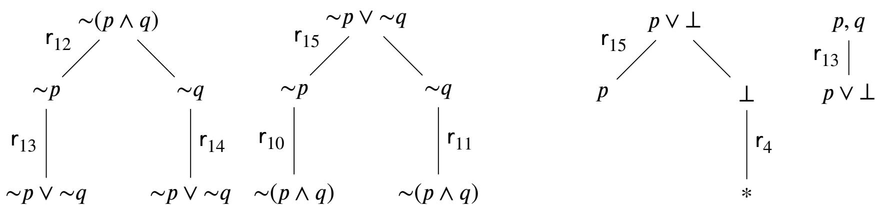


Figure 2: Proofs in $\mathsf{R}_{\mathcal{B}}$ witnessing that ${\sim}(p \wedge q) \lhd\!\rhd_{\mathcal{B}} {\sim}p \vee {\sim}q$ and $p \vee \bot \lhd\!\rhd_{\mathcal{B}} p, q$.

2.4 Implication connectives and criteria for implicative expansions

In the present study, we will follow some very specific research paths with the goal of adding an implication to given logics; we will refer to the resulting logics as *implicative expansions*. First of all, it is important to define precisely what we mean by an implication connective on SET-SET and SET-FMLA logics. Throughout this subsection, let Σ be an arbitrary signature with a binary connective $\vee$ and denote by $\Sigma_{\Rightarrow}$ the signature resulting from adding to Σ the binary connective $\Rightarrow$.

Definition 2.10. *The connective* $\Rightarrow$ *is an* implication in a SET-SET logic $\rhd$ over $\Sigma_{\Rightarrow}$ *provided, for all* $\Phi, \Psi, \{\varphi, \psi\} \subseteq L_{\Sigma}(P)$*, with* Ψ *finite,*

$$\Phi, \varphi \rhd \psi, \Psi \text{ if, and only if, } \Phi \rhd \varphi \Rightarrow \bigvee \{\psi\} \cup \Psi.$$

Note that the above definition reduces in SET-FMLA to the *deduction-detachment theorem* (DDT) that holds, for example, in intuitionistic and classical logics, and this is indeed what we will take to be an implication in SET-FMLA, in this paper. What we did above was a suitable generalization of the DDT for SET-SET logics, using it to abstractly characterize what we expect of an implication connective (namely, the internalization of the consequence relation).

Based on the behaviour of implication in classical logic, we also consider the following stronger notion of implication:

Definition 2.11. *The connective* $\Rightarrow$ *is a* classic-like implication in a SET-SET logic $\rhd$ over $\Sigma_{\Rightarrow}$ *provided, for all* $\Phi, \Psi, \{\varphi, \psi\} \subseteq L_{\Sigma}(P)$,

$$\Phi, \varphi \rhd \psi, \Psi \text{ if, and only if, } \Phi \rhd \varphi \Rightarrow \psi, \Psi.$$

It should be clear enough that, for logics with a well-behaved disjunction, classic-like implications are implications in the sense of Definition 2.10.

In this paper, we shall require two minimal criteria on implicative extensions:

(I1) The connective being added must qualify as an implication (Definition 2.10);

(I2) The expansion must be conservative.

We will prove below some facts regarding the connections among the above notions of implication, residuation and characterizability via a single PNmatrix.

Proposition 2.12. *Let $\Rightarrow^{\mathbf{A}}$ be the residuum of $\wedge^{\mathbf{A}}$ in each $\mathbf{A} \in \mathsf{K}$, where K is a class of $\Sigma_{\Rightarrow}$-algebras, with $\Sigma_{\Rightarrow} \supseteq \Sigma^{\mathsf{bL}}$, whose $\{\wedge, \top\}$-reducts are $\wedge$-semilattices with a greatest element assigned to $\top$. Then $\Rightarrow$ is an implication in $\rhd^{\leq}_{\mathsf{K}}$ (Definition 2.4).*

Proof. Let $\Phi, \Psi, \{\varphi\} \subseteq L_{\Sigma}(P)$ with $\Psi \neq \varnothing$ finite. From the left to the right, suppose that $\Phi, \varphi \rhd^{\leq}_{\mathsf{K}} \Psi$. Then $\Phi' \rhd^{\leq}_{\mathsf{K}} \Psi'$, for finite $\Phi' \subseteq \Phi \cup \{\varphi\}$ and $\Psi' \subseteq \Psi$, and thus (a): $\Phi'', \varphi \rhd^{\leq}_{\mathsf{K}} \Psi'$, for $\Phi' \cup \{\varphi\} = \Phi'' \cup \{\varphi\}$ and $\varphi \notin \Phi''$. Let $\mathbf{A} \in \mathsf{K}$. By (a), we have that $\bigwedge \Phi'' \wedge \varphi \leq \bigvee \Psi'$ is valid in $\mathbf{A}$, thus $\bigwedge \Phi'' \wedge \varphi \leq \bigvee \Psi$ is valid in $\mathbf{A}$; hence, by residuation, $\bigwedge \Phi'' \leq \varphi \Rightarrow \bigvee \Psi$ is valid in $\mathbf{A}$, thus $\Phi \rhd^{\leq}_{\mathsf{K}} \varphi \Rightarrow \bigvee \Psi$. From the right to the left, suppose that $\Phi \rhd^{\leq}_{\mathsf{K}} \varphi \Rightarrow \bigvee \Psi$. Then, without loss of generality, we have that $\Phi' \rhd^{\leq}_{\mathsf{K}} \varphi \Rightarrow \bigvee \Psi$, for some finite $\Phi' \subseteq \Phi$. Let $\mathbf{A} \in \mathsf{K}$. We have that $\bigwedge \Phi' \leq \varphi \Rightarrow (\bigvee \Psi)$ is valid in $\mathbf{A}$. By residuation, we have $\bigwedge \Phi' \wedge \varphi \leq \bigvee \Psi$ also valid in $\mathbf{A}$, from which we easily obtain that $\Phi, \varphi \rhd^{\leq}_{\mathsf{K}} \Psi$. □

Observe that the above result extends to $\vdash^{\leq}_{\mathsf{K}}$ (recall Definition 2.7) since it is the SET-FMLA companion of $\rhd^{\leq}_{\mathsf{K}}$. This also looks like the right moment to introduce intuitionistic-like implications:

Definition 2.13. *A* Heyting implication *in an algebra $\mathbf{A} \in \mathsf{K}$, with K as described in Proposition 2.12, is an implication that corresponds to the residuum of $\wedge^{\mathbf{A}}$.*

Inspired by Avron [2020], we also consider the following additional criteria to guide our investigations:

(A1) The expanded logic is determined by the single PNmatrix $\langle \mathbf{A}, D \rangle$, and, for all $a, b \in A$, we have $a \Rightarrow^{\mathbf{A}} b \subseteq D$ if, and only if, either $a \notin D$ or $b \in D$.

(A2) The expanded logic is self-extensional (Definition 2.3).

Note that, for SET-SET logics, the criterion (A1) is strong to the point of forcing the referred implication to be classic-like:

Proposition 2.14. *Let $\mathfrak{M} := \langle \mathbf{A}, D \rangle$ be a $\Sigma_{\Rightarrow}$-PNmatrix. Then $\Rightarrow$ is a classic-like implication in $\rhd_{\mathfrak{M}}$ if, and only if, $\mathfrak{M}$ satisfies* (A1).

Proof. We have that $\Phi, \varphi \blacktriangleright_{\mathfrak{M}} \psi, \Psi$ if, and only if, $v[\Phi \cup \{\varphi\}] \subseteq D$ and $v[\{\psi\} \cup \Psi] \subseteq \overline{D}$ for some valuation v if, and only if, $v[\Phi] \subseteq D$ and $v(\psi) \in \overline{D}$ and $v(\varphi \Rightarrow \psi) \in \overline{D}$ for some valuation v if, and only if, $\Phi \blacktriangleright_{\mathfrak{M}} \varphi \Rightarrow \psi, \Psi$. □

3 Perfect Paradefinite Algebras and their logics

In this section, we present the main definitions concerning perfect paradefinite algebras and the logics associated to them. Most of the material come from Gomes et al. [2022], where these objects were first introduced and investigated.

Definition 3.1. *Given a Σ^{PP}-algebra whose Σ^{DM}-reduct is a De Morgan algebra, we say that it constitutes a* perfect paradefinite algebra *(*PP-algebra*) if it satisfies the equations:*

(PP1) $\circ\circ x \approx \top$ **(PP2)** $\circ x \approx \circ{\sim}x$ **(PP3)** $\circ\top \approx \top$ **(PP4)** $x \wedge {\sim}x \wedge \circ x \approx \bot$
(PP5) $\circ(x \wedge y) \approx (\circ x \vee \circ y) \wedge (\circ x \vee {\sim}y) \wedge (\circ y \vee {\sim}x)$

Example 6. *An example of PP-algebra is $\mathbf{PP}_6 := \langle \mathcal{V}_6, \cdot^{\mathbf{PP}_6} \rangle$, the Σ^{PP}-algebra defined as $\mathbf{IS}_6$ in Example* 2*, differing only in that, instead of containing an interpretation for ∇, it contains the following interpretation for $\circ$:*

$$\circ^{\mathbf{PP}_6} a := \begin{cases} \hat{\mathbf{f}} & a \in \mathcal{V}_6 \backslash \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\} \\ \hat{\mathbf{t}} & a \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\} \end{cases}$$

Other examples are the algebras $\mathbf{PP}_i$, the subalgebras of $\mathbf{PP}_6$ having, respectively, the same lattice structures of the algebras $\mathbf{IS}_i$, for $2 \leq i \leq 5$, exhibited in Figure 1b.

We denote by $\mathbb{PP}$ the variety of PP-algebras. This variety is term-equivalent to the variety of involutive Stone algebras [Gomes et al., 2022, Thm. 3.6] — in particular, $\nabla x := x \vee {\sim}{\circ} x$. Also, it holds that $\mathbb{PP} = \mathbb{V}(\{\mathbf{PP_6}\})$ [Gomes et al., 2022, Prop. 3.8]. As it occurs with IS-algebras, we may define, in the language of PP-algebras, a pseudo-complement satisfying the Stone equation; to that effect, it suffices to set $\neg x := {\sim}x \wedge \circ x$ (alternatively, one might now set $\neg x := {\sim}\nabla x$, as usual).

We shall denote by $\mathcal{PP}^{\triangleright}_{\leq}$ and $\mathcal{PP}_{\leq}$, respectively, the SET-SET and SET-FMLA order-preserving logics induced by $\mathbb{PP}$ (cf. Subsection 2.2). In addition, we shall denote by $\mathcal{PP}^{\triangleright}_{\top}$ and $\mathcal{PP}_{\top}$, respectively, the SET-SET and SET-FMLA $\top$-assertional logics induced by $\mathbb{PP}$. We know that $\mathcal{PP}^{\triangleright}_{\leq} = \triangleright_{\langle \mathbf{PP_6}, \uparrow\mathbf{b}\rangle}$ and thus $\mathcal{PP}_{\leq} = \vdash_{\langle \mathbf{PP_6}, \uparrow\mathbf{b}\rangle}$ [Gomes et al., 2022, Theorem 3.11].

Taking a proof-theoretical perspective, from [Gomes et al., 2022, Cor. 4.3] we know that $\mathcal{PP}^{\triangleright}_{\leq}$ is axiomatized by a $\{p, {\sim}p, \circ p\}$-analytic SET-SET calculus, which we now recall:

Definition 3.2. *Let $\mathsf{R}_{\mathcal{PP}^{\triangleright}_{\leq}}$ be the following* SET-SET *calculus:*

$$\frac{}{\top}\,\mathsf{r}_1 \qquad \frac{{\sim}\top}{}\,\mathsf{r}_2 \qquad \frac{}{{\sim}\bot}\,\mathsf{r}_3 \qquad \frac{\bot}{}\,\mathsf{r}_4 \qquad \frac{p}{{\sim}{\sim}p}\,\mathsf{r}_5 \qquad \frac{{\sim}{\sim}p}{p}\,\mathsf{r}_6$$

$$\frac{p \wedge q}{p}\,\mathsf{r}_7 \qquad \frac{p \wedge q}{q}\,\mathsf{r}_8 \qquad \frac{p, q}{p \wedge q}\,\mathsf{r}_9 \qquad \frac{{\sim}p}{{\sim}(p \wedge q)}\,\mathsf{r}_{10} \qquad \frac{{\sim}q}{{\sim}(p \wedge q)}\,\mathsf{r}_{11} \qquad \frac{{\sim}(p \wedge q)}{{\sim}p, {\sim}q}\,\mathsf{r}_{12}$$

$$\frac{p}{p \vee q}\,\mathsf{r}_{13} \qquad \frac{q}{p \vee q}\,\mathsf{r}_{14} \qquad \frac{p \vee q}{p, q}\,\mathsf{r}_{15} \qquad \frac{{\sim}p, {\sim}q}{{\sim}(p \vee q)}\,\mathsf{r}_{16} \qquad \frac{{\sim}(p \vee q)}{{\sim}p}\,\mathsf{r}_{17} \qquad \frac{{\sim}(p \vee q)}{{\sim}q}\,\mathsf{r}_{18}$$

$$\frac{}{\circ\bot}\,\mathsf{r}_{19} \qquad \frac{}{\circ\top}\,\mathsf{r}_{20} \qquad \frac{}{\circ\circ p}\,\mathsf{r}_{21} \qquad \frac{\circ p}{\circ{\sim}p}\,\mathsf{r}_{22} \qquad \frac{\circ{\sim}p}{\circ p}\,\mathsf{r}_{23} \qquad \frac{\circ p}{p, {\sim}p}\,\mathsf{r}_{24} \qquad \frac{\circ p, p, {\sim}p}{}\,\mathsf{r}_{25}$$

$$\frac{\circ p}{\circ(p \wedge q), p}\,\mathsf{r}_{26} \qquad \frac{\circ q}{\circ(p \wedge q), q}\,\mathsf{r}_{27} \qquad \frac{\circ(p \wedge q), q}{\circ p}\,\mathsf{r}_{28} \qquad \frac{\circ(p \wedge q), p}{\circ q}\,\mathsf{r}_{29} \qquad \frac{\circ p, \circ q}{\circ(p \wedge q)}\,\mathsf{r}_{30} \qquad \frac{\circ(p \wedge q)}{\circ p, \circ q}\,\mathsf{r}_{31}$$

$$\frac{\circ p, \circ q}{\circ(p \vee q)}\,\mathsf{r}_{32} \qquad \frac{\circ(p \vee q)}{\circ p, \circ q}\,\mathsf{r}_{33} \qquad \frac{\circ p, p}{\circ(p \vee q)}\,\mathsf{r}_{34} \qquad \frac{\circ q, q}{\circ(p \vee q)}\,\mathsf{r}_{35} \qquad \frac{\circ(p \vee q)}{\circ p, q}\,\mathsf{r}_{36} \qquad \frac{\circ(p \vee q)}{\circ q, p}\,\mathsf{r}_{37}$$

In the mentioned paper, the above calculus was transformed into a SET-FMLA axiomatization for $\mathcal{PP}_{\leq}$, using a technique that we shall detail and employ in Section 6.

It is perhaps worth calling attention to the contribution played by rules r_{24} and r_{25} in making the perfection operator, $\circ$, restore 'classicality', as described in the so-called Derivability Adjustment Theorems (for a semantical perspective, see Marcos [2005b, Sec. 2], and more specifically Gomes et al. [2022, Thm. 3.29]).

Finally, we have that the $\top$-assertional logics $\mathcal{PP}^{\triangleright}_{\top}$ and $\mathcal{PP}_{\top}$ are determined by a single three-valued matrix. In fact, it can be shown that such logics are term-equivalent to the three-valued Łukasiewicz logic (in SET-SET and SET-FMLA, respectively) — where $\nabla x := x \vee {\sim}{\circ}x$, as above.

Proposition 3.3 (Gomes, Greati, Marcelino, Marcos, and Rivieccio [2022, Prop. 3.12]). $\mathcal{PP}^{\triangleright}_{\top} = \triangleright^{\top}_{\mathbb{V}(\mathbf{PP}_3)} = \triangleright_{\langle \mathbf{PP}_3, \{\hat{\mathbf{t}}\}\rangle}$, *and thus* $\mathcal{PP}_{\top} = \vdash^{\top}_{\mathbb{V}(\mathbf{PP}_3)} = \vdash_{\langle \mathbf{PP}_3, \{\hat{\mathbf{t}}\}\rangle}$ *(recall the definition of* $\mathbf{PP}_3$ *in Example 6).*

We have at this point all the relevant facts about the logics of PP-algebras we are interested in. Let us move to the main goal of the paper: adding an implication to them.

4 Conservatively expanding $\mathcal{PP}^{\triangleright}_{\leq}$ and $\mathcal{PP}_{\leq}$ by adding a classic-like implication to their matrix

The first path we shall consider amounts to modifying the logical matrix of $\mathcal{PP}^{\triangleright}_{\leq}$, namely $\langle \mathbf{PP_6}, \uparrow\mathbf{b}\rangle$, by enriching its algebra with a new multioperation $\Rightarrow_{\mathsf{A}}$, thus obtaining a multialgebra $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{A}}}$ for which both criteria (A1) and (A2) mentioned in Section 1 hold, that is:

(A1) $a \Rightarrow_{\mathsf{A}} b \subseteq \uparrow\mathbf{b}$ if, and only if, either $a \notin \uparrow\mathbf{b}$ or $b \in \uparrow\mathbf{b}$.

(A2) The resulting logic $\rhd_{\langle \mathbf{PP_6}^{\Rightarrow_\mathsf{A}}, \uparrow\mathbf{b}\rangle}$ is self-extensional.

This path soon leads to a dead end, since:

Theorem 4.1. *There is no multialgebra $\mathbf{PP_6}^{\Rightarrow_\mathsf{A}}$ simultaneously satisfying conditions* (A1) *and* (A2).

Proof. Let $\Rightarrow_\mathsf{A}$ be an implication defined in $\mathbf{PP_6}$ that satisfies (A1) and $\mathfrak{M}^{\Rightarrow_\mathsf{A}} := \langle \mathbf{PP_6}^{\Rightarrow_\mathsf{A}}, \uparrow\mathbf{b}\rangle$. Then (a): we have $\rhd_{\mathfrak{M}^{\Rightarrow_\mathsf{A}}} p \vee (p \Rightarrow \bot)$, because, for every valuation v, either $v(p) \in \uparrow\mathbf{b}$ — in which case we have $v(p \vee (p \Rightarrow \bot)) \in v(p) \vee v(p \Rightarrow \bot) \subseteq \uparrow\mathbf{b}$ as well — or $v(p) \notin \uparrow\mathbf{b}$, which gives us $v(p \Rightarrow q) \in v(p) \Rightarrow v(\bot) \subseteq \uparrow\mathbf{b}$ by (A1).

Note that, for a valuation v such that $v(p) = \mathbf{b}$, using (A1) we have $v(p \Rightarrow \bot) \in \mathbf{b} \Rightarrow \hat{\mathbf{f}} \not\subseteq \uparrow\mathbf{b}$, which means that $v(p \Rightarrow \bot)$ may take a value in $\{\mathbf{n}, \mathbf{f}, \hat{\mathbf{f}}\}$. Hence, this being the case, $v(p \vee (p \Rightarrow \bot)) \in \mathbf{b} \vee v(p \Rightarrow \bot) \in \{\mathbf{b}, \mathbf{t}\}$, which entails $v(\circ(p \vee (p \Rightarrow \bot))) = \hat{\mathbf{f}}$. Thus $\blacktriangleright_{\mathfrak{M}^{\Rightarrow_\mathsf{A}}} \circ(p \vee (p \Rightarrow \bot))$.

This prevents the logic from being self-extensional. Indeed, assuming (A2), from (a) we would have that $p \vee (p \Rightarrow \bot)$ and $\top$ are logically equivalent, thus we would be able to conclude that $\circ(p \vee (p \Rightarrow \bot))$ and $\circ\top$ are logically equivalent too. Since $\rhd_{\mathfrak{M}^{\Rightarrow_\mathsf{A}}} \circ\top$, we would conclude that $\rhd_{\mathfrak{M}^{\Rightarrow_\mathsf{A}}} \circ(p \vee (p \Rightarrow \bot))$, against what we have shown. □

In view of the preceding result, we proceed in this section by pursuing (A1), thus necessarily admitting non-self-extensional logics. In the next sections we will instead see some possibilities that arise when we opt for abandoning (A1).

The space of binary multioperations over $\mathcal{V}_6$ satisfying (A1) is finite but very large — to be precise, there are $(2^3)^9 \cdot (2^3)^{27}$ of them, each choice consisting in a refinement of the multioperation $\Rightarrow_\mathsf{full}$ defined as:

$$\Rightarrow_\mathsf{full} (a, b) := \begin{cases} \uparrow\mathbf{b} & \text{if } a \notin \uparrow\mathbf{b} \text{ or } b \in \uparrow\mathbf{b} \\ \mathcal{V}_6 \backslash \uparrow\mathbf{b} & \text{otherwise.} \end{cases}$$

Denote by $\mathbf{PP_6}^{\Rightarrow_\mathsf{full}}$ the multialgebra obtained from $\mathbf{PP_6}$ by expanding its signature with $\Rightarrow$ and interpreting this connective as $\Rightarrow_\mathsf{full}$. Let $\mathfrak{M}^{\Rightarrow_\mathsf{full}} := \langle \mathbf{PP_6}^{\Rightarrow_\mathsf{full}}, \uparrow\mathbf{b}\rangle$. We will soon see that the logic induced by this matrix plays an important role regarding the conservative expansions of $\mathcal{PP}^{\rhd}_{\leq}$ by a classic-like implication; but let us first present an analytic axiomatization for it.

Definition 4.2. *Let $\mathsf{R}_{\mathfrak{M}^{\Rightarrow_\mathsf{full}}}$ be the calculus given by all inference rules in $\mathsf{R}_{\mathcal{PP}^{\rhd}_{\leq}}$ plus the following three inference rules:*

$$\frac{q}{p \Rightarrow q}\,\mathsf{r}_1^{\mathcal{CL}} \qquad \frac{}{p, p \Rightarrow q}\,\mathsf{r}_2^{\mathcal{CL}} \qquad \frac{p, p \Rightarrow q}{q}\,\mathsf{r}_3^{\mathcal{CL}}$$

In a monadic PNmatrix $\mathfrak{M} := \langle \mathbf{A}, D\rangle$, a set $\mathcal{S}_a$ of unary formulas *isolates* $a \in A$ in case, for every $b \neq a \in A$, there is a separator in $\mathcal{S}_a$ for a and b. A *discriminator for* $\mathfrak{M}$ is a family $\{(\Omega_a, \mho_a)\}_{a \in A}$ such that $\Omega_a \cup \mho_a$ isolates a and $\mathrm{S}^{\mathbf{A}}(a) \subseteq D$ if $\mathrm{S} \in \Omega_a$, and $\mathrm{S}^{\mathbf{A}}(a) \subseteq \overline{D}$ if $\mathrm{S} \in \mho_a$. Discriminators play an essential role in the axiomatization of monadic PN-matrices, as we shall see in the next results.

Theorem 4.3. $\mathsf{R}_{\mathfrak{M}^{\Rightarrow_\mathsf{full}}}$ *is* $\{p, {\sim}p, \circ p\}$*-analytic and axiomatizes* $\rhd_{\mathfrak{M}^{\Rightarrow_\mathsf{full}}}$.

Proof. We consider the axiomatization method presented by Marcelino and Caleiro [2021, Theorem 3.5 and Theorem 3.12], which can be applied because $\mathfrak{M}^{\Rightarrow_\mathsf{full}}$ is monadic, with same discriminator as that of $\mathfrak{M}_6$ (see Figure 1). The method can be seen essentially as a process of refining fully indeterministic six-valued interpretations of the connectives (that is, interpretations where $\mathcal{V}_6$ appears at every entry) by imposing soundness of some collections of inference rules, until obtaining the desired interpretations (in our case, the ones of $\mathfrak{M}^{\Rightarrow_\mathsf{full}}$). For the connectives $\wedge, \vee, \sim$ and $\circ$, the method produces the calculus $\mathsf{R}_{\mathcal{PP}^{\rhd}_{\leq}}$ after simplifications (Gomes et al. [2022]). Thus, since the method is modular on the connectives, we only need to see what happens when we run it on the new connective $\Rightarrow_\mathsf{full}$ and then add the resulting rules to $\mathsf{R}_{\mathcal{PP}^{\rhd}_{\leq}}$. The rules that are imposed will have the following shape:

$$\frac{\Omega_a(p), \Omega_b(q), \Omega_c(p \Rightarrow q)}{\mho_a(p), \mho_b(q), \mho_c(p \Rightarrow q)}$$

for each $c \in \mathcal{V}_6 \backslash (a \Rightarrow_\mathsf{full} b)$ and each $a, b \in \mathcal{V}_6$, where the sets Ω_d and $\mho_d$ for each $d \in \mathcal{V}_6$ are given by the following table:

d	Ω_d	$\mho_d$
$\hat{\mathbf{f}}$	$\circ p$	p
$\mathbf{f}$	$\sim p$	$\circ p, p$
$\mathbf{n}$	$\varnothing$	$p, \circ p, \sim p$
$\mathbf{b}$	$p, \sim p$	$\circ p$
$\mathbf{t}$	p	$\circ p, \sim p$
$\hat{\mathbf{t}}$	$p, \circ p$	$\varnothing$

Table 1: Discriminator for $\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}$. They give, for example, $\Omega_{\mathbf{b}} = \{p, \sim p\}$ and $\mho_{\mathbf{n}} = \{p, \circ p, \sim p\}$.

These rules at first do not seem to relate to the three rules for $\Rightarrow$ in $\mathsf{R}_{\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}}$. However, we will see now that each of them is a 'dilution' of one of the latter (that is, they are obtained from the latter by adding formulas on the antecedents and succedents). Then, because these three rules for implication are sound, it follows trivially that we can use only them and discard its dilutions without harm for completeness and for analyticity.

First, consider the entries of $a \Rightarrow_{\mathsf{full}} b$ for $b \in {\uparrow}\mathbf{b}$. The values $c \in \mathcal{V}_6 \backslash (a \Rightarrow_{\mathsf{full}} b)$ are precisely $\mathbf{n}$, $\mathbf{f}$ and $\hat{\mathbf{f}}$. Then $q \in \Omega_b(q)$ and $p \Rightarrow q \in \mho_c(p \Rightarrow q)$, thus the above rules are all dilutions of $\mathsf{r}_1^{\mathcal{CL}}$.

Second, consider the entries of $a \Rightarrow_{\mathsf{full}} b$ for $a \notin {\uparrow}\mathbf{b}$. The values $c \in \mathcal{V}_6 \backslash (a \Rightarrow_{\mathsf{full}} b)$ are precisely $\mathbf{n}$, $\mathbf{f}$ and $\hat{\mathbf{f}}$. Then $p \in \mho_a(p)$ and $p \Rightarrow q \in \mho_c(p \Rightarrow q)$, thus the above rules are all dilutions of $\mathsf{r}_2^{\mathcal{CL}}$.

Finally, consider the entries of $a \Rightarrow_{\mathsf{full}} b$ for $a \in {\uparrow}\mathbf{b}$ and $b \notin {\uparrow}\mathbf{b}$. The values $c \in \mathcal{V}_6 \backslash (a \Rightarrow_{\mathsf{full}} b)$ are precisely $\mathbf{b}$, $\mathbf{t}$ and $\hat{\mathbf{t}}$. Then $p \in \Omega_a(p)$, $p \Rightarrow q \in \Omega_c(p \Rightarrow q)$ and $q \in \mho_b(q)$, thus the above rules are all dilutions of $\mathsf{r}_3^{\mathcal{CL}}$. □

Remark 4.4. *The above result could have been obtained by another strategy, observing that $\rhd_{\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}}$ is induced by the* strict product *[Caleiro and Marcelino, 2023, Definition 10] of $\mathfrak{M}_6$ and the matrix $\mathfrak{M}_2$ over the signature containing only $\Rightarrow$, in which the algebra has carrier $\{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ and interprets $\Rightarrow$ as in classical logic ($a \Rightarrow^{\mathfrak{M}_2} b = \hat{\mathbf{f}}$ if, and only if, $a = \hat{\mathbf{t}}$ and $b = \hat{\mathbf{f}}$), and the designated set is $\{\hat{\mathbf{t}}\}$. By Caleiro and Marcelino [2023, Theorem 12], this implies that $\rhd_{\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}}$ is the* disjoint fibring *of $\mathcal{PP}_{\leq}^{\rhd}$ and $\rhd_{\mathfrak{M}_2}$ (that is, the smallest* SET-SET *logic in the signature $\Sigma_{\Rightarrow}$ extending both logics). One can then show that, because both logics have analytic* SET-SET *axiomatizations, it is enough to merge both calculi in order to axiomatize their disjoint fibring. As it is well-known that $\rhd_{\mathfrak{M}_2}$ is axiomatized by the $\{p\}$-analytic calculus given by the three rules for implication in Definition 4.2, the desired result follows.*

From the above, we have that $\rhd_{\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}}$ is special in the sense of being the smallest conservative expansion of $\mathcal{PP}_{\leq}^{\rhd}$ in which the added implication is classic-like (cf. Definition 2.11):

Proposition 4.5. *Let $\rhd$ be a conservative expansion of $\mathcal{PP}_{\leq}^{\rhd}$ over $\Sigma_{\Rightarrow}^{\mathsf{PP}}$ in which $\Rightarrow$ is a classic-like implication. Then $\rhd_{\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}} \subseteq \rhd$.*

Proof. It is easy to see that any classic-like implication must satisfy the three rules for implication in the calculus for $\rhd_{\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}}$, and this is all we need for the present result. □

We observe that each proper refinement of $\Rightarrow_{\mathsf{full}}$ produces a proper extension of $\rhd_{\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}}$. In fact, following again the axiomatization method in Caleiro and Marcelino [2019], for each such extension we can use the monadicity of $\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}$ to obtain SET-SET rules that hold in it but not in $\mathsf{R}_{\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}}$, and they will be precisely the rules that need to be added to the latter to obtain a $\{p, \sim p, \circ p\}$-analytic calculus for these extensions.

Proposition 4.6. *Let $\mathfrak{M}$ be obtained from $\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}$ by refining $\Rightarrow_{\mathsf{full}}$. Then $\mathsf{R}_{\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}}$ with the following rules provide a $\{p, \sim p, \circ p\}$-analytic calculus for $\mathfrak{M}$:*

$$\frac{\Omega_a(p), \Omega_b(q), \Omega_c(p \Rightarrow q)}{\mho_a(p), \mho_b(q), \mho_c(p \Rightarrow q)}$$

for each $c \in (a \Rightarrow_{\mathsf{full}} b) \backslash (a \Rightarrow^{\mathfrak{M}} b)$ and each $a, b \in \mathcal{V}_6$.

Proof. Directly from the method in Marcelino and Caleiro [2021, Theorem 3.5 and Theorem 3.12]. □

For an example of the latter axiomatization technique in action, if we remove the value $\mathbf{n}$ from the entry $\mathbf{b} \Rightarrow_{\mathsf{full}} \hat{\mathbf{f}}$, we axiomatize the resulting logic by adding the following rule:

$$\frac{p, \sim p, \circ q, \sim q,}{\circ p, q, p \Rightarrow q, \sim(p \Rightarrow q), \circ(p \Rightarrow q)},$$

which clearly does not hold in $\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}$ under a valuation v with $v(p) = \mathbf{b}$, $v(q) = \hat{\mathbf{f}}$ and $v(p \Rightarrow q) = \mathbf{n}$, but holds in the new logic precisely because this valuation was forbidden when we deleted $\mathbf{n}$ from that entry.

As another application of this technique, we show how to axiomatize the logic $\mathrm{LET}^+_{\mathrm{K}}$ of Coniglio and Rodrigues [2023] with a $\{p, {\sim}p, \circ p\}$-analytic calculus, as it fits precisely in this setting, that is, it is determined by a refinement of $\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}$. In fact, its implication is given by the following truth table:

$\Rightarrow^{\mathrm{LET}^+_{\mathrm{K}}}$	$\hat{\mathbf{f}}$	$\mathbf{f}$	$\mathbf{n}$	$\mathbf{b}$	$\mathbf{t}$	$\hat{\mathbf{t}}$
$\hat{\mathbf{f}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\hat{\mathbf{t}}$
$\mathbf{n}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\hat{\mathbf{t}}$
$\mathbf{b}$	$\hat{\mathbf{f}}$	$\mathbf{f}$	$\mathbf{n}$	$\mathbf{b}$	$\mathbf{t}$	$\hat{\mathbf{t}}$
$\mathbf{t}$	$\hat{\mathbf{f}}$	$\mathbf{f}$	$\mathbf{n}$	$\mathbf{b}$	$\mathbf{t}$	$\hat{\mathbf{t}}$
$\hat{\mathbf{t}}$	$\hat{\mathbf{f}}$	$\mathbf{f}$	$\mathbf{n}$	$\mathbf{b}$	$\mathbf{t}$	$\hat{\mathbf{t}}$

Then, to obtain the desired axiomatization, it is enough to add to $\mathsf{R}_{\mathfrak{M}^{\Rightarrow_{\mathsf{full}}}}$ the following inference rules:

$$\frac{{\sim}(p \Rightarrow q)}{p} \quad \frac{{\sim}(p \Rightarrow q)}{{\sim}q} \quad \frac{p, {\sim}q}{{\sim}(p \Rightarrow q)} \quad \frac{\circ(p \Rightarrow q)}{\circ p, \circ q} \quad \frac{\circ(p \Rightarrow q)}{\circ p, p, q}$$

$$\frac{\circ(p \Rightarrow q), p}{\circ q} \quad \frac{\circ p}{\circ(p \Rightarrow q), p} \quad \frac{p, \circ q}{\circ(p \Rightarrow q)} \quad \frac{\circ q, q}{\circ(p \Rightarrow q)}$$

The attentive reader will notice that they are not quite the same rules as the ones produced by the recipe in Proposition 4.6. In fact, they are simplifications thereof, following the streamlining procedures described in Marcelino and Caleiro [2021]. Still, it is not hard to see that these rules produce the desired refinements of $\Rightarrow_{\mathsf{full}}$ in a similar way as we did in the previous example.

5 Logics of PP-algebras expanded with a Heyting implication

In the light of Theorem 4.1, as we proceed to expand our logic with an implication we shall necessarily have to drop either (A1) or (A2). In this section we explore the first option, that is, we stick to (A2) while dropping (A1). Having fixed a (deterministic) implication operator (say, on $\mathbf{PP_6}$), a straightforward way to ensure that the resulting logic will be self-extensional (cf. Proposition 2.8) is to consider, as in the implication-less case, the SET-FMLA consequence relation that preserves the lattice order of $\mathbf{PP_6}$ (or, to be more precise, of the resulting class of algebras augmented with an implication). Indeed, as shown by Jansana [2006, Sec. 3], under certain assumptions every self-extensional logic turns out to be the consequence associated to a suitably defined partial order.

We thus shall follow this route, which still leaves us free to choose among the implication operators for $\mathbf{PP_6}$. Since on $\mathbf{PP_6}$ we cannot define a classic-like implication suitable for our purposes — i.e., one satisfying both (A1) and (A2) — we suggest to introduce a Heyting implication. From an algebraic point of view, such an operator is readily available. Indeed, since $\mathbf{PP_6}$ has a (finite) distributive lattice reduct, the meet operation has a residuum (we will denote it by $\Rightarrow_{\mathsf{H}}$), which is precisely the relative pseudo-complement operation.

Definition 5.1. *Let* $\mathbf{PP_6}^{\Rightarrow_{\mathsf{H}}}$ *be the algebra obtained by expanding* $\mathbf{PP_6}$ *with the operation* $\Rightarrow_{\mathsf{H}}$ *defined as follows:*

$$a \Rightarrow_{\mathsf{H}} b := \max\{c \in \mathcal{V}_6 \mid a \wedge^{\mathbf{PP_6}} c \leq b\}, \textit{ for all } a, b \in \mathcal{V}_6.$$

In accordance with the above definition, the truth-table of the implication in $\mathbf{PP_6}^{\Rightarrow_{\mathsf{H}}}$ looks as follows:

$\Rightarrow_{\mathsf{H}}$	$\hat{\mathbf{f}}$	$\mathbf{f}$	$\mathbf{n}$	$\mathbf{b}$	$\mathbf{t}$	$\hat{\mathbf{t}}$
$\hat{\mathbf{f}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$
$\mathbf{f}$	$\hat{\mathbf{f}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$
$\mathbf{n}$	$\hat{\mathbf{f}}$	$\mathbf{b}$	$\hat{\mathbf{t}}$	$\mathbf{b}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$
$\mathbf{b}$	$\hat{\mathbf{f}}$	$\mathbf{n}$	$\mathbf{n}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$
$\mathbf{t}$	$\hat{\mathbf{f}}$	$\mathbf{f}$	$\mathbf{n}$	$\mathbf{b}$	$\hat{\mathbf{t}}$	$\hat{\mathbf{t}}$
$\hat{\mathbf{t}}$	$\hat{\mathbf{f}}$	$\mathbf{f}$	$\mathbf{n}$	$\mathbf{b}$	$\mathbf{t}$	$\hat{\mathbf{t}}$

$\mathbf{PP_6}^{\Rightarrow_{\mathsf{H}}}$ is obviously (i.e., has a reduct which is) a Heyting algebra. Furthermore, since it also carries a De Morgan negation, it may be called a *De Morgan-Heyting algebra* according to Sankappanavar [1987], or a *symmetric Heyting* *algebra* according to Monteiro [1980][1]. Indeed, since $\mathbf{PP_6}^{\Rightarrow_H}$ (as $\mathbf{PP_6}$) also has a Stone lattice reduct (i.e. a pseudo-complemented distributive lattice satisfying $\neg x \vee \neg\neg x \approx \top$), we may be a little more specific, observing that $\mathbf{PP_6}^{\Rightarrow_H}$ is also, in Monteiro's terminology, a *Stonean symmetric Heyting algebra* [Monteiro, 1980, Ch. IV, Def. 1.1]. These observations will be exploited in Section 7.

Remark 5.2. *Before we proceed any further, one may wonder whether we are really adding something new to $\mathbf{PP_6}$. In other words, was the implication $\Rightarrow_H$ already term-definable in this algebra? The answer is negative. To see why, observe that $\mathbf{PP_6}$ is a subdirectly irreducible algebra having a unique non-trivial congruence θ, which is the one that identifies (only) the elements in the set $\{\mathbf{t}, \mathbf{f}, \mathbf{b}, \mathbf{n}\}$. By adding the Heyting implication, we obtain a simple algebra, in which θ is no longer a congruence (indeed, $\mathbf{b}\theta\mathbf{n}$, but it is not the case that $(\mathbf{b} \Rightarrow_H \mathbf{b})\theta(\mathbf{b} \Rightarrow_H \mathbf{n})$).*

Before moving to the logics of order associated to the new algebra, we could first consider the SET-SET logic determined by $\langle \mathbf{PP_6}^{\Rightarrow_H}, \uparrow\mathbf{b}\rangle$, which is guaranteed to conservatively expand $\mathcal{PP}_{\leq}^{\triangleright}$ (with the implication $\Rightarrow_H$ being the residuum of $\wedge$). Since $\mathfrak{M}_6$ is monadic, we can axiomatize $\Rightarrow_H$ straight away (similarly as we did in the proof of Theorem 4.3) following the method of Marcelino and Caleiro [2021]:

Theorem 5.3. *The* SET-SET $\Sigma_{\Rightarrow}$*-logic induced by $\mathfrak{M}_6$ expanded with $\Rightarrow_H$ is axiomatized by the $\{p, {\sim}p, \circ p\}$-analytic* SET-SET *calculus given by $\mathsf{R}_{\mathcal{PP}_{\leq}^{\triangleright}}$ plus the following rules:*

$$\frac{q}{p \Rightarrow q} \qquad \frac{p, p \Rightarrow q}{q} \qquad \frac{{\sim}(p \Rightarrow q)}{{\sim}q} \qquad \frac{{\sim}q}{{\sim}(p \Rightarrow q), {\sim}p}$$

$$\frac{}{p \Rightarrow q, \circ q, p} \qquad \frac{p \Rightarrow q}{\circ(p \Rightarrow q), {\sim}q, q} \qquad \frac{p \Rightarrow q, \circ q}{\circ p, q} \qquad \frac{{\sim}(p \Rightarrow q), {\sim}p}{\circ(p \Rightarrow q)}$$

$$\frac{{\sim}p}{\circ(p \Rightarrow q), p} \qquad \frac{\circ(p \Rightarrow q), \circ p, p}{\circ q} \qquad \frac{\circ(p \Rightarrow q), p}{\circ q, q} \qquad \frac{\circ p}{p \Rightarrow q, p}$$

$$\frac{\circ q}{\circ(p \Rightarrow q)} \qquad \frac{q}{{\sim}(p \Rightarrow q), \circ(p \Rightarrow q), \circ p}$$

Note however that $\langle \mathbf{PP_6}^{\Rightarrow_H}, \uparrow\mathbf{b}\rangle$ does not satisfy (A1): pick for instance $a := \mathbf{f}$ and $b := \hat{\mathbf{f}}$. Self-extensionality (A2) is also not satisfied. To see why, note that the formula ${\sim}(p \Rightarrow q) \wedge {\sim}(q \Rightarrow p)$ and $\bot$ are logically equivalent. However, $\nabla\bot$ and $\nabla({\sim}(p \Rightarrow q) \wedge {\sim}(q \Rightarrow p))$ are not — recall that $\nabla x := x \vee {\sim}\circ x$. In fact, the latter and $\bot$ are logically equivalent, while this does not hold for $\nabla\bot$. To see this, consider a valuation such that $v(p) = \mathbf{b}$ and $v(q) = \mathbf{n}$. Then $v(\nabla({\sim}(p \Rightarrow q) \wedge {\sim}(q \Rightarrow p))) = \nabla^{\mathbf{PP_6}^{\Rightarrow_H}}({\sim}^{\mathbf{PP_6}^{\Rightarrow_H}}(\mathbf{b} \Rightarrow^{\mathbf{PP_6}^{\Rightarrow_H}} \mathbf{n}) \wedge^{\mathbf{PP_6}^{\Rightarrow_H}} {\sim}^{\mathbf{PP_6}^{\Rightarrow_H}}(\mathbf{n} \Rightarrow^{\mathbf{PP_6}^{\Rightarrow_H}} \mathbf{b})) = \hat{\mathbf{t}}$. Actually, as we are going to show in Proposition 5.7, there is only one self-extensional SET-FMLA logic determined by a class of matrices based on the algebra $\mathbf{PP_6}^{\Rightarrow_H}$ and principal filters.

Let us now resume our discussion about the logics of order. Having obtained a new algebra $\mathbf{PP_6}^{\Rightarrow_H}$, we can consider the variety it generates, denoted $\mathbb{V}(\mathbf{PP_6}^{\Rightarrow_H})$, and the order-preserving logics associated to it. We denote by $\mathcal{PP}_{\leq}^{\triangleright,\Rightarrow_H}$ and $\mathcal{PP}_{\leq}^{\Rightarrow_H}$, respectively, the SET-SET and SET-FMLA order-preserving logics associated to $\mathbb{V}(\mathbf{PP_6}^{\Rightarrow_H})$ (cf. Subsection 2.2 for the precise definitions). By the residuation property, we clearly have that $\Rightarrow_H$ is an implication in these logics (cf. Definition 2.10 and Proposition 2.12). Furthermore:

Proposition 5.4. *$\mathcal{PP}_{\leq}^{\triangleright,\Rightarrow_H}$ and $\mathcal{PP}_{\leq}^{\Rightarrow_H}$ are conservative expansions of $\mathcal{PP}_{\leq}^{\triangleright}$ and $\mathcal{PP}_{\leq}$, respectively. Moreover, they are self-extensional.*

Proof. That both logics are conservative expansions of $\mathcal{PP}_{\leq}$ follows directly from their matrix characterizations (Proposition 2.5), while self-extensionality follows from Proposition 2.8. □

We may actually simplify the family of matrices that characterize these logics (see Proposition 2.5) by removing some redundancies:

Proposition 5.5. *The logics $\mathcal{PP}_{\leq}^{\triangleright,\Rightarrow_H}$ and $\mathcal{PP}_{\leq}^{\Rightarrow_H}$ are determined by the class of matrices $\{\langle \mathbf{PP_6}^{\Rightarrow_H}, \uparrow\mathbf{f}\rangle, \langle \mathbf{PP_6}^{\Rightarrow_H}, \uparrow\mathbf{b}\rangle, \langle \mathbf{PP_6}^{\Rightarrow_H}, \uparrow\hat{\mathbf{t}}\rangle\}$. Moreover, all these matrices are reduced.*

[1]Monteiro's work suggests another natural candidate for an implication (this one, already term-definable) in $\mathbf{PP_6}$. This is the 'implication faible' $\Rightarrow_W$ introduced in [Monteiro, 1980, Ch. IV, Def. 4.1], which can be given by the following term: $x \Rightarrow_W y := {\sim}x \vee {\sim}\circ x \vee y$.

Proof. We begin by item (1). Any matrix $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, D\rangle$ with $D \neq \mathcal{V}_6$ is reduced, because $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$ is a simple algebra. However, two of the matrices appearing in Proposition 5.4 may be safely omitted. Obviously this holds for $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, \uparrow\hat{\mathbf{f}}\rangle$, which defines a trivial logic. Note further that $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, \uparrow\mathbf{b}\rangle$ and $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, \uparrow\mathbf{n}\rangle$ are isomorphic, and thus determine the same logic; so one of them can also be omitted. Finally, the matrix with set of designated values $\uparrow\mathbf{t}$ can be safely omitted as we only need to consider prime filters. Thus only the matrices listed in the statement remain. Item (2) follows from the observation that $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ is the SET-FMLA companion of the two SET-SET order-preserving logics. ☐

There is another SET-SET logic that has $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ as SET-FMLA companion, namely the one determined by the class of matrices based on the algebra $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$ having as designated sets the principal filters of $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$, which we denote by $\mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}$. By a similar argument as above, we have that $\mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}$ is determined by the class of matrices $\{\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, \uparrow\mathbf{f}\rangle, \langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, \uparrow\mathbf{b}\rangle, \langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, \uparrow\mathbf{t}\rangle, \langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, \uparrow\hat{\mathbf{t}}\rangle\}$ — that is, the matrices from Proposition 5.5 together with the matrix $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, \uparrow\mathbf{t}\rangle$. In this logic, however, $\Rightarrow$ is not an implication in our sense, since $\rhd_{\mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}} (p \vee q) \Rightarrow (p \vee q)$, but $p \vee q \blacktriangleright_{\mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}} p, q$. Still, we will include $\mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}$ in our considerations in view of its close relationship with $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ and because our techniques will also apply very naturally to it, as we shall see.

The above results do not clarify whether one could find a single matrix to characterize those logics (as it happens with $\mathbf{PP}_{\mathbf{6}}$): the next result rules out this possibility.

Theorem 5.6. *The logic $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ is not determined by a single logical matrix. The same holds also for the logics $\mathcal{PP}_{\leq}^{\rhd,\Rightarrow_{\mathsf{H}}}$ and $\mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}$.*

Proof. Since the two SET-SET logics share the same SET-FMLA companion $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$, it is enough to prove that the latter is not determined by a single logical matrix. For that, we use the fact that a SET-FMLA logic that fails the property of *cancellation* fails to be determined by a single logical matrix (Wójcicki [1988], Shoesmith and Smiley [1978]). A Σ-logic $\vdash$ respects cancellation when $\Phi, \{\Psi_i \,:\, i \in I\} \vdash \varphi$ implies $\Phi \vdash \varphi$ for all $\{\varphi\}, \Phi, \Psi_i \subseteq L_\Sigma(P)$, $i \in I$, such that

1. $\mathsf{props}(\Phi \cup \{\varphi\}) \cap \mathsf{props}(\bigcup_{i\in I} \Psi_i) = \varnothing$
2. $\mathsf{props}(\Psi_i) \cap \mathsf{props}(\Psi_j) = \varnothing$, $i \neq j$
3. for all $i \in I$, we have $\Psi_i \nvdash \psi$ for some $\psi \in L_\Sigma(P)$.

Consider now $I = \{1\}$, $\Phi = \varnothing$, $\Psi_1 = \{p \wedge {\sim}p \wedge q \wedge {\sim}q \wedge {\sim}{\circ}(p \Rightarrow q)\}$ and $\varphi = r \vee {\sim}r$. Conditions (1) and (2) are obviously satisfied. Condition (3) is satisfied too, just consider the matrix $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, \uparrow\mathbf{f}\rangle$ to see that $\Psi_1 \nvdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} \bot$. Moreover, we have clearly that $\nvdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} r \vee {\sim}r$ (it suffices to consider the matrix $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, \uparrow\mathbf{b}\rangle$). Finally, one can check that $p \wedge {\sim}p \wedge q \wedge {\sim}q \wedge {\sim}{\circ}(p \Rightarrow q) \vdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} r \vee {\sim}r$. ☐

As earlier anticipated, we now proceed to show that $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ is actually the only self-extensional SET-FMLA logic we can have when semantics of classes of matrices based on the algebra $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$ and principal filters are concerned.

Proposition 5.7. *For $\mathcal{V} \subseteq \mathcal{V}_6$, let $\vdash_{\mathcal{V}}$ be the* SET-FMLA *logic determined by $\left\{\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, \uparrow a\rangle \mid a \in \mathcal{V}\right\}$. If the logic $\vdash_{\mathcal{V}}$ is self-extensional, then $\vdash_{\mathcal{V}} = \mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$.*

Proof. By Jansana [2006, Thm. 3.7], the finitary self-extensional extensions of $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ are in a one-to-one correspondence with the subvarieties of $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$. The matrices in $\{\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, \uparrow a\rangle \mid a \in \mathcal{V}\}$ are reduced models of $\vdash_{\mathcal{V}}$, implying that $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$ belongs to the subvariety $\mathsf{K} \subseteq \mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$ associated with $\vdash_{\mathcal{V}}$. Then $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}) \subseteq \mathsf{K}$, so $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}) = \mathsf{K}$, and thus $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}} = \vdash_{\mathcal{V}}$. ☐

Adopting the terminology of Jansana [2006], we may say that $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ is *semilattice-based* relative to $\wedge$ and to $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$. This observation allows us to obtain further information: for instance, we know by Jansana [2006, Thm. 3.12] that the class of algebra reducts of reduced matrices for $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ is precisely $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$. Semilattice-based logics are often non-algebraizable but have an algebraizable companion, i.e. an extension that shares the same algebraic models. We establish this for $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ below.

Proposition 5.8. *$\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ is equivalential but not algebraizable. The following is a set of equivalence formulas for it:*

$$\Xi(x,y) := \{x \Rightarrow y, y \Rightarrow x, \circ(x \Rightarrow y), \circ(y \Rightarrow x)\}$$

or, equivalently (setting $\Delta x := x \wedge \circ x$),

$$\Xi(x,y) := \{\Delta(x \Rightarrow y), \Delta(y \Rightarrow x)\}.$$

Proof. To prove that $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ is equivalential, it suffices to verify that the following conditions are met (see Font [2016, Thm. 6.60]):

1. $\vdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} \Xi(x,x)$
2. $x, \Xi(x,y) \vdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} y$
3. $\Xi(x,y) \vdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} \Xi(\circ x, \circ y)$ and $\Xi(x,y) \vdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} \Xi({\sim}x, {\sim}y)$
4. $\Xi(x_1,y_1), \Xi(x_2,y_2) \vdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} \Xi(x_1 © x_2, y_1 © y_2)$ for every binary connective © in the signature.

The verification of the above conditions is straightforward from the matrix characterization of $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$. Now, if $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ were algebraizable, then the set of designated elements would be equationally definable on every reduced matrix (see Font [2016, Def. 6.90]). But this is not the case, because the matrices $\{\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, \uparrow a\rangle \mid a \in \mathcal{V}_6 \setminus \{\hat{\mathbf{f}}\}\}$, all based on the same algebra, are all reduced, and have distinct sets of designated elements. □

The algebraizable companion of $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ is the $\top$-assertional logic associated to the variety $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$, which we denote by $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$. The reduced models of this logic are all matrices of the form $\langle \mathbf{A}, \{\top^{\mathbf{A}}\}\rangle$, where $\mathbf{A}$ is an algebra in $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$. One easily verifies that $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ satisfies the following rules — any of them being in fact sufficient to distinguish $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ from $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ (given that they are sound in the former but not in the latter):

$$\frac{p}{\Delta p} \qquad \frac{p}{\circ p} \qquad \frac{p \Rightarrow q}{{\sim}q \Rightarrow {\sim}p} \qquad \frac{p,\ p \Rightarrow_{\mathsf{W}} q}{q}$$

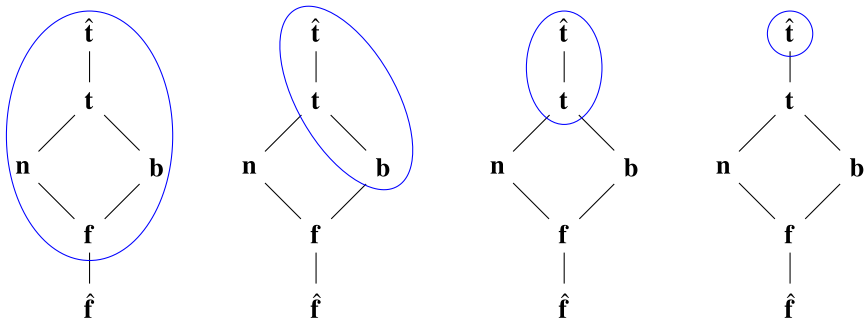
where $p \Rightarrow_{\mathsf{W}} q := {\sim}p \vee {\sim}\circ p \vee q$ and $\Delta p := p \wedge \circ p$.

5.1 Characterizability by single finite PNmatrices

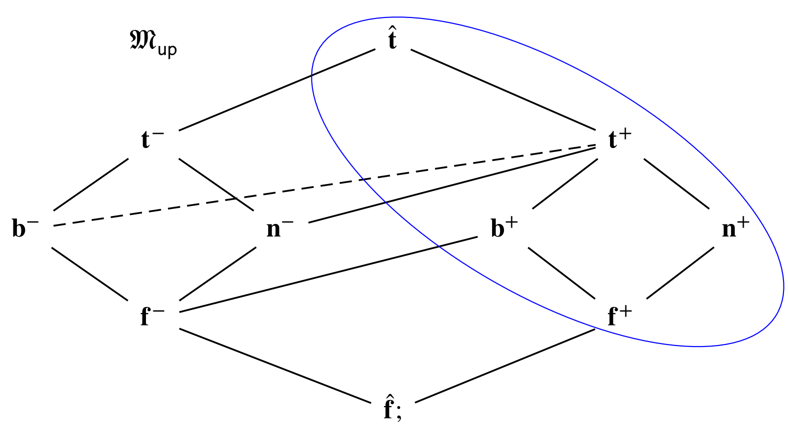
We saw in Theorem 5.6 that the logics we introduced in this section are not characterized by any single logical matrix. In this subsection, we will demonstrate the power of partial non-deterministic matrices by showing that $\mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}$ and $\mathcal{PP}_{\leq}^{\rhd,\Rightarrow_{\mathsf{H}}}$ are characterized by a single finite PNmatrix. In consequence, $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ will be characterized by either of them. The essential idea is that the collections of matrices that characterize these logics can be packaged into a single structure using partiality.

The construction we will present makes use of the notion of total components of a Σ-PNmatrix, which we now proceed to introduce. Let $\mathfrak{M} := \langle \mathbf{A}, D\rangle$ be a Σ-PNmatrix. For $X \subseteq A$, denote by $\mathfrak{M}_X$ the Σ-PNmatrix $\langle \mathbf{A}_X, D \cap X\rangle$, where $\mathbf{A}_X := \langle A \cap X, \cdot^{\mathbf{A}_X}\rangle$ is a Σ-multialgebra such that $©^{\mathbf{A}_X}(a_1, \ldots, a_k) := ©^{\mathbf{A}}(a_1, \ldots, a_k) \cap X$ for all $a_1, \ldots, a_k \in X$, $k \in \omega$ and $© \in \Sigma_k$. This PNmatrix is called *the restriction of* $\mathfrak{M}$ *to* X. We say that $X \neq \varnothing$ is a *total component* of $\mathfrak{M}$ whenever $\mathfrak{M}_X$ is total. A total component X is *maximal* if adding any other value to X leads to a component that is not total. Denote by $\mathbb{T}(\mathfrak{M})$ the collection of maximal total components of $\mathfrak{M}$. Then we have that $\rhd_{\mathfrak{M}} = \rhd_{\{\mathfrak{M}_X \mid X \in \mathbb{T}(\mathfrak{M})\}}$ (Caleiro and Marcelino [2023]). The latter observation is key to us: the matrices induced by the maximal total components of the PNmatrices we will construct are be precisely the ones in the classes that determine the logics $\mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}$ and $\mathcal{PP}_{\leq}^{\rhd,\Rightarrow_{\mathsf{H}}}$.

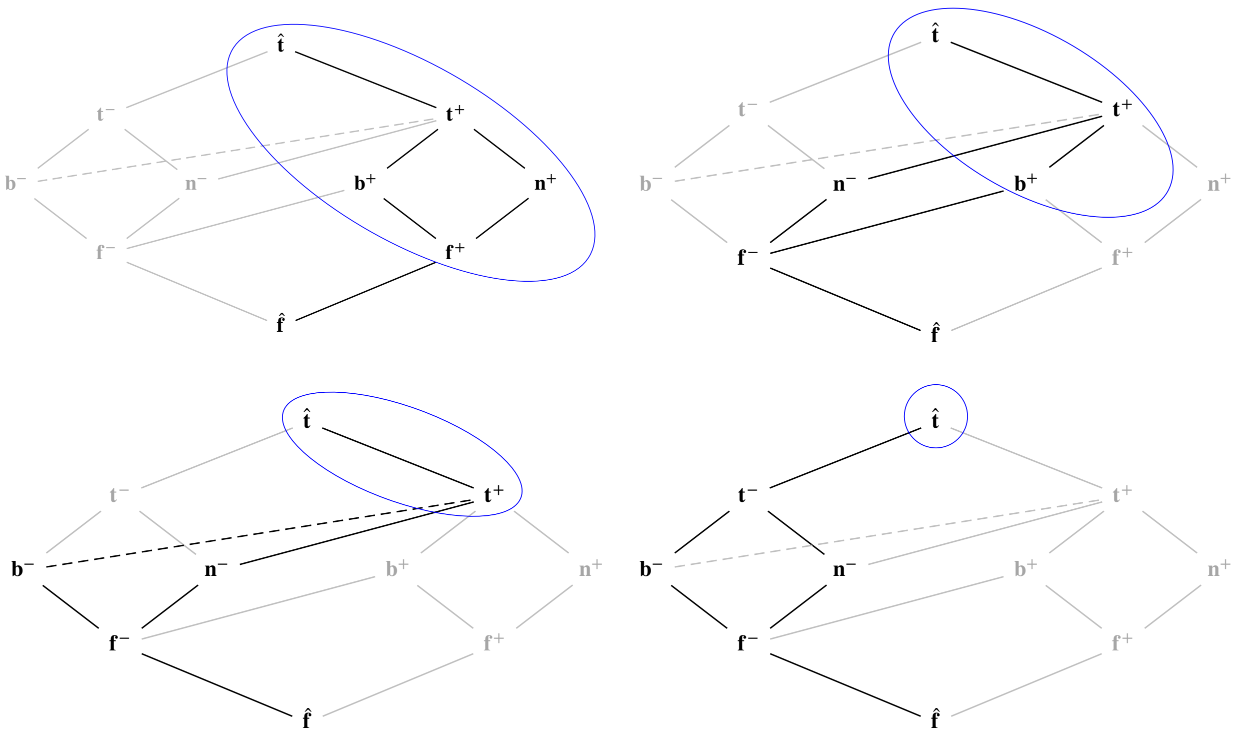
We display below diagrams of the four matrices that determine the logic $\mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}$ (in each case, the sets of designated elements are highlighted with an ellipse).



Below we depict the structure of the PNmatrix $\mathfrak{M}_{\mathsf{up}}$ we propose for this logic, whose principle of construction is the combination of the above matrices, such that each of them consists of a total component of the PNmatrix. Note: There is nothing special about the dashed edge, it is pictured as dashed only because it crosses other edges.



We try to make the above idea clearer in the following pictures, which show how to identify each of the four matrices inside $\mathfrak{M}_{\mathsf{up}}$.



The PNmatrix for $\mathcal{PP}_{\leq}^{\triangleright,\Rightarrow_{\mathsf{H}}}$, which we dub $\mathfrak{M}_{\leq}$, is essentially the same, the only difference being the absence of the dashed line.

After this informal presentation of the general approach, we proceed to a precise definition of the PNmatrices $\mathfrak{M}_{\mathsf{up}}$ and $\mathfrak{M}_{\leq}$, and prove that they indeed determine, respectively, the logics $\mathcal{PP}_{\mathsf{up}}^{\triangleright,\Rightarrow_{\mathsf{H}}}$ and $\mathcal{PP}_{\leq}^{\triangleright,\Rightarrow_{\mathsf{H}}}$.

Definition 5.9. *Let* $\mathcal{V}_{10} := \{\hat{\mathbf{f}}, \mathbf{f}^-, \mathbf{n}^-, \mathbf{b}^-, \mathbf{t}^-, \mathbf{f}^+, \mathbf{n}^+, \mathbf{b}^+, \mathbf{t}^+, \hat{\mathbf{t}}\}$ *and* $D := \{\mathbf{f}^+, \mathbf{n}^+, \mathbf{b}^+, \mathbf{t}^+, \hat{\mathbf{t}}\} \subseteq \mathcal{V}_{10}$*. Consider the ternary relations* $\mathsf{inc}_{\mathsf{up}}(a, b, c)$ *and* $\mathsf{inc}_{\leq}(a, b, c)$ *on* $\mathcal{V}_{10}$ *defined as follows:*

$$\begin{aligned}
&\mathsf{inc}_{\mathsf{up}}(a, b, c) \textit{ iff} \\
&\qquad X \subseteq \{a, b, c\}, \textit{for some } X \in \{\{d^-, d^+\} : d \in \{\mathbf{f}, \mathbf{n}, \mathbf{b}, \mathbf{t}\}\} \cup \\
&\qquad\quad \{\{\mathbf{b}^-, \mathbf{f}^+\}, \{\mathbf{n}^-, \mathbf{f}^+\}, \{\mathbf{b}^+, \mathbf{t}^-\}, \{\mathbf{n}^+, \mathbf{t}^-\}, \{\mathbf{n}^+, \mathbf{b}^+, \mathbf{f}^-\}\} \\
&\mathsf{inc}_{\leq}(a, b, c) \textit{ iff} \\
&\qquad X \subseteq \{a, b, c\}, \textit{for some } X \in \{\{d^-, d^+\} : d \in \{\mathbf{f}, \mathbf{n}, \mathbf{b}, \mathbf{t}\}\} \cup \\
&\qquad\quad \{\{\mathbf{b}^-, \mathbf{f}^+\}, \{\mathbf{n}^-, \mathbf{f}^+\}, \{\mathbf{b}^+, \mathbf{t}^-\}, \{\mathbf{n}^+, \mathbf{t}^-\}, \{\mathbf{n}^+, \mathbf{b}^+, \mathbf{f}^-\}, \{\mathbf{n}^-, \mathbf{b}^-, \mathbf{t}^+\}\}
\end{aligned}$$

Consider also the function $f : \mathcal{V}_{10} \to \{\hat{\mathbf{f}}, \mathbf{f}, \mathbf{n}, \mathbf{b}, \mathbf{t}, \hat{\mathbf{t}}\}$ *given by* $f(\hat{\mathbf{f}}) := \hat{\mathbf{f}}$*,* $f(\hat{\mathbf{t}}) := \hat{\mathbf{t}}$*,* $f(a^i) := a$ *for* $a \in \{\mathbf{f}, \mathbf{n}, \mathbf{b}, \mathbf{t}\}$*. We define the PNmatrices* $\mathfrak{M}_{\mathsf{up}} := \langle \mathbf{A}_{\mathsf{up}}, D\rangle$ *and* $\mathfrak{M}_{\leq} := \langle \mathbf{A}_{\leq}, D\rangle$*, with* $\mathbf{A}_{\mathsf{up}} := \langle \mathcal{V}_{10}, \cdot_{\mathsf{up}}\rangle$ *and* $\mathbf{A}_{\leq} := \langle \mathcal{V}_{10}, \cdot_{\leq}\rangle$ *such that*

- *for* $\copyright \in \{\sim, \circ\}$*,*

$$\copyright_{\mathsf{up}}(a) := \{b \in \mathcal{V}_{10} : \copyright_{\mathbf{PP}_6^{\Rightarrow_{\mathsf{H}}}}(f(a)) = f(b),\ \textit{not } \mathsf{inc}_{\mathsf{up}}(a, a, b)\}$$
$$\copyright_{\leq}(a) := \{b \in \mathcal{V}_{10} : \copyright_{\mathbf{PP}_6^{\Rightarrow_{\mathsf{H}}}}(f(a)) = f(b),\ \textit{not } \mathsf{inc}_{\leq}(a, a, b)\}$$

- *for* $\copyright \in \{\wedge, \vee, \Rightarrow\}$*,*

$$\copyright_{\mathsf{up}}(a, b) := \{c \in \mathcal{V}_{10} : \copyright_{\mathbf{PP}_6^{\Rightarrow_{\mathsf{H}}}}(f(a), f(b)) = f(c),\ \textit{not } \mathsf{inc}_{\mathsf{up}}(a, b, c)\}$$
$$\copyright_{\leq}(a, b) := \{c \in \mathcal{V}_{10} : \copyright_{\mathbf{PP}_6^{\Rightarrow_{\mathsf{H}}}}(f(a), f(b)) = f(c),\ \textit{not } \mathsf{inc}_{\leq}(a, b, c)\}$$

Proposition 5.10. $\mathcal{PP}_{\mathsf{up}}^{\triangleright,\Rightarrow_{\mathsf{H}}}$ *is determined by* $\mathfrak{M}_{\mathsf{up}}$ *and* $\mathcal{PP}_{\leq}^{\triangleright,\Rightarrow_{\mathsf{H}}}$ *is determined by* $\mathfrak{M}_{\leq}$*.*

Proof. We have that if $\{a, b, c\} \subseteq X$ and $\mathsf{inc}_{\mathsf{up}}(a, b, c)$ then X is not contained in a total component of $\mathfrak{M}_{\mathsf{up}}$. In fact,

- if $b_1, b_2 \in X$ with
$$\{b_1, b_2\} \in \{\{a^-, a^+\} : a \in \{\mathbf{f}, \mathbf{n}, \mathbf{b}, \mathbf{t}\}\} \cup \{\{\mathbf{b}^-, \mathbf{f}^+\}, \{\mathbf{n}^-, \mathbf{f}^+\}, \{\mathbf{b}^+, \mathbf{t}^-\}, \{\mathbf{n}^+, \mathbf{t}^-\}\}$$
then $b_1 \wedge_{\mathsf{up}} b_2 = \varnothing$.
- if $\{\mathbf{n}^+, \mathbf{b}^+, \mathbf{f}^-\} \subseteq X$, then $\mathbf{n}^+ \wedge_{\mathsf{up}} \mathbf{b}^+ = \{\mathbf{f}^+\}$ and $\mathbf{f}^- \wedge_{\mathsf{up}} \mathbf{f}^+ = \varnothing$.

The maximal total components of $\mathfrak{M}_{\mathsf{up}}$ are

$$\mathbb{T}(\mathfrak{M}_{\mathsf{up}}) = \{\{\hat{\mathbf{f}}, \mathbf{f}^-, \mathbf{n}^-, \mathbf{b}^-, \mathbf{t}^-, \hat{\mathbf{t}}\}, \{\hat{\mathbf{f}}, \mathbf{f}^-, \mathbf{n}^-, \mathbf{b}^-, \mathbf{t}^+, \hat{\mathbf{t}}\}, \{\hat{\mathbf{f}}, \mathbf{f}^-, \mathbf{n}^-, \mathbf{b}^+, \mathbf{t}^+, \hat{\mathbf{t}}\}, \{\hat{\mathbf{f}}, \mathbf{f}^+, \mathbf{n}^+, \mathbf{b}^+, \mathbf{t}^+, \hat{\mathbf{t}}\}\}$$

since for every $X \in \mathbb{T}(\mathfrak{M}_{\mathsf{up}})$ and every $a, b, c \in X$ we have that $\mathsf{inc}_{\mathsf{up}}(a, b, c)$ is not the case. Thus the restriction of $\mathfrak{M}_{\mathsf{up}}$ to X is isomorphic to (that is, it is the same up to renaming of truth values) some matrix with set of designated values $D_X = {\uparrow} a$ for $a \in \mathcal{V}_6$. The isomorphism is given by the restriction of f to X.

Similarly, the maximal total components of $\mathfrak{M}_{\leq}$ are

$$\mathbb{T}(\mathfrak{M}_{\leq}) = \{\{\hat{\mathbf{f}}, \mathbf{f}^-, \mathbf{n}^-, \mathbf{b}^-, \mathbf{t}^-, \hat{\mathbf{t}}\}, \{\hat{\mathbf{f}}, \mathbf{f}^-, \mathbf{n}^-, \mathbf{b}^+, \mathbf{t}^+, \hat{\mathbf{t}}\}, \{\hat{\mathbf{f}}, \mathbf{f}^+, \mathbf{n}^+, \mathbf{b}^+, \mathbf{t}^+, \hat{\mathbf{t}}\}\}$$

This is so because $\mathsf{inc}_{\leq}(a, b, c)$ iff $\mathsf{inc}_{\mathsf{up}}(a, b, c)$ or $\{\mathbf{n}^-, \mathbf{b}^-, \mathbf{t}^+\} \subseteq X$, and the fact that if $\{\mathbf{n}^-, \mathbf{b}^-, \mathbf{t}^+\} \subseteq X$ then X is not in a total component of $\mathfrak{M}_{\leq}$, since $\mathbf{n}^- \vee_{\mathsf{up}} \mathbf{b}^- = \{\mathbf{t}^-\}$ and $\mathbf{t}^- \wedge_{\mathsf{up}} \mathbf{t}^+ = \varnothing$. $\square$

6 Hilbert-style axiomatizations

Our goal here is to present analytic SET-SET Hilbert-style axiomatizations for the implicative extensions $\mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}$ and $\mathcal{PP}_{\leq}^{\rhd,\Rightarrow_{\mathsf{H}}}$, as well as a SET-FMLA axiomatization for $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$. Unfortunately, the PNmatrices from the previous section do not allow us to extract automatically an analytic SET-SET calculus for the corresponding logics using the technology of Caleiro and Marcelino [2019], employed in Gomes et al. [2022] and in the previous sections of the present paper, in view of the following:

Proposition 6.1. *Neither $\mathfrak{M}_{\leq}$ nor $\mathfrak{M}_{\leq}$ is monadic.*

Proof. Note that no unary formula can separate $\mathbf{n}^+$ from $\mathbf{b}^+$, nor $\mathbf{n}^-$ from $\mathbf{b}^-$. Indeed, one can show inductively on the structure of a unary formula $\varphi(p)$ that, given valuations v_a, v_b such that $v_a(p) = a^s$ and $v_b(p) = b^s$ for $s \in \{+, -\}$, we have that $v_x(\varphi) \in \{\hat{\mathbf{f}}, x^s, \hat{\mathbf{t}}\}$, and either $v_a(\varphi) = v_b(\varphi)$ or $v_a(\varphi) = a^s$ and $v_b(\varphi) = b^s$. $\square$

We need, therefore, to delve into specific details of the logical matrices introduced above, and extract what is important to characterize their algebraic and logical structures in terms of formulas and SET-SET rules of inference. We will do that and obtain analytic axiomatizations for the SET-SET logics and then, taking advantage of the fact that in $\mathcal{PP}_{\leq}^{\rhd,\Rightarrow_{\mathsf{H}}}$ we have a disjunction (a notion we will soon make precise), we will convert the calculus for $\mathcal{PP}_{\leq}^{\rhd,\Rightarrow_{\mathsf{H}}}$ into a SET-FMLA axiomatization for $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$.

6.1 Analytic axiomatizations for $\mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}$ and $\mathcal{PP}_{\leq}^{\rhd,\Rightarrow_{\mathsf{H}}}$

We begin by the analytic SET-SET axiomatizations. What follows is a succession of definitions introducing groups of rules of inference that capture particular aspects of the collections of logical matrices determining $\mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}$ and $\mathcal{PP}_{\leq}^{\rhd,\Rightarrow_{\mathsf{H}}}$. We check the soundness of each of them and ultimately arrive to the desired completeness results. Throughout the proofs, we will make use of the following abbreviations:

Definition 6.2. *Set ${\uparrow} p := \circ({\sim} p \Rightarrow p)$ and ${\downarrow} p := \circ(p \Rightarrow {\sim} p)$.*

Note that the above definitions introduce new connectives by means of abbreviations, so we have $\mathsf{sub}({\uparrow} p) = \{p, {\sim} p, {\sim} p \Rightarrow p, \circ({\sim} p \Rightarrow p)\}$ (the case of $\mathsf{sub}({\downarrow} p)$ is similar).

Definition 6.3. *Let* $\mathsf{R}_{\Diamond}$ *be the* SET-SET *calculus given by the following inference rules:*

$$\frac{}{\uparrow p, \downarrow p}\,\mathsf{r}^{\Diamond}_{\uparrow\mathsf{or}\downarrow} \qquad \frac{}{\circ(p \Rightarrow p)}\,\mathsf{r}^{\Diamond}_{\mathsf{id}} \qquad \frac{\circ(p \Rightarrow q), \circ(q \Rightarrow r)}{\circ q, \circ(p \Rightarrow r)}\,\mathsf{r}^{\Diamond}_{\mathsf{trans}}$$

$$\frac{}{\downarrow p, \circ q, \circ(q \Rightarrow p)}\,\mathsf{r}^{\Diamond}_{\leq\mathbf{t}} \qquad \frac{}{\uparrow p, \circ(p \Rightarrow q)}\,\mathsf{r}^{\Diamond}_{\geq\mathbf{f}}$$

$$\frac{\uparrow p, \circ(p \Rightarrow q)}{\circ p, \uparrow q}\,\mathsf{r}^{\Diamond}_{\mathsf{incclass}_1} \qquad \frac{\downarrow q, \circ(p \Rightarrow q)}{\circ q, \downarrow p}\,\mathsf{r}^{\Diamond}_{\mathsf{incclass}_2} \qquad \frac{\uparrow p, \downarrow q, \circ(p \Rightarrow q)}{\circ q, \circ(q \Rightarrow p)}\,\mathsf{r}^{\Diamond}_{\mathsf{incclass}_3}$$

$$\frac{\downarrow p, \uparrow r}{\circ p, \circ(p \Rightarrow q), \circ(p \Rightarrow r), \circ(q \Rightarrow r)}\,\mathsf{r}^{\Diamond}_{\mathsf{just}_2}$$

Proposition 6.4. *The rules of* $\mathsf{R}_{\Diamond}$ *are sound for any matrix* $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, D\rangle$ *with* D *containing* $\hat{\mathbf{t}}$ *and not containing* $\hat{\mathbf{f}}$.

Proof. We proceed rule by rule.

$\mathsf{r}^{\Diamond}_{\uparrow\mathsf{or}\downarrow}$**:** If $v(\uparrow p) \notin D$, then $v(\uparrow p) = \hat{\mathbf{f}}$, so $v(p) = \mathbf{f}$ and thus $v(\downarrow p) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}^{\Diamond}_{\mathsf{id}}$**:** Clearly, $v(\circ(p \Rightarrow p)) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}^{\Diamond}_{\mathsf{trans}}$**:** If $v(\circ q) \notin D$, we have $v(q) \neq \hat{\mathbf{f}}$, hence $v(\circ(p \Rightarrow q)) = \hat{\mathbf{t}}$ and $v(\circ(q \Rightarrow r)) = \hat{\mathbf{t}}$, thus $v(p) \leq v(q) \leq v(r)$, and therefore $v(\circ(p \Rightarrow r)) = \hat{\mathbf{t}}$.

$\mathsf{r}^{\Diamond}_{\leq\mathbf{t}}$**:** If $v(\downarrow p) \notin D$, then $v(p) = \mathbf{t}$. If $v(\circ q) \notin D$, then $\mathbf{f} \leq v(q) \leq \mathbf{t}$ and therefore $v(\circ(q \Rightarrow p)) = \hat{\mathbf{t}}$.

$\mathsf{r}^{\Diamond}_{\geq\mathbf{f}}$**:** If $v(\uparrow p) \notin D$ then $v(p) = \mathbf{f}$ and therefore $v(\circ(p \Rightarrow q)) = \hat{\mathbf{t}}$.

$\mathsf{r}^{\Diamond}_{\mathsf{incclass}_1}$**:** If $v(\uparrow p) \in D$ and $v(\circ p) \notin D$ then $v(p) \in \{\mathbf{b}, \mathbf{n}, \mathbf{t}\}$ and if $v(\circ(p \Rightarrow q)) \in D$ then $v(q) \in \{\hat{\mathbf{f}}, \mathbf{b}, \mathbf{n}, \mathbf{t}, \hat{\mathbf{t}}\}$ and therefore $v(\uparrow q) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}^{\Diamond}_{\mathsf{incclass}_2}$**:** If $v(\downarrow q) \in D$ and $v(\circ q) \notin D$ then $v(q) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}\}$ and if $v(\circ(p \Rightarrow q)) \in D$ then $v(p) \in \{\hat{\mathbf{f}}, \mathbf{f}, \mathbf{b}, \mathbf{n}\}$ and therefore $v(\downarrow p) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}^{\Diamond}_{\mathsf{incclass}_3}$**:** If $v(\uparrow p), v(\downarrow q) \in D$ and $v(\circ q) \notin D$ then $v(p) \neq \mathbf{f}$ and $v(q) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}\}$. Further, if $v(\circ(p \Rightarrow q)) \in D$ then $v(p) \leq v(q)$ and so $v(q \Rightarrow p) \in D$.

$\mathsf{r}^{\Diamond}_{\mathsf{just}_2}$**:** If $v(\downarrow p), v(\uparrow r) \in D$ and $v(\circ p) \notin D$ then $v(p) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}\}$ and $v(r) \neq \mathbf{f}$. If $v(\circ(p \Rightarrow q)), v(\circ(p \Rightarrow q)) \notin D$, then $v(p \Rightarrow q), v(q \Rightarrow r) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$. Then $v(p), v(q), v(r) \in \{\mathbf{b}, \mathbf{n}\}$. Hence, either $v(p) = v(q)$, in which case $v(p \Rightarrow q) = \hat{\mathbf{t}} \in D$, or $v(q) = v(r)$, in which case $v(q \Rightarrow r) = \hat{\mathbf{t}} \in D$ (both cases contradicting the assumptions) or $v(p) = v(r)$, in which case $v(p \Rightarrow r) = \hat{\mathbf{t}} \in D$, as desired. □

Definition 6.5. *Let* $\mathsf{R}_{\Rightarrow}$ *be the* SET-SET *calculus given by the following inference rules:*

$$\frac{\circ q}{\circ(p \Rightarrow q)}\,\mathsf{r}^{\Rightarrow}_{1} \qquad \frac{q}{p \Rightarrow q}\,\mathsf{r}^{\Rightarrow}_{2} \qquad \frac{p, p \Rightarrow q}{q}\,\mathsf{r}^{\Rightarrow}_{3} \qquad \frac{\circ p, p, \circ(p \Rightarrow q)}{\circ q}\,\mathsf{r}^{\Rightarrow}_{4} \qquad \frac{\circ p, p, \downarrow(p \Rightarrow q)}{\downarrow q}\,\mathsf{r}^{\Rightarrow}_{5}$$

$$\frac{\circ p, p, \uparrow(p \Rightarrow q)}{\uparrow q}\,\mathsf{r}^{\Rightarrow}_{6} \qquad \frac{\uparrow q}{\uparrow(p \Rightarrow q)}\,\mathsf{r}^{\Rightarrow}_{7} \qquad \frac{\downarrow q}{\downarrow(p \Rightarrow q)}\,\mathsf{r}^{\Rightarrow}_{8} \qquad \frac{}{\circ(q \Rightarrow (p \Rightarrow q))}\,\mathsf{r}^{\Rightarrow}_{9} \qquad \frac{\circ p}{p, \circ(p \Rightarrow q)}\,\mathsf{r}^{\Rightarrow}_{10}$$

$$\frac{\circ p}{p, p \Rightarrow q}\,\mathsf{r}^{\Rightarrow}_{11} \qquad \frac{\circ q, p \Rightarrow q}{q, \circ p}\,\mathsf{r}^{\Rightarrow}_{12} \qquad \frac{\circ(p \Rightarrow q)}{\circ q, p \Rightarrow q}\,\mathsf{r}^{\Rightarrow}_{13} \qquad \frac{}{\downarrow p, \circ(p \Rightarrow q), \circ((p \Rightarrow q) \Rightarrow q)}\,\mathsf{r}^{\Rightarrow}_{14}$$

$$\frac{\uparrow p, \circ(p \Rightarrow q)}{\circ p, \uparrow q}\,\mathsf{r}^{\Rightarrow}_{15} \qquad \frac{\downarrow p}{\circ p, \uparrow(p \Rightarrow q)}\,\mathsf{r}^{\Rightarrow}_{16} \qquad \frac{}{\circ p, \downarrow(p \Rightarrow q)}\,\mathsf{r}^{\Rightarrow}_{17}$$

$$\frac{\uparrow p, \circ(p \Rightarrow (p \Rightarrow q))}{\circ p, \uparrow q}\,\mathsf{r}^{\Rightarrow}_{18} \qquad \frac{\uparrow q}{\circ(p \Rightarrow q), \circ((p \Rightarrow q) \Rightarrow q)}\,\mathsf{r}^{\Rightarrow}_{19}$$

Proposition 6.6. *The rules of* $\mathsf{R}_{\Rightarrow}$ *are sound for any matrix* $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, D\rangle$ *with* $D = {\uparrow}a$ *for some* $a > \hat{\mathbf{f}}$.

Proof. We proceed rule by rule.

$\mathsf{r}_1^{\Rightarrow}$**:** If $v(\circ q) \in D$ then $v(q) \in \{\hat{\mathbf{t}}, \hat{\mathbf{f}}\}$, hence $v(\circ(p \Rightarrow q)) \in D$.

$\mathsf{r}_2^{\Rightarrow}$**:** If $v(q) \in D$, then $v(q) \geq a$ and by analysing the table of $\Rightarrow_{\mathsf{H}}$ we conclude that $v(p \Rightarrow q) \geq v(q) \geq a$ and thus $v(p \Rightarrow q) \in D$.

$\mathsf{r}_3^{\Rightarrow}$**:** If $v(p), v(p \Rightarrow q) \in D$ then $v(p) \geq a$ and $v(p \Rightarrow q) \geq a$, and by analysing the table of $\Rightarrow_{\mathsf{H}}$ we conclude that $v(q) \geq a$ and thus $v(q) \in D$.

$\mathsf{r}_4^{\Rightarrow}$**:** If $v(\circ p), v(p) \in D$ and $v(\circ q) \notin D$, then $v(p) = \hat{\mathbf{t}}$ and $v(q) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$ and $v(p \Rightarrow q) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$, so $v(\circ(p \Rightarrow q)) = \hat{\mathbf{f}} \notin D$.

$\mathsf{r}_5^{\Rightarrow}$**:** If $v(\circ p), v(p) \in D$ and $v({\downarrow} q) \notin D$, then $v(p) = \hat{\mathbf{t}}$ and $v(q) = \mathbf{t}$, thus $v(p \Rightarrow q) = \mathbf{t}$ and we are done.

$\mathsf{r}_6^{\Rightarrow}$**:** If $v(\circ p), v(p) \in D$ and $v({\uparrow} q) \notin D$, then $v(p) = \hat{\mathbf{t}}$ and $v(q) = \mathbf{f}$, thus $v(p \Rightarrow q) = \mathbf{f}$ and we are done.

$\mathsf{r}_7^{\Rightarrow}$**:** If $v({\uparrow} q) \in D$, then $v(q) \neq \mathbf{f}$. But then $v(p \Rightarrow q) \neq \mathbf{f}$, and $v({\uparrow}(p \Rightarrow q)) = \hat{\mathbf{t}}$.

$\mathsf{r}_8^{\Rightarrow}$**:** Similar to the proof for $\mathsf{r}_7^{\Rightarrow}$.

$\mathsf{r}_9^{\Rightarrow}$**:** If $v(\circ(p \Rightarrow (q \Rightarrow p))) \notin D$, then $v(p \Rightarrow (q \Rightarrow p)) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$. Then $v(p) \in \{\mathbf{b}, \mathbf{n}, \mathbf{t}, \hat{\mathbf{t}}\}$ and $v(q \Rightarrow p) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$. So, $v(q) \in \{\mathbf{b}, \mathbf{n}, \mathbf{t}, \hat{\mathbf{t}}\}$ and $v(p) \in \{\mathbf{b}, \mathbf{n}, \mathbf{t}\}$. If $v(p) = \mathbf{b}$ and $v(q \Rightarrow p) = \mathbf{b}$, then $v(p \Rightarrow (q \Rightarrow p)) = \hat{\mathbf{t}}$. If $v(p) = \mathbf{n}$, the proof is similar. If $v(p) = \mathbf{t}$, then $v(q \Rightarrow p) = \hat{\mathbf{t}}$ or $v(q) = \hat{\mathbf{t}}$. In all cases we reach a contradiction.

$\mathsf{r}_{10}^{\Rightarrow}, \mathsf{r}_{11}^{\Rightarrow}$**:** If $v(p), v(\circ p) \notin D$, then $v(p) = \hat{\mathbf{f}}$, thus $v(p \Rightarrow q) = v(\circ(p \Rightarrow q)) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}_{12}^{\Rightarrow}$**:** If $v(\circ q) \in D$ and $v(q) \notin D$, then $v(q) = \hat{\mathbf{f}}$. If $v(\circ p) \notin D$, then $v(p) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$. But then $v(p \Rightarrow q) = \mathbf{f} \notin D$.

$\mathsf{r}_{13}^{\Rightarrow}$**:** If $v(\circ(p \Rightarrow q)) \in D$, then $v(p \Rightarrow q) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$. Further, if $v(\circ q) \notin D$, then $v(q) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$, thus $v(p \Rightarrow q) = \hat{\mathbf{t}}$.

$\mathsf{r}_{14}^{\Rightarrow}$**:** If $v(\circ(p \Rightarrow q)) \notin D$, we have $v(p \Rightarrow q) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$. If $v({\downarrow} p) \notin D$, then $v(p) = \mathbf{t}$, and $v(p \Rightarrow q) = v(q)$, and we are done.

$\mathsf{r}_{15}^{\Rightarrow}$**:** If $v(\circ p) \notin D$, then $v(p) \in \{\mathbf{b}, \mathbf{n}, \mathbf{t}\}$. If $v({\uparrow} q) \notin D$, then $v(q) = \mathbf{f}$. Thus $v(p \Rightarrow q) \in \{\mathbf{b}, \mathbf{n}, \mathbf{f}\}$, so $v(\circ(p \Rightarrow q)) = \hat{\mathbf{f}}$.

$\mathsf{r}_{16}^{\Rightarrow}$**:** If $v(\circ p) \notin D$ and $v({\downarrow} p) \in D$, then $v(p) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}\}$ and $v(p \Rightarrow q) \neq \mathbf{f}$, and thus $v({\uparrow}(p \Rightarrow q)) = \hat{\mathbf{t}}$.

$\mathsf{r}_{17}^{\Rightarrow}$**:** If $v(\circ p) \notin D$, then $v(p) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$. Thus $v(p \Rightarrow q) \neq \mathbf{t}$, so $v({\downarrow}(p \Rightarrow q)) = \hat{\mathbf{t}}$.

$\mathsf{r}_{18}^{\Rightarrow}$**:** If $v({\uparrow} p) \in D$ and $v(\circ p) \notin D$, then $v(p) \in \{\mathbf{b}, \mathbf{n}, \mathbf{t}\}$. If $v({\uparrow} q) \notin D$, then $v(q) = \mathbf{f}$. If $v(p) = \mathbf{b}$, then $v(p \Rightarrow q) = \mathbf{n}$, and then $v(p \Rightarrow (p \Rightarrow q)) = \mathbf{n}$. Similarly, we have $v(p \Rightarrow (p \Rightarrow q)) = \mathbf{b}$ if $v(p) = \mathbf{n}$. If $v(p) = \mathbf{t}$, then $v(p \Rightarrow q) = \mathbf{f}$, and $v(p \Rightarrow (p \Rightarrow q)) = \mathbf{f}$. In any case, $v(\circ(p \Rightarrow (p \Rightarrow q))) = \hat{\mathbf{f}}$.

$\mathsf{r}_{19}^{\Rightarrow}$**:** If $v(\circ(p \Rightarrow q)) \notin D$, we have $v(p \Rightarrow q) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$. If $v({\uparrow} q) \in D$, we have $v(q) \neq \mathbf{f}$. The case $v(p \Rightarrow q) = \mathbf{f}$ is impossible. If $v(p \Rightarrow q) = \mathbf{b}$, we have $v(q) = \mathbf{b}$, and clearly $v(\circ((p \Rightarrow q) \Rightarrow q)) = \hat{\mathbf{t}}$. The case $v(p \Rightarrow q) = \mathbf{n}$ is similar as the previous case. If $v(p \Rightarrow q) = \mathbf{t}$, then $v(p) = \hat{\mathbf{t}}$, thus $v(\circ((p \Rightarrow q) \Rightarrow q)) = \hat{\mathbf{t}}$. ☐

Definition 6.7. *Let* $\mathsf{R}_{\sim}$ *be the* SET-SET *calculus given by the following inference rules:*

$$\frac{\circ p}{p, {\sim} p}\,\mathsf{r}_1^{\sim} \qquad \frac{\circ p, p, {\sim} p}{\quad}\,\mathsf{r}_2^{\sim} \qquad \frac{\circ p}{\circ {\sim} p}\,\mathsf{r}_3^{\sim} \qquad \frac{\circ {\sim} p}{\circ p}\,\mathsf{r}_4^{\sim}$$

$$\frac{{\uparrow}{\sim} p}{{\downarrow} p}\,\mathsf{r}_5^{\sim} \qquad \frac{{\downarrow}{\sim} p}{{\uparrow} p}\,\mathsf{r}_6^{\sim} \qquad \frac{{\downarrow} p}{{\uparrow}{\sim} p}\,\mathsf{r}_7^{\sim} \qquad \frac{{\uparrow} p}{{\downarrow}{\sim} p}\,\mathsf{r}_8^{\sim}$$

Proposition 6.8. *The rules of* $\mathsf{R}_{\sim}$ *are sound for any matrix* $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, D\rangle$ *with* $D = {\uparrow}a$ *for some* $a > \hat{\mathbf{f}}$.

Proof. We proceed rule by rule.

$\mathsf{r}^{\sim}_{1}$: If $v(\circ p) \in D$ and $v(p) \notin D$ then $v(p) = \hat{\mathbf{f}}$ and $v({\sim}p) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}^{\sim}_{2}$: If $v(p), v(\circ p) \in D$ then $v(p) = \hat{\mathbf{t}}$ and $v({\sim}p) = \hat{\mathbf{f}} \notin D$.

$\mathsf{r}^{\sim}_{3}$: If $v(\circ p) \in D$ then $v(p), v({\sim}p) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ and so $v(\circ{\sim}p) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}^{\sim}_{4}$: If $v(\circ{\sim}p) \in D$ then $v({\sim}p), v(p) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ and so $v(\circ p) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}^{\sim}_{5}$: If $v(\downarrow p) \notin D$ then $v(p) = \mathbf{t}$, so $v({\sim}p) = \mathbf{f}$ and hence $v(\uparrow{\sim}p) = \hat{\mathbf{f}} \notin D$.

$\mathsf{r}^{\sim}_{6}$: If $v(\uparrow p) \notin D$ then $v(p) = \mathbf{f}$, so $v({\sim}p) = \mathbf{t}$ and hence $v(\downarrow{\sim}p) = \hat{\mathbf{f}} \notin D$.

$\mathsf{r}^{\sim}_{7}$: If $v(\downarrow p) \in D$ then $v(p) \neq \mathbf{t}$, hence $v({\sim}p) \neq \mathbf{f}$ and so $v(\uparrow{\sim}p) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}^{\sim}_{8}$: If $v(\uparrow p) \in D$ then $v(p) \neq \mathbf{f}$, hence $v({\sim}p) \neq \mathbf{t}$ and so $v(\downarrow{\sim}p) = \hat{\mathbf{t}} \in D$. ☐

Definition 6.9. *Let* $\mathsf{R}_\circ$ *be the* SET-SET *calculus given by the following inference rule:*

$$\frac{}{\circ\circ p}\,\mathsf{r}_\circ$$

Proposition 6.10. *The rules of* $\mathsf{R}_\circ$ *are sound for any matrix* $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, D\rangle$ *with* $\hat{\mathbf{t}} \in D$.

Proof. Easily, $v(\circ\circ p) = \hat{\mathbf{t}} \in D$. ☐

Definition 6.11. *Let* $\mathsf{R}_\wedge$ *be the* SET-SET *calculus given by the following inference rules:*

$$\frac{\circ p, \circ q}{\circ(p \wedge q)}\,\mathsf{r}^{\wedge}_{1} \qquad \frac{p, q}{p \wedge q}\,\mathsf{r}^{\wedge}_{2} \qquad \frac{p \wedge q}{q}\,\mathsf{r}^{\wedge}_{3} \qquad \frac{p, \circ(p \wedge q)}{\circ q}\,\mathsf{r}^{\wedge}_{4} \qquad \frac{\circ p}{\circ(q \Rightarrow (p \wedge q))}\,\mathsf{r}^{\wedge}_{5}$$

$$\frac{}{\circ((p \wedge q) \Rightarrow q)}\,\mathsf{r}^{\wedge}_{6} \qquad \frac{p \wedge q}{p}\,\mathsf{r}^{\wedge}_{7} \qquad \frac{q, \circ(p \wedge q)}{\circ p}\,\mathsf{r}^{\wedge}_{8} \qquad \frac{\circ q}{\circ(p \Rightarrow (p \wedge q))}\,\mathsf{r}^{\wedge}_{9}$$

$$\frac{}{\circ((p \wedge q) \Rightarrow p)}\,\mathsf{r}^{\wedge}_{10} \qquad \frac{\circ p}{p, \circ(p \wedge q)}\,\mathsf{r}^{\wedge}_{11} \qquad \frac{\circ q}{q, \circ(p \wedge q)}\,\mathsf{r}^{\wedge}_{12} \qquad \frac{\circ(p \wedge q)}{\circ p, \circ q}\,\mathsf{r}^{\wedge}_{13}$$

$$\frac{\circ(p \Rightarrow q)}{\circ(p \Rightarrow (p \wedge q))}\,\mathsf{r}^{\wedge}_{14} \qquad \frac{\circ(q \Rightarrow p)}{\circ(q \Rightarrow (p \wedge q))}\,\mathsf{r}^{\wedge}_{15} \qquad \frac{\downarrow p, \uparrow(p \wedge q)}{\circ p, \circ(p \Rightarrow q)}\,\mathsf{r}^{\wedge}_{16}$$

Proposition 6.12. *The rules of* $\mathsf{R}_\wedge$ *are sound for any matrix* $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, D\rangle$ *with* $D = {\uparrow}a$ *for some* $a > \hat{\mathbf{f}}$.

Proof. We proceed rule by rule.

$\mathsf{r}^{\wedge}_{1}$: If $v(\circ p), v(\circ q) \in D$, then $v(p), v(q) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$, thus $v(p \wedge q) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$, and $v(\circ(p \wedge q)) = \hat{\mathbf{t}}$.

$\mathsf{r}^{\wedge}_{2}$: It follows because principal filters are closed under meets.

$\mathsf{r}^{\wedge}_{3}, \mathsf{r}^{\wedge}_{7}$: If $v(p \wedge q) \geq a$, then from $v(p) \geq v(p \wedge q)$ and $v(q) \geq v(p \wedge q)$ we have that $v(p), v(q) \in D$.

$\mathsf{r}^{\wedge}_{4}, \mathsf{r}^{\wedge}_{8}$: If $v(\circ q) \notin D$, then $v(q) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$. If $v(\circ(p \wedge q)) \in D$, then $v(p \wedge q) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$. Thus if $v(p \wedge q) = \hat{\mathbf{t}}$, then $v(p) = v(q) = \hat{\mathbf{t}}$, absurd. Otherwise, we must have $v(p) = \hat{\mathbf{f}}$. The proof is similar for the other rule.

$\mathsf{r}^{\wedge}_{5}, \mathsf{r}^{\wedge}_{9}$: If $v(\circ p) \in D$, then $v(p) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$. Then either $v(q \Rightarrow (p \wedge q)) = v(q \Rightarrow q) = \hat{\mathbf{t}}$ or $v(q \Rightarrow (p \wedge q)) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$. In both cases we are done. The proof is similar for the other rule.

$\mathsf{r}^{\wedge}_{6}, \mathsf{r}^{\wedge}_{10}$: We have that $v(p \wedge q) \leq v(p)$ and $v(p \wedge q) \leq v(q)$ hence $v((p \wedge q) \Rightarrow q) = v(\circ((p \wedge q) \Rightarrow q)) = v((p \wedge q) \Rightarrow p) = v(\circ((p \wedge q) \Rightarrow p)) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}^{\wedge}_{11}, \mathsf{r}^{\wedge}_{12}$: If $v(\circ p) \in D$ and $v(p) \notin D$, or $v(\circ q) \in D$ and $v(q) \notin D$, then either $v(p) = \hat{\mathbf{f}}$ or $v(q) = \hat{\mathbf{f}}$. In any case we have that $v(p \wedge q) = \hat{\mathbf{f}}$ and $v(\circ(p \wedge q)) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}^{\wedge}_{13}$**:** If $v(\circ(p \wedge q)) \in D$, then $v(p \wedge q) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$. If $v(p) = v(q) = \hat{\mathbf{t}}$, clearly $v(\circ p) = v(\circ q) = \hat{\mathbf{t}}$. If $v(p) = \hat{\mathbf{f}}$, then $v(\circ p) = \hat{\mathbf{t}}$. Analogously if $v(q) = \hat{\mathbf{f}}$.

$\mathsf{r}^{\wedge}_{14}, \mathsf{r}^{\wedge}_{15}$**:** Suppose $\circ(p \Rightarrow q) \in D$. Then either $v(q) = \hat{\mathbf{f}}$, in which case $v((p \vee q) \Rightarrow q), v((q \vee p) \Rightarrow q) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ and thus $v(\circ((p \vee q) \Rightarrow q)) = v(\circ((q \vee p) \Rightarrow q)) = \hat{\mathbf{t}} \in D$; or $v(p) \leq v(q)$, and so $v(p \vee q) = v(q \vee p) = v(q)$, hence $v(\circ((p \vee q) \Rightarrow q)) = v(\circ((q \vee p) \Rightarrow q)) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}^{\wedge}_{16}$**:** If $v(\circ p) \notin D$ and $v(\downarrow p) \in D$, then $v(p) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}\}$. If $v(\uparrow(p \wedge q)) \in D$, then $v(p \wedge q) \neq \mathbf{f}$. We may safely focus on cases in which $v(p) \neq v(q)$. If $v(p) = \mathbf{f}$, then $v(p \Rightarrow q) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$. If $v(p) = \mathbf{b}$, we have $v(q) \notin \{\mathbf{f}, \mathbf{b}, \mathbf{n}\}$ from the above assumptions, and this gives $v(p \Rightarrow q) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$. Similarly if $v(p) = \mathbf{n}$. □

Definition 6.13. *Let* $\mathsf{R}_{\vee}$ *be the* SET-SET *calculus given by the following inference rules:*

$$\frac{\circ p, \circ q}{\circ(p \vee q)}\,\mathsf{r}^{\vee}_{1} \qquad \frac{\circ p, p \vee q}{p, q}\,\mathsf{r}^{\vee}_{2} \qquad \frac{q}{p \vee q}\,\mathsf{r}^{\vee}_{3} \qquad \frac{\circ(p \vee q)}{p, \circ q}\,\mathsf{r}^{\vee}_{4} \qquad \frac{}{\circ(q \Rightarrow (p \vee q))}\,\mathsf{r}^{\vee}_{5}$$

$$\frac{\circ p}{p, \circ((p \vee q) \Rightarrow q)}\,\mathsf{r}^{\vee}_{6} \qquad \frac{p}{p \vee q}\,\mathsf{r}^{\vee}_{7} \qquad \frac{\circ(p \vee q)}{q, \circ p}\,\mathsf{r}^{\vee}_{8} \qquad \frac{}{\circ(p \Rightarrow (p \vee q))}\,\mathsf{r}^{\vee}_{9}$$

$$\frac{\circ q}{q, \circ((p \vee q) \Rightarrow p)}\,\mathsf{r}^{\vee}_{10} \qquad \frac{p, \circ p}{\circ(p \vee q)}\,\mathsf{r}^{\vee}_{11} \qquad \frac{q, \circ q}{\circ(p \vee q)}\,\mathsf{r}^{\vee}_{12} \qquad \frac{\circ(p \vee q)}{\circ p, \circ q}\,\mathsf{r}^{\vee}_{13}$$

$$\frac{\circ(p \Rightarrow q)}{\circ((p \vee q) \Rightarrow q)}\,\mathsf{r}^{\vee}_{14} \qquad \frac{\circ(q \Rightarrow p)}{\circ((p \vee q) \Rightarrow p)}\,\mathsf{r}^{\vee}_{15} \qquad \frac{\uparrow q, \downarrow(p \vee q)}{\circ p, \circ(p \Rightarrow q)}\,\mathsf{r}^{\vee}_{16}$$

Proposition 6.14. *The rules of* $\mathsf{R}_{\vee}$ *are sound for any matrix* $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, D\rangle$ *with* $D = {\uparrow}a$ *for some* $a > \hat{\mathbf{f}}$.

Proof. We proceed rule by rule.

$\mathsf{r}^{\vee}_{1}$**:** If $v(\circ p), v(\circ q) \in D$, then $v(p), v(q) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$, thus $v(p \vee q) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$, and we are done.

$\mathsf{r}^{\vee}_{2}$**:** If $v(\circ p) \in D$ and $v(p) \notin D$, $v(p) = \hat{\mathbf{f}}$. Then $v(p \vee q) = v(q)$ and we are done.

$\mathsf{r}^{\vee}_{3}, \mathsf{r}^{\vee}_{7}$**:** As $v(q) \leq v(p \vee q)$, if $v(q) \in D$, then $v(p \vee q) \in D$. Similarly for the other rule.

$\mathsf{r}^{\vee}_{4}, \mathsf{r}^{\vee}_{8}$**:** If $v(\circ(p \vee q)) \in D$, then $v(p \vee q) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$. If $v(\circ q) \notin D$, $v(q) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$. Then $v(p) = \hat{\mathbf{t}}$. Similarly for the other rule.

$\mathsf{r}^{\vee}_{5}, \mathsf{r}^{\vee}_{9}$**:** We have that $v(p \vee q) \geq v(p)$ and $v(p \vee q) \geq v(q)$, hence $v(q \Rightarrow (p \vee q)) = v(\circ(q \Rightarrow (p \vee q))) = v(p \Rightarrow (p \vee q)) = v(\circ(p \Rightarrow (p \vee q))) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}^{\vee}_{6}, \mathsf{r}^{\vee}_{10}$**:** If $v(\circ p) \in D$ and $v(p) \notin D$ then $v(p) = \hat{\mathbf{f}}$ and $v(p \vee q) = v(q)$, so $v((p \vee q) \Rightarrow q) = \hat{\mathbf{t}}$. Similarly if $v(\circ q) \in D$ and $v(q) \notin D$.

$\mathsf{r}^{\vee}_{11}, \mathsf{r}^{\vee}_{12}$**:** If $v(p), v(\circ p) \in D$, or $v(q), v(\circ q) \in D$, then either $v(p) = \hat{\mathbf{t}}$ or $v(q) = \hat{\mathbf{t}}$. In any case we have that $v(p \vee q) = v(\circ(p \vee q)) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}^{\vee}_{13}$**:** If $v(\circ p), v(\circ q) \notin D$, we have $v(p), v(q) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$, thus $v(p \vee q) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$, and $v(\circ(p \vee q)) \notin D$.

$\mathsf{r}^{\vee}_{14}, \mathsf{r}^{\vee}_{15}$**:** Suppose $\circ(p \Rightarrow q) \in D$. Then either $v(q) = \hat{\mathbf{f}}$, in which case $v((p \vee q) \Rightarrow q), v((q \vee p) \Rightarrow q) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ and thus $v(\circ((p \vee q) \Rightarrow q)) = v(\circ((q \vee p) \Rightarrow q)) = \hat{\mathbf{t}} \in D$; or $v(p) \leq v(q)$, so $v(p \vee q) = v(q \vee p) = v(q)$, hence $v(\circ((p \vee q) \Rightarrow q)) = v(\circ((q \vee p) \Rightarrow q)) = \hat{\mathbf{t}} \in D$.

$\mathsf{r}^{\vee}_{16}$**:** If $v(\circ p) \notin D$ and $v(\uparrow q) \in D$, $v(p) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$ and $v(q) \neq \mathbf{f}$. If $v(p) = v(q)$, we are done, so suppose $v(p) \neq v(q)$. If $v(\downarrow(p \vee q)) \in D$, then $v(p \vee q) \neq \mathbf{t}$. If $v(p) = \mathbf{f}$, then $v(p \Rightarrow q) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$, and we are done. If $v(p) = \mathbf{b}$, the only non-obvious case is $v(q) = \mathbf{n}$, but this is impossible as $v(p \vee q) \neq \mathbf{t}$. If $v(p) = \mathbf{n}$, the proof is similar as the previous case. If $v(p) = \mathbf{t}$, the only non-obvious cases are if $v(q) \in \{\mathbf{b}, \mathbf{n}\}$, but they are impossible again because $v(p \vee q) \neq \mathbf{t}$. □

Definition 6.15. *Let* $\mathsf{R}_{\bot\top}$ *be the* SET-SET *calculus given by the following inference rules:*

$$\frac{}{\top}\,\mathsf{r}_1^{\top} \qquad \frac{}{\circ\top}\,\mathsf{r}_2^{\top} \qquad \frac{\bot}{}\,\mathsf{r}_1^{\bot} \qquad \frac{}{\circ\bot}\,\mathsf{r}_2^{\bot}$$

Proposition 6.16. *The rules of* $\mathsf{R}_{\top\bot}$ *are sound for any matrix* $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, D\rangle$ *with* $D = {\uparrow}a$ *for some* $a > \hat{\mathbf{f}}$.

Proof. Obvious. □

The final rules we introduce encode the differences between the logics $\mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}$ and $\mathcal{PP}_{\leq}^{\rhd,\Rightarrow_{\mathsf{H}}}$.

Definition 6.17. *Consider the following inference rules:*

$$\frac{p, {\downarrow}p, q}{\circ p, \circ(p \Rightarrow q), \circ r, r}\,\mathsf{r}_{D_\wedge} \qquad \frac{p, \circ(p \Rightarrow q)}{\circ q, q}\,\mathsf{r}_{D_\leq} \qquad \frac{r, {\uparrow}q}{{\downarrow}r, \circ(p \Rightarrow q), p, q}\,\mathsf{r}_{D\neq\uparrow\mathbf{t}}$$

Proposition 6.18. *In the matrix* $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, D\rangle$*: (i) the rules* $\mathsf{r}_{D_\wedge}$ *and* $\mathsf{r}_{D_\leq}$ *are sound when* $D = {\uparrow}a$ *for* $a > \hat{\mathbf{f}}$*; (ii) the rule* $\mathsf{r}_{D\neq\uparrow\mathbf{t}}$ *is sound when* D *is a prime filter; (iii) the rule* $\mathsf{r}_{D\neq\uparrow\mathbf{t}}$ *is not sound when* $D = {\uparrow}\mathbf{t}$.

Proof. For items (i) and (ii), we prove soundness rule by rule.

$\mathsf{r}_{D_\wedge}$**:** If $v(\circ(p \Rightarrow q)) \notin D$, then $v(p \Rightarrow q) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$. From that and the assumption that $v(\circ p) \notin D$ and $v({\downarrow}p) \in D$, we have $v(p) \in \{\mathbf{b}, \mathbf{n}\}$. Also, we obtain $v(q) \in \{\mathbf{f}, \mathbf{n}, \mathbf{b}\}$. Suppose $v(p), v(q) \in D$. If $v(q) = \mathbf{f}$, then $D = {\uparrow}\mathbf{f}$, and supposing $v(\circ r) \notin D$, we have $v(r) \in D$. If $v(q) = \mathbf{b}$, we must have $v(p) = \mathbf{n}$. Then again $D = {\uparrow}\mathbf{f}$, for the same reason as before. The case of $v(q) = \mathbf{n}$ is analogous.

$\mathsf{r}_{D_\leq}$**:** If $v(\circ(p \Rightarrow q)) \in D$, $v(p) \geq a$ and $v(\circ q) < a$, then $v(\circ(p \Rightarrow q)) = \hat{\mathbf{t}}$.

$\mathsf{r}_{D\neq\uparrow\mathbf{t}}$**:** If $v({\downarrow}r) \notin D$, we have $v(r) = \mathbf{t}$. If $v({\uparrow}q) \in D$, we have $v(q) \neq \mathbf{f}$. If $v(\circ(p \Rightarrow q)) \notin D$, we have $v(p \Rightarrow q) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$. This gives $v(p) \in \{\mathbf{b}, \mathbf{n}, \mathbf{t}\}$ and $v(q) \in \{\mathbf{b}, \mathbf{n}, \mathbf{t}\}$. We only consider the cases $v(p) \neq v(q)$. If $v(p) = \mathbf{b}$, then $v(q) = \mathbf{n}$, and supposing $v(p), v(q) \notin D$, given that D is prime we must have $D = {\uparrow}\hat{\mathbf{t}}$, and $v(r) \notin D$. If $v(p) = \mathbf{n}$, the proof is similar to the previous case. If $v(p) = \mathbf{t}$, again we have $D = {\uparrow}\hat{\mathbf{t}}$, and we are done.

For item (iii), a valuation v such that $v(r) = \mathbf{t}$, $v(p) = \mathbf{n}$ and $v(q) = \mathbf{b}$ shows that $\mathsf{r}_{D\neq\uparrow\mathbf{t}}$ is not sound when $D = {\uparrow}\mathbf{t}$. □

We are ready to define our SET-SET axiomatizations and prove the desired completeness results.

Definition 6.19. *From the previous definitions, let*

1. $\mathsf{R}_{\mathsf{up}} := \mathsf{R}_{\Diamond} \cup \mathsf{R}_D \cup \mathsf{R}_{\Rightarrow} \cup \mathsf{R}_{\circ} \cup \mathsf{R}_{\sim} \cup \mathsf{R}_{\wedge} \cup \mathsf{R}_{\vee} \cup \mathsf{R}_{\top\bot} \cup \{\mathsf{r}_{D_\wedge}, \mathsf{r}_{D_\leq}\}$;
2. $\mathsf{R}_{\leq} := \mathsf{R}_{\mathsf{up}} \cup \{\mathsf{r}_{D\neq\uparrow\mathbf{t}}\}$.

Theorem 6.20. R_{up} *axiomatizes* $\mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}$ *and* $\mathsf{R}_{\leq}$ *axiomatizes* $\mathcal{PP}_{\leq}^{\rhd,\Rightarrow_{\mathsf{H}}}$*. Moreover, both* R_{up} *and* $\mathsf{R}_{\leq}$ *are* Θ*-analytic calculi, for* $\Theta := \{p, \circ p, \circ(p \Rightarrow q), {\uparrow}p, {\downarrow}p\}$.

Proof. That $\rhd_{\mathsf{R}_{\mathsf{up}}} \subseteq \mathcal{PP}_{\mathsf{up}}^{\rhd,\Rightarrow_{\mathsf{H}}}$ and $\rhd_{\mathsf{R}_{\leq}} \subseteq \mathcal{PP}_{\leq}^{\rhd,\Rightarrow_{\mathsf{H}}}$ (i.e. soundness) follows easily using the previous propositions. To check completeness for $\mathsf{R} \in \{\mathsf{R}_{\mathsf{up}}, \mathsf{R}_{\leq}\}$ we need to show that from $\Phi \blacktriangleright_{\mathsf{R}} \Psi$ we can find a valuation over the algebra $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$ and a principal filter that testifies it. Let $\Lambda := \mathsf{sub}(\Phi \cup \Psi)$. Recall that from $\Phi \blacktriangleright_{\mathsf{R}} \Psi$, by cut for sets, there is a partition $(\Omega, \overline{\Omega})$ of $\Theta(\Lambda)$ such that $\Omega \blacktriangleright_{\mathsf{R}} \overline{\Omega}$. We will build a partial valuation f on Λ and show that for its extension to the whole language it holds that $f(\Lambda \cap \Omega) \subseteq D$ and $f(\Lambda \cap \overline{\Omega}) \subseteq \mathcal{V}_6 \backslash D$ for suitable D (either principal or prime depending on the case). Further, analyticity will follow from the fact that we will be using only instances of the rules in R with formulas in Λ, thus only formulas in $\Theta(\Lambda) := \{\psi^\sigma : \psi \in \Theta, \sigma : \{p, q\} \to \Lambda\}$ will appear along this proof.

Let

$$\begin{aligned}
\Lambda_{\hat{\mathbf{f}}} &:= \{\varphi \in \Lambda : \varphi \in \overline{\Omega}, \circ\varphi \in \Omega\} \\
\Lambda_{\hat{\mathbf{t}}} &:= \{\varphi \in \Lambda : \varphi, \circ\varphi \in \Omega\} \\
\Lambda_{\Diamond} &:= \{\varphi \in \Lambda : \circ\varphi \in \overline{\Omega}\} \\
\Lambda_{\mathbf{t}} &:= \{\varphi \in \Lambda_{\Diamond} : \downarrow\varphi \in \overline{\Omega}\} \\
\Lambda_{\mathbf{f}} &:= \{\varphi \in \Lambda_{\Diamond} : \uparrow\varphi \in \overline{\Omega}\} \\
\Lambda_{\mathsf{mid}} &:= \{\varphi \in \Lambda_{\Diamond} : \uparrow\varphi, \downarrow\varphi \in \Omega\}
\end{aligned}$$

In what follows, consider the relation $\equiv_{\Diamond} \subseteq \Lambda_{\Diamond} \times \Lambda_{\Diamond}$ given by $\varphi \equiv_{\Diamond} \psi$ if, and only if, $\circ(\varphi \Rightarrow \psi), \circ(\psi \Rightarrow \varphi) \in \Omega$.

Lemma 6.21. *From $\mathsf{R}^{\Diamond} \subseteq \mathsf{R}$, we obtain that*

1. $\Lambda_{\Diamond} = \Lambda_{\mathbf{f}} \cup \Lambda_{\mathbf{t}} \cup \Lambda_{\mathsf{mid}}$*;*
2. *the relation $\equiv_{\Diamond}$ is an equivalence relation that partitions $\Lambda_{\Diamond}$ into at most four equivalence classes. In particular, each set $\Lambda_{\mathbf{f}}$ and $\Lambda_{\mathbf{t}}$ consists of formulas corresponding to a single $\equiv_{\Diamond}$-class, and if $\Lambda_{\mathsf{mid}} \neq \emptyset$ then it is partitioned into exactly two $\equiv_{\Diamond}$-classes, which we name $\Lambda_{\mathbf{b}}$ and $\Lambda_{\mathbf{n}}$, such that, for $\varphi, \psi \in \Lambda_{\mathsf{mid}}$ and $a \in \{\mathbf{b}, \mathbf{n}\}$, $\varphi, \psi \in \Lambda_a$ if, and only if, $\circ(\varphi \Rightarrow \psi) \in \Omega$;*
3. *for $\varphi \in \Lambda_a$ and $\psi \in \Lambda_b$ with $a, b \in \{\mathbf{f}, \mathbf{n}, \mathbf{b}, \mathbf{t}\}$, $a \leq b$ if, and only if, $\circ(\varphi \Rightarrow \psi) \in \Omega$.*

Proof. Note that since $\Omega \cap \overline{\Omega} = \emptyset$ we have that $\Lambda_a \cap \Lambda_b = \emptyset$ for $a \neq b$ and $a, b \in \{\mathbf{f}, \mathbf{t}, \mathsf{mid}, \hat{\mathbf{t}}, \hat{\mathbf{f}}\}$. By $\mathsf{r}^{\sim}_{\uparrow\mathsf{or}\downarrow}$ we know that $\Lambda_{\Diamond} = \Lambda_{\mathbf{f}} \cup \Lambda_{\mathbf{t}} \cup \Lambda_{\mathsf{mid}}$, and this is item (1).

For all that follows, recall that $\circ\varphi \in \overline{\Omega}$ for any $\varphi \in \Lambda_{\Diamond}$, by definition of $\Lambda_{\Diamond}$. That $\equiv_{\Diamond}$ is an equivalence relation follows by the fact that the definition is symmetric and the presence of the rules r_{id} and $\mathsf{r}_{\mathsf{trans}}$.

We show now that formulas in $\Lambda_{\mathbf{t}}$ and $\Lambda_{\mathbf{f}}$ correspond to the same $\equiv_{\Diamond}$-class. If $\varphi, \psi \in \Lambda_{\mathbf{t}}$, then $\downarrow\varphi, \downarrow\psi \in \overline{\Omega}$ and by $\mathsf{r}^{\Diamond}_{\leq\mathbf{t}}$ we obtain that $\varphi \equiv_{\Diamond} \psi$. Similarly, if $\varphi, \psi \in \Lambda_{\mathbf{f}}$, then $\uparrow\varphi, \uparrow\psi \in \overline{\Omega}$ and by $\mathsf{r}^{\Diamond}_{\geq\mathbf{f}}$ we obtain that $\varphi \equiv_{\Diamond} \psi$.

We will show that there are subsets of Λ_{mid} corresponding to a partition of this set. Our candidates are just the classes $[\varphi]_{\equiv_{\Diamond}}$. They are clearly disjoint and their union yields Λ_{mid}, so it is enough to show that they are all subsets of Λ_{mid}. In fact, if $\psi \in [\varphi]_{\equiv_{\Diamond}}$, we have $\circ\psi \in \overline{\Omega}$ and $\circ(\varphi \Rightarrow \psi), \circ(\psi \Rightarrow \varphi) \in \Omega$. Then we obtain $\circ\psi \in \overline{\Omega}$ and $\uparrow\psi, \downarrow\psi \in \Omega$ by the rules $\mathsf{r}^{\Diamond}_{\mathsf{incclass}_1}$ and $\mathsf{r}^{\Diamond}_{\mathsf{incclass}_2}$. Moreover, for $\varphi, \psi \in \Lambda_{\mathsf{mid}}$, we have that $\circ(\varphi \Rightarrow \psi) \in \Omega$ if, and only if, $\varphi \equiv_{\Diamond} \psi$. The difficult direction is left-to-right, and it follows by $\mathsf{r}^{\Diamond}_{\mathsf{incclass}_3}$.

Still, Λ_{mid} might be partitioned into more than two $\equiv_{\Diamond}$-equivalence classes. We avoid this with the rule $\mathsf{r}^{\sim}_{\mathsf{just}_2}$, which, together with the fact proved in the previous paragraph, forbids the existence of more than two $\equiv_{\Diamond}$-equivalence classes in Λ_{mid}.

Finally, for item (3), let $\varphi \in \Lambda_a$ and $\psi \in \Lambda_b$ with $a, b \in \{\mathbf{f}, \mathbf{n}, \mathbf{b}, \mathbf{t}\}$. From left-to-right, suppose that $a \leq b$. We want to prove $\circ(\varphi \Rightarrow \psi) \in \Omega$. Note that we cannot have $a = \mathbf{b}$ and $b = \mathbf{n}$, nor $a = \mathbf{n}$ and $b = \mathbf{b}$. The only cases we need to consider, then, are:

1. If $a = b$, we already have that each Λ_c, with $c \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$ forms an $\equiv_{\Diamond}$-class, so $\circ(\varphi \Rightarrow \psi) \in \Omega$ by the definition of $\equiv_{\Diamond}$.
2. If $a = \mathbf{f}$, we obtain $\circ(\varphi \Rightarrow \psi) \in \Omega$ by $\mathsf{r}^{\Diamond}_{\geq\mathbf{f}}$.
3. If $b = \mathbf{t}$, we obtain $\circ(\varphi \Rightarrow \psi) \in \Omega$ by $\mathsf{r}^{\Diamond}_{\leq\mathbf{t}}$.

From right-to-left, work contrapositively. Suppose that $a \not\leq b$. We want to conclude that $\circ(\varphi \Rightarrow \psi) \in \overline{\Omega}$. We only need to consider the following cases:

1. If $a = \mathbf{t}$, we have $b < \mathbf{t}$. Thus $\downarrow\psi \in \Omega$. By $\mathsf{r}^{\Diamond}_{\mathsf{incclass}_2}$, we have $\circ(\varphi \Rightarrow \psi) \in \overline{\Omega}$.

2. If $a = \mathbf{b}$ or $a = \mathbf{n}$, and $b = \mathbf{f}$, we obtain $\circ(\varphi \Rightarrow \psi) \in \overline{\Omega}$ by $\mathsf{r}^{\Diamond}_{\mathsf{incclass}_1}$.

3. If $\{a, b\} = \{\mathbf{b}, \mathbf{n}\}$, we have that $\circ(\varphi \Rightarrow \psi), \circ(\psi \Rightarrow \varphi) \in \overline{\Omega}$ by the definition of the sets $\Lambda_{\mathbf{b}}$ and $\Lambda_{\mathbf{n}}$.

Note that in this proof the mentioned rules were instantiated only with formulas in Λ and all such instances have their formulas in $\Theta(\Lambda)$. □

Our candidate for partial valuation is $f : \Lambda \to \mathcal{V}_6$ given by $f(\varphi) := a$ if $\varphi \in \Lambda_a$. One can check without difficulty that this function is well defined (as a matter of fact, the sets Λ_a defined above are pairwise disjoint). We will prove that f is indeed the desired valuation using a succession of lemmas. Let us begin with proving that f is a partial homomorphism.

Lemma 6.22. *Suppose* $\mathsf{R}_{\Rightarrow}, \mathsf{R}_{\Diamond} \subseteq \mathsf{R}$. *If* $\{\varphi, \psi, \varphi \Rightarrow \psi\} \subseteq \Lambda$, *then* $f(\varphi \Rightarrow \psi) = f(\varphi) \Rightarrow_{\mathsf{H}} f(\psi)$.

Proof. By cases on the values of $f(\varphi)$ and $f(\psi)$:

1. If $f(\psi) = \hat{\mathbf{t}}$, we have $\circ\psi, \psi \in \Omega$. We want $f(\varphi \Rightarrow \psi) = \hat{\mathbf{t}}$, which follows from $\mathsf{r}^{\Rightarrow}_1$ and $\mathsf{r}^{\Rightarrow}_2$.

2. If $f(\varphi) = \hat{\mathbf{t}}$, we have $\circ\varphi, \varphi \in \Omega$ and proceed by cases on the value of $f(\psi)$. Note that we want to show that $f(\varphi \Rightarrow \psi) = f(\psi)$:
   (a) The case $f(\psi) = \hat{\mathbf{t}}$ was already covered.
   (b) If $f(\psi) = \hat{\mathbf{f}}$, it follows from $\mathsf{r}^{\Rightarrow}_1$ and $\mathsf{r}^{\Rightarrow}_3$.
   (c) If $f(\psi) = \mathbf{t}$, it follows from $\mathsf{r}^{\Rightarrow}_4$ and $\mathsf{r}^{\Rightarrow}_5$.
   (d) If $f(\psi) = \mathbf{f}$, it follows from $\mathsf{r}^{\Rightarrow}_4$ and $\mathsf{r}^{\Rightarrow}_6$.
   (e) If $f(\psi) \in \{\mathbf{b}, \mathbf{n}\}$, it follows by $\mathsf{r}^{\Rightarrow}_4$, $\mathsf{r}^{\Rightarrow}_7$ and $\mathsf{r}^{\Rightarrow}_8$ that $f(\varphi \Rightarrow \psi) \in \{\mathbf{b}, \mathbf{n}\}$. Then, by $\mathsf{r}^{\Rightarrow}_9$ and Lemma 6.21 we have $f(\varphi \Rightarrow \psi) = f(\psi)$.

3. If $f(\varphi) = \hat{\mathbf{f}}$, we have $\circ\varphi \in \Omega$ and $\varphi \in \overline{\Omega}$. We obtain that $f(\varphi \Rightarrow \psi) = \hat{\mathbf{t}}$ by $\mathsf{r}^{\Rightarrow}_{10}$ and $\mathsf{r}^{\Rightarrow}_{11}$.

4. If $f(\psi) = \hat{\mathbf{f}}$, the case $f(\varphi) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$ was already treated, so we consider $f(\varphi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$ and want to prove $f(\varphi \Rightarrow \psi) = \hat{\mathbf{f}}$. From the available information, we have that $\circ\psi \in \Omega$, $\psi \in \overline{\Omega}$, $\circ\varphi \in \overline{\Omega}$. Then for any of the possibilities for $f(\varphi)$, the result follows by $\mathsf{r}^{\Rightarrow}_1$ and $\mathsf{r}^{\Rightarrow}_{12}$.

5. In case $f(\varphi), f(\psi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$, we make intensive use of Lemma 6.21.
   (a) If $f(\varphi) \leq f(\psi)$, we want to show $f(\varphi \Rightarrow \psi) = \hat{\mathbf{t}}$. By Lemma 6.21, we already have $\circ(\varphi \Rightarrow \psi) \in \Omega$. The result then follows by $\mathsf{r}^{\Rightarrow}_{13}$.
   (b) If $f(\psi) \leq f(\varphi)$ and $f(\varphi) \nleq f(\psi)$, we have $\circ(\varphi \Rightarrow \psi) \in \overline{\Omega}$ and $\circ(\psi \Rightarrow \varphi) \in \Omega$ by Lemma 6.21, and consider the following subcases:
      i. If $f(\varphi) = \mathbf{t}$, we want $f(\varphi \Rightarrow \psi) = f(\psi)$, which follows by $\mathsf{r}^{\Rightarrow}_9$ and $\mathsf{r}^{\Rightarrow}_{14}$.
      ii. If $f(\varphi) \in \{\mathbf{n}, \mathbf{b}\}$ and $f(\psi) = \mathbf{f}$, we want $f(\varphi \Rightarrow \psi) = \mathbf{b}$. This follows from by $\mathsf{r}^{\Rightarrow}_{15}$, $\mathsf{r}^{\Rightarrow}_{16}$, $\mathsf{r}^{\Rightarrow}_{17}$ and $\mathsf{r}^{\Rightarrow}_{18}$. The first two force $f(\varphi \Rightarrow \psi) \in \{\mathbf{b}, \mathbf{n}\}$, while the third forces $f(\varphi) \neq f(\varphi \Rightarrow \psi)$.
   (c) Otherwise, $\{f(\varphi), f(\psi)\} = \{\mathbf{b}, \mathbf{n}\}$, thus we want $f(\varphi \Rightarrow \psi) = f(\psi)$, which is achieved by $\mathsf{r}^{\Rightarrow}_9$ and $\mathsf{r}^{\Rightarrow}_{19}$ together with Lemma 6.21.

Note that in this proof the mentioned rules were instantiated only with formulas in Λ and all such instances have their formulas in $\Theta(\Lambda)$. □

Lemma 6.23. *Suppose* $\mathsf{R}_{\sim}, \mathsf{R}_{\Diamond} \subseteq \mathsf{R}$. *If* $\{\varphi, {\sim}\varphi\} \subseteq \Lambda$, *then* $f({\sim}\varphi) = {\sim}^{\mathbf{PP}_6^{\Rightarrow_\mathsf{H}}}(f(\varphi))$.

Proof. By cases on the value of $f(\varphi)$, to show that $f({\sim}\varphi) = {\sim}^{\mathbf{PP}_6^{\Rightarrow_\mathsf{H}}}(f(\varphi))$:

1. If $f(\varphi) = \hat{\mathbf{t}}$, then $\varphi, \circ\varphi \in \Omega$, so ${\sim}\varphi \in \overline{\Omega}$ and $\circ{\sim}\varphi \in \Omega$ by $\mathsf{r}^{\sim}_2$ and $\mathsf{r}^{\sim}_3$.

2. If $f(\varphi) = \hat{\mathbf{f}}$, then $\circ\varphi \in \Omega$ and $\varphi \in \overline{\Omega}$, so $\sim\varphi \in \Omega$ and $\circ{\sim}\varphi \in \Omega$ by $\mathsf{r}^{\sim}_1$ and $\mathsf{r}^{\sim}_3$.

3. If $f(\varphi) = \mathbf{t}$, then $\circ\varphi \in \overline{\Omega}$ and $\downarrow\varphi \in \overline{\Omega}$. Thus, by $\mathsf{r}^{\sim}_4$ and $\mathsf{r}^{\sim}_5$ we have that $\circ{\sim}\varphi \in \overline{\Omega}$ and $\uparrow{\sim}\varphi \in \overline{\Omega}$.

4. If $f(\varphi) = \mathbf{f}$, then $\circ{\sim}\varphi \in \overline{\Omega}$ and $\uparrow\varphi \in \overline{\Omega}$. Thus by $\mathsf{r}^{\sim}_4$ and $\mathsf{r}^{\sim}_6$ we have $\circ{\sim}\varphi \in \overline{\Omega}$ and $\downarrow{\sim}\varphi \in \overline{\Omega}$.

5. If $f(\varphi) \in \{\mathbf{b}, \mathbf{n}\}$, then $\circ\varphi \in \overline{\Omega}$ and $\uparrow\varphi, \downarrow\varphi \in \Omega$. By $\mathsf{r}^{\sim}_4$, $\mathsf{r}^{\sim}_7$ and $\mathsf{r}^{\sim}_8$, we have $\circ{\sim}\varphi \in \overline{\Omega}$ and $\uparrow{\sim}\varphi, \downarrow{\sim}\varphi \in \Omega$. This gives us that $f({\sim}\varphi) \in \{\mathbf{b}, \mathbf{n}\}$. Also, since we have $\circ(\varphi \Rightarrow {\sim}\varphi) = \downarrow\varphi \in \Omega$, we must have $f({\sim}\varphi) = f(\varphi)$ by Lemma 6.21.

Note that in this proof the mentioned rules were instantiated only with formulas in Λ and all such instances have their formulas in $\Theta(\Lambda)$. ☐

Lemma 6.24. *Suppose* $\mathsf{R}_\circ \subseteq \mathsf{R}$. *If* $\{\varphi, \circ\varphi\} \subseteq \Lambda$, *then* $f(\circ\varphi) = \circ^{\mathbf{PP}_6^{\Rightarrow\mathsf{H}}}(f(\varphi))$.

Proof. Note that by r°_1 we always have $\circ\circ\varphi \in \Omega$. Hence, if $f(\varphi) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$, then $\circ\varphi \in \Omega$ and thus $f(\circ\varphi) = \hat{\mathbf{t}} = \circ^{\mathbf{PP}_6^{\Rightarrow\mathsf{H}}}(f(\varphi))$. If $f(\varphi) \notin \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$, then $\circ\varphi \in \overline{\Omega}$, thus $f(\circ\varphi) = \hat{\mathbf{f}} = \circ^{\mathbf{PP}_6^{\Rightarrow\mathsf{H}}}(f(\varphi))$. ☐

Lemma 6.25. *Suppose* $\mathsf{R}_\wedge, \mathsf{R}_\Diamond \subseteq \mathsf{R}$. *If* $\{\varphi, \psi, \varphi \wedge \psi\} \subseteq \Lambda$, *then* $f(\varphi \wedge \psi) = f(\varphi) \wedge^{\mathbf{PP}_6^{\Rightarrow\mathsf{H}}} f(\psi)$.

Proof. We proceed by cases on the values of $f(\varphi)$ and $f(\psi)$:

1. If $f(\varphi) = \hat{\mathbf{t}}$, we have $\circ\varphi, \varphi \in \Omega$. We want $f(\varphi \wedge \psi) = f(\psi)$.
   (a) If $f(\psi) = \hat{\mathbf{t}}$, we have $\circ\psi, \psi \in \Omega$. Then we have $\circ(\varphi \wedge \psi), \varphi \wedge \psi \in \Omega$ by $\mathsf{r}^{\wedge}_1$ and $\mathsf{r}^{\wedge}_2$.
   (b) If $f(\psi) = \hat{\mathbf{f}}$, we have $\circ\psi \in \Omega$ and $\psi \in \overline{\Omega}$. Then we have $\circ(\varphi \wedge \psi) \in \Omega$ and $\varphi \wedge \psi \in \overline{\Omega}$ by $\mathsf{r}^{\wedge}_1$ and $\mathsf{r}^{\wedge}_3$.
   (c) If $f(\psi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$, we have $\circ\psi \in \overline{\Omega}$ and that $f(\varphi \wedge \psi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$ by $\mathsf{r}^{\wedge}_4$. Then we have that $\circ(\psi \Rightarrow (\varphi \wedge \psi)), \circ((\varphi \wedge \psi) \Rightarrow \psi) \in \Omega$ by $\mathsf{r}^{\wedge}_5$ and $\mathsf{r}^{\wedge}_6$. This, together with Lemma 6.21, gives the desired result.

2. If $f(\psi) = \hat{\mathbf{t}}$, the argument is very similar to the previous one. The case $f(\varphi) = \hat{\mathbf{t}}$ was already covered.
   (a) If $f(\varphi) = \hat{\mathbf{f}}$, we have $\circ\varphi \in \Omega$ and $\varphi \in \overline{\Omega}$. Then we have $\circ(\varphi \wedge \psi) \in \Omega$ and $\varphi \wedge \psi \in \overline{\Omega}$ by $\mathsf{r}^{\wedge}_1$ and $\mathsf{r}^{\wedge}_7$.
   (b) If $f(\varphi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$, we have $\circ\varphi \in \overline{\Omega}$ and also that $f(\varphi \wedge \psi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$, in view of $\mathsf{r}^{\wedge}_8$. Then we have that $\circ(\varphi \Rightarrow (\varphi \wedge \psi)), \circ((\varphi \wedge \psi) \Rightarrow \varphi) \in \Omega$ by $\mathsf{r}^{\wedge}_9$ and $\mathsf{r}^{\wedge}_{10}$. Then, by Lemma 6.21, we obtain the desired result.

3. If $f(\varphi) = \hat{\mathbf{f}}$, we have $\circ\varphi \in \Omega$ and $\varphi \in \overline{\Omega}$. By $\mathsf{r}^{\wedge}_{11}$ and $\mathsf{r}^{\wedge}_7$, we have $\circ(\varphi \wedge \psi) \in \Omega$ and $\varphi \wedge \psi \in \overline{\Omega}$, as desired.

4. If $f(\psi) = \hat{\mathbf{f}}$, the argument is similar as above, and follows by $\mathsf{r}^{\wedge}_{12}$ and $\mathsf{r}^{\wedge}_3$.

5. If $f(\varphi), f(\psi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$, we have $\circ\varphi, \circ\psi \in \overline{\Omega}$. Thus we have $\circ(\varphi \wedge \psi) \in \overline{\Omega}$ by $\mathsf{r}^{\wedge}_{13}$ and thus $f(\varphi \wedge \psi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$, as desired. Consider then the following cases:
   (a) If $f(\varphi) \leq f(\psi)$, we want to obtain $f(\varphi \wedge \psi) = f(\varphi)$. By Lemma 6.21, it is enough to conclude $\circ(\varphi \Rightarrow (\varphi \wedge \psi)), \circ((\varphi \wedge \psi) \Rightarrow \varphi) \in \Omega$, which follows by $\mathsf{r}^{\wedge}_{10}$ and $\mathsf{r}^{\wedge}_{14}$.
   (b) If $f(\psi) \leq f(\varphi)$, we want to obtain $f(\varphi \wedge \psi) = f(\psi)$. Similarly as in the previous item, by Lemma 6.21, it is enough to conclude $\circ(\psi \Rightarrow (\varphi \wedge \psi)), \circ((\varphi \wedge \psi) \Rightarrow \psi) \in \Omega$, which follows by $\mathsf{r}^{\wedge}_{15}$ and $\mathsf{r}^{\wedge}_6$.
   (c) Otherwise, we have $f(\varphi), f(\psi) \in \{\mathbf{b}, \mathbf{n}\}$ and $f(\varphi) \neq f(\psi)$. Note that by Lemma 6.21 we obtain $\circ(\varphi \Rightarrow \psi) \in \overline{\Omega}$. We want to obtain here $f(\varphi \wedge \psi) = \mathbf{f}$, that is, we want to have $\uparrow(\varphi \vee \psi) \in \overline{\Omega}$, which is achieved by $\mathsf{r}^{\wedge}_{16}$.

Note that the mentioned rules were instantiated only with formulas in Λ and all such instances have their formulas in $\Theta(\Lambda)$. ☐

Lemma 6.26. *Suppose* $\mathsf{R}_{\vee}, \mathsf{R}_{\Diamond} \subseteq \mathsf{R}$. *If* $\{\varphi, \psi, \varphi \vee \psi\} \subseteq \Lambda$, *then* $f(\varphi \vee \psi) = f(\varphi) \vee^{\mathbf{PP}_6^{\Rightarrow\mathsf{H}}} f(\psi)$.

Proof. We proceed by cases on the values of $f(\varphi)$ and $f(\psi)$:

1. If $f(\varphi) = \hat{\mathbf{f}}$, we know that $\varphi \in \overline{\Omega}$ and $\circ\varphi \in \Omega$. We want to obtain that $f(\varphi \vee \psi) = f(\psi)$, since $f(\psi) = \hat{\mathbf{f}} \vee^{\mathbf{PP}_6^{\Rightarrow\mathsf{H}}} f(\psi)$.
   (a) If $f(\psi) = \hat{\mathbf{f}}$, the result follows by $\mathsf{r}_1^{\vee}$ and $\mathsf{r}_2^{\vee}$.
   (b) If $f(\psi) = \hat{\mathbf{t}}$, the result follows by $\mathsf{r}_1^{\vee}$ and $\mathsf{r}_3^{\vee}$.
   (c) If $f(\psi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$, we have $\circ\psi \in \overline{\Omega}$ and that $f(\varphi \vee \psi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$ by $\mathsf{r}_4^{\vee}$. Then we have that $\circ(\psi \Rightarrow (\varphi \vee \psi)), \circ((\varphi \vee \psi) \Rightarrow \psi) \in \Omega$ by $\mathsf{r}_5^{\vee}$ and $\mathsf{r}_6^{\vee}$, which, with Lemma 6.21, gives the desired result.
2. If $f(\psi) = \hat{\mathbf{f}}$, we know that $\psi \in \overline{\Omega}$ and $\circ\psi \in \Omega$. We want to obtain that $f(\varphi \vee \psi) = f(\varphi)$, since $f(\varphi) = f(\varphi) \vee^{\mathbf{PP}_6^{\Rightarrow\mathsf{H}}} \hat{\mathbf{f}}$.
   (a) The case $f(\varphi) = \hat{\mathbf{f}}$ was covered already.
   (b) If $f(\varphi) = \hat{\mathbf{t}}$, the result follows by $\mathsf{r}_1^{\vee}$ and $\mathsf{r}_7^{\vee}$.
   (c) If $f(\varphi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$, we have $\circ\varphi \in \overline{\Omega}$ and that $f(\varphi \vee \psi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$ in view of $\mathsf{r}_8^{\vee}$. So we have that $\circ(\varphi \Rightarrow (\varphi \vee \psi)), \circ((\varphi \vee \psi) \Rightarrow \varphi) \in \Omega$ by $\mathsf{r}_9^{\vee}$ and $\mathsf{r}_{10}^{\vee}$. By Lemma 6.21, then, we obtain the desired result.
3. If either $f(\varphi) = \hat{\mathbf{t}}$ or $f(\psi) = \hat{\mathbf{t}}$, we know that either $\circ\varphi, \varphi \in \Omega$ or $\circ\psi, \psi \in \Omega$. In any case, we obtain $\circ(\varphi \vee \psi), \varphi \vee \psi \in \Omega$ by $\mathsf{r}_{11}^{\vee}$, $\mathsf{r}_7^{\vee}$, $\mathsf{r}_{12}^{\vee}$ and $\mathsf{r}_3^{\vee}$, and therefore $f(\varphi \vee \psi) = \hat{\mathbf{t}}$, as desired.
4. If $f(\varphi), f(\psi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$, we know that $\circ\varphi, \circ\psi \in \overline{\Omega}$, and so $\circ(\varphi \vee \psi) \in \overline{\Omega}$, by $\mathsf{r}_{13}^{\vee}$, thus $f(\varphi \vee \psi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$. Consider now the following cases:
   (a) If $f(\varphi) \leq f(\psi)$, we want to obtain $f(\varphi \vee \psi) = f(\psi)$. By Lemma 6.21, we know that $\circ(\varphi \Rightarrow \psi) \in \Omega$. But then $\circ(\psi \Rightarrow (\varphi \vee \psi)), \circ((\varphi \vee \psi) \Rightarrow \psi) \in \Omega$, by $\mathsf{r}_5^{\vee}$ and $\mathsf{r}_{14}^{\vee}$, which gives us the desired result by Lemma 6.21 again.
   (b) If $f(\psi) \leq f(\varphi)$, we want to obtain $f(\varphi \vee \psi) = f(\varphi)$. By Lemma 6.21, we know that $\circ(\psi \Rightarrow \varphi) \in \Omega$. But then $\circ(\varphi \Rightarrow (\varphi \vee \psi)), \circ((\varphi \vee \psi) \Rightarrow \varphi) \in \Omega$, by $\mathsf{r}_9^{\vee}$ and $\mathsf{r}_{15}^{\vee}$, which gives us the desired result by Lemma 6.21 again.
   (c) Otherwise, we have $f(\varphi), f(\psi) \in \{\mathbf{b}, \mathbf{n}\}$ and $f(\varphi) \neq f(\psi)$. Note that by Lemma 6.21 we obtain $\circ(\varphi \Rightarrow \psi) \in \overline{\Omega}$. We want to obtain here $f(\varphi \vee \psi) = \mathbf{t}$, that is, we want to have $\downarrow(\varphi \vee \psi) \in \overline{\Omega}$, and this is achieved by $\mathsf{r}_{16}^{\vee}$.

Note that the mentioned rules were instantiated only with formulas in Λ and all such instances have their formulas in $\Theta(\Lambda)$. $\square$

Lemma 6.27. *Suppose* $\mathsf{R}_{\top\bot} \subseteq \mathsf{R}$. *If* $\top \in \Lambda$, *then* $f(\top) = \hat{\mathbf{t}}$; *if* $\bot \in \Lambda$, *then* $f(\bot) = \hat{\mathbf{f}}$.

Proof. Obvious from the rules of $\mathsf{R}_{\top\bot}$ and the definition of f. $\square$

Lemma 6.28. *If* $\{\mathsf{r}_{D_\wedge}, \mathsf{r}_{D_\leq}\}, \mathsf{R}_{\Diamond} \subseteq \mathsf{R}$, *we have* $f[\Omega \cap \Lambda] = \uparrow a \cap f[\Lambda]$ *for some* $a > \hat{\mathbf{f}}$; *and if we also have* $\mathsf{r}_{D \neq \uparrow \mathbf{t}} \in \mathsf{R}$, *then* $f[\Omega \cap \Lambda] = \uparrow a \cap f[\Lambda]$ *for some* $\hat{\mathbf{f}} < a \neq \mathbf{t}$.

Proof. First of all, we show that, for $\varphi_1, \ldots, \varphi_n \in \Omega \cap \Lambda$, if $\bigwedge_i f(\varphi_i) \in f[\Lambda]$, then $\bigwedge_i f(\varphi_i) \in f[\Omega \cap \Lambda]$. By induction on n, consider the base case $n = 2$. The cases $f(\varphi) \leq f(\psi)$ and $f(\psi) \leq f(\varphi)$ are obvious, as $f(\varphi) \wedge f(\psi)$ will coincide either with $f(\varphi)$ or with $f(\psi)$ and they are in $f[\Omega \cap \Lambda]$ by assumption. The tricky case thus is when $\{f(\varphi), f(\psi)\} = \{\mathbf{b}, \mathbf{n}\}$. Suppose that for some $\theta \in \Lambda$ we have $f(\theta) = f(\varphi) \wedge f(\psi) = \mathbf{f}$. Then, by $\mathsf{r}_{D_\wedge}$, we must have $\theta \in \Omega$, and so $f(\theta) = \mathbf{f} \in f[\Omega \cap \Lambda]$. In the inductive step, suppose that $b := (f(\varphi_1) \wedge \ldots \wedge f(\varphi_n)) \wedge f(\varphi_{n+1}) \in f[\Lambda]$, for $\varphi_1, \ldots, \varphi_{n+1} \in \Omega \cap \Lambda$. Then either $b = (f(\varphi_1) \wedge \ldots \wedge f(\varphi_n))$, $b = f(\varphi_{n+1})$ or $b = \mathbf{f}$ and $\{f(\varphi_1) \wedge \ldots \wedge f(\varphi_n), f(\varphi_{n+1})\} = \{\mathbf{b}, \mathbf{n}\}$. In the first case, use the induction hypothesis. The second case is obvious. For the third, use $\mathsf{r}_{D_\wedge}$ as we did in the base case.

Second, we show that, for $\varphi \in \Omega \cap \Lambda$ and $\psi \in \Lambda$, if $f(\varphi) \leq f(\psi)$, then $\psi \in \Omega$. By cases on the value of $f(\varphi)$ (note that $f(\varphi) \neq \mathbf{f}$ as $\varphi \in \Omega$):

1. If $f(\varphi) = \hat{\mathbf{t}}$, we must have $f(\psi) = \hat{\mathbf{t}}$, thus $\psi \in \Omega$.
2. If $f(\varphi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$, we have either $f(\psi) = \hat{\mathbf{t}}$, and thus $\psi \in \Omega$, or $f(\psi) \in \{\mathbf{f}, \mathbf{b}, \mathbf{n}, \mathbf{t}\}$. In that case, by Lemma 6.21, we have $\circ(\varphi \Rightarrow \psi) \in \Omega$. Then $\psi \in \Omega$ follows by $\mathsf{r}_{D_{\leq}}$.

Let $a := \bigwedge f[\Omega \cap \Lambda]$. Clearly, $a \neq \hat{\mathbf{f}}$. Since $a \leq b$ for each $b \in f[\Omega \cap \Lambda]$, we must have $f[\Omega \cap \Lambda] \subseteq {\uparrow}a \cap f[\Lambda]$. It remains to show that ${\uparrow}a \cap f[\Lambda] \subseteq f[\Omega \cap \Lambda]$. Suppose that there is $\theta \in \overline{\Omega} \cap \Lambda$ such that (a): $a \leq f(\theta)$. By cases:

1. If $f(\theta) \leq f(\varphi)$ for all $\varphi \in \Omega \cap \Lambda$, then $f(\theta) = a$, and thus $\theta \in \Omega$ by the first claim we proved above.
2. If $f(\theta) \nleq f(\psi)$ for some $\psi \in \Omega$, we have the following subcases:
   (a) If $f(\psi) < f(\theta)$, by the second claim we proved above we must have $\theta \in \Omega \cap \Lambda$.
   (b) If $f(\psi) \nleq f(\theta)$, we have $f(\psi) = \mathbf{b}$ and $f(\theta) = \mathbf{n}$ or vice-versa. By (a), we have $a \in \{\mathbf{f}, \mathbf{n}\}$. The case $a = \mathbf{n}$ was treated in (1), so we only consider $a = \mathbf{f}$. This means that either $\mathbf{b}, \mathbf{n} \in f[\Omega \cap \Lambda]$ or $\mathbf{f} \in f[\Omega \cap \Lambda]$, and the result follows from the second claim proved above.

Now, in case the rule $\mathsf{r}_{D \neq \uparrow \mathbf{t}}$ is present in the calculus, we are able to show that if $\varphi, \psi \in \Omega \cap \Lambda$ and $f(\varphi) \vee f(\psi) \in f[\Omega \cap \Lambda]$, then either $f(\varphi) \in f[\Omega \cap \Lambda]$ or $f(\psi) \in f[\Omega \cap \Lambda]$, which essentially excludes the possibility of $a = \mathbf{t}$. Suppose that for $\theta \in \Lambda$, $f(\theta) = f(\varphi) \vee f(\psi) \in f[\Omega \cap \Lambda]$. Then either $f(\theta) = f(\varphi)$, or $f(\theta) = f(\psi)$ or $f(\theta) = \mathbf{t}$ and $\{f(\varphi), f(\psi)\} = \{\mathbf{b}, \mathbf{n}\}$. The first two cases are obvious. The third one follows because the rule $\mathsf{r}_{D \neq \uparrow \mathbf{t}}$ forces $\varphi \in \Omega$ in this situation.

Note that the mentioned rules were instantiated only with formulas in Λ and all such instances have their formulas in $\Theta(\Lambda)$. $\square$

The above lemmas show that f is a partial homomorphism over $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$ (which can of course be extended to a full homomorphism) and, in view of the above lemma, by considering a set of designated values of the form ${\uparrow}a$ for appropriate a we obtain a countermodel for $\Phi \rhd_{\mathcal{PP}_{\mathsf{up}}^{\rhd, \Rightarrow_{\mathsf{H}}}} \Psi$ or for $\Phi \rhd_{\mathcal{PP}_{\leq}^{\rhd, \Rightarrow_{\mathsf{H}}}} \Psi$ as desired. $\square$

6.2 SET-FMLA axiomatization for $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$

The SET-SET calculus developed in the preceding subsection for the logic $\mathcal{PP}_{\leq}^{\rhd, \Rightarrow_{\mathsf{H}}}$ induces a SET-FMLA logic for its SET-FMLA companion $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$. We begin by defining this calculus, then indicate why it is complete for $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$.

Definition 6.29. *Let* R *be a* SET-SET *calculus. We define* $\mathsf{R}^{\vee}$ *as the* SET-FMLA *calculus*

$$\left\{ \frac{p}{p \vee q}, \frac{p \vee q}{q \vee p}, \frac{p \vee (q \vee r)}{(p \vee q) \vee r} \right\} \cup \left\{ \mathsf{r}^{\vee} \mid \mathsf{r} \in \mathsf{R} \right\}$$

where $\mathsf{r}^{\vee}$ *is* $\frac{\varnothing}{\varphi}$ *if* $\mathsf{r} = \frac{\varnothing}{\varphi}$, $\frac{\Phi \vee p_0}{(\bigvee \Psi) \vee p_0}$ *if* $\mathsf{r} = \frac{\Phi}{\Psi}$, *and* $\frac{\Phi \vee p_0}{p_0}$ *if* $\mathsf{r} = \frac{\Phi}{\varnothing}$, *for* p_0 *is a propositional variable not occurring in the rules that belong to* R.

Using the above recipe is straightforward, and this gives the reason why we decided to not spell out the whole axiomatization here. Before introducing the completeness result, we define what it means for a SET-FMLA logic to have a disjunction. A SET-FMLA logic $\vdash$ over Σ *has a disjunction* provided that $\Phi, \varphi \vee \psi \vdash \phi$ if, and only if, $\Phi, \varphi \vdash \phi$ and $\Phi, \psi \vdash \phi$ (for $\vee$ a binary connective in Σ). The completeness result is immediate from the fact that $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ has a disjunction, in view of the following result:

Lemma 6.30 (Shoesmith and Smiley [1978, Thm. 5.37])**.** *Let* R *be a* SET-SET *calculus over a signature containing a binary connective* $\vee$*. If* $\vdash_{\mathsf{R}}$ *has a disjunction, then* $\vdash_{\mathsf{R}^{\vee}} = \vdash_{\mathsf{R}}$.

From this it immediately follows that:

Theorem 6.31. $\vdash_{\mathsf{R}_{\leq}^{\vee}} = \mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$.

7 Algebraic study of $\mathcal{PP}_{\leq}^{\Rightarrow_\mathsf{H}}$ and $\mathcal{PP}_{\top}^{\Rightarrow_\mathsf{H}}$

In this section we look at the class of algebras that corresponds to $\mathcal{PP}_{\leq}^{\Rightarrow_\mathsf{H}}$ and $\mathcal{PP}_{\top}^{\Rightarrow_\mathsf{H}}$ according to the general theory of algebraization of logics (Font [2016]). In order to facilitate the exposition, here we will write a Σ-algebra $\mathbf{A} := \langle A, \cdot^{\mathbf{A}} \rangle$ as $\langle A; ©_1^{\mathbf{A}}, \dots, ©_n^{\mathbf{A}} \rangle$, with $©_i \in \Sigma$ for each $1 \leq i \leq n$. We will further omit the superscript $\mathbf{A}$ from the interpretations of the connectives whenever there is no risk of confusion.

We begin by recalling that, as observed earlier, the algebra $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_\mathsf{H}}$ is a symmetric Heyting algebra in Monteiro's sense.

Definition 7.1 (Monteiro [1980, Def. 1.2, p. 61])**.** *A* symmetric Heyting algebra (SHA) *is a* $\Sigma_{\Rightarrow}^{\mathsf{DM}}$*-algebra* $\langle A; \wedge, \vee, \Rightarrow, {\sim}, \bot, \top \rangle$ *such that:*

(i) $\langle A; \wedge, \vee, \Rightarrow, \bot, \top \rangle$ *is a Heyting algebra.*

(ii) $\langle A; \wedge, \vee, {\sim}, \bot, \top \rangle$ *is a De Morgan algebra.*

As mentioned earlier, SHAs are alternatively known as *De Morgan-Heyting algebras* in the terminology introduced by Sankappanavar [1987]. The logical counterpart of SHAs is Moisil's 'symmetric modal logic', which is the expansion of the Hilbert-Bernays positive logic (the conjunction-disjunction-implication fragment of intuitionistic logic) by the addition of a De Morgan negation. One might expect Moisil's logic to be closely related to $\mathcal{PP}_{\leq}^{\Rightarrow_\mathsf{H}}$. In fact, as we shall see, the logic $\mathcal{PP}_{\top}^{\Rightarrow_\mathsf{H}}$ considered earlier (the $\top$-assertional companion of $\mathcal{PP}_{\leq}^{\Rightarrow_\mathsf{H}}$) may be viewed as an axiomatic extension of Moisil's logic; whereas we may obtain $\mathcal{PP}_{\leq}^{\Rightarrow_\mathsf{H}}$ from Moisil's logic provided we extend it by appropriate axioms but also drop the *contraposition* rule schema ($\mathsf{r}_{12}^{\mathsf{M}}$ below). The following is a Hilbert-style calculus for Moisil's logic (see Monteiro [1980, p. 60]):

$$\frac{}{p \Rightarrow (q \Rightarrow p)}\ \mathsf{r}_1^{\mathsf{M}} \qquad \frac{}{(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))}\ \mathsf{r}_2^{\mathsf{M}}$$

$$\frac{}{(p \wedge q) \Rightarrow p}\ \mathsf{r}_3^{\mathsf{M}} \qquad \frac{}{(p \wedge q) \Rightarrow q}\ \mathsf{r}_4^{\mathsf{M}} \qquad \frac{}{(p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow (p \Rightarrow (q \wedge r)))}\ \mathsf{r}_5^{\mathsf{M}}$$

$$\frac{}{p \Rightarrow (p \vee q)}\ \mathsf{r}_6^{\mathsf{M}} \qquad \frac{}{q \Rightarrow (p \vee q)}\ \mathsf{r}_7^{\mathsf{M}} \qquad \frac{}{(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow ((p \vee q) \Rightarrow r))}\ \mathsf{r}_8^{\mathsf{M}}$$

$$\frac{}{p \Rightarrow {\sim}{\sim}p}\ \mathsf{r}_9^{\mathsf{M}} \qquad \frac{}{{\sim}{\sim}p \Rightarrow p}\ \mathsf{r}_{10}^{\mathsf{M}}$$

$$\frac{p, p \Rightarrow q}{q}\ \mathsf{r}_{11}^{\mathsf{M}} \qquad \frac{p \Rightarrow q}{{\sim}q \Rightarrow {\sim}p}\ \mathsf{r}_{12}^{\mathsf{M}}$$

The Lindenbaum-Tarski algebras of Moisil's logic are precisely the symmetric Heyting algebras (see Monteiro [1980, Thm. 2.3, p. 62]). Using this result, it is easy to obtain the following:

Proposition 7.2. *Moisil's logic is algebraizable (in the sense of Blok and Pigozzi [1989]) with the same translations as positive logic (namely, equivalence formulas* $\{x \Rightarrow y, y \Rightarrow x\}$ *and defining equation* $x \approx \top$*). Its equivalent algebraic semantics is the variety of symmetric Heyting algebras.*

Having verified that $\mathcal{PP}_{\top}^{\Rightarrow_\mathsf{H}}$ is an axiomatic extension of Moisil's logic (see our axiomatization below), we will immediately obtain that $\mathcal{PP}_{\top}^{\Rightarrow_\mathsf{H}}$ is algebraizable with the translations mentioned in the preceding proposition; the equivalent algebraic semantics of $\mathcal{PP}_{\top}^{\Rightarrow_\mathsf{H}}$ is then bound to be a sub(quasi)variety of SHAs. To obtain $\mathcal{PP}_{\top}^{\Rightarrow_\mathsf{H}}$ from the above axiomatizations of Moisil's logic, we need to add the following axioms (we use $\neg p$ as an abbreviation of $p \Rightarrow {\sim}(p \Rightarrow p)$ and $\circ p$ as an abbreviation of $\neg p \vee \neg{\sim}p$)):

$$\frac{}{\neg p \Rightarrow {\sim}\neg\neg p}\ \mathsf{r}_1^{\top} \qquad \frac{}{{\sim}\neg\neg p \Rightarrow \neg p}\ \mathsf{r}_2^{\top}$$

$$\frac{}{(\circ(p_1 \Rightarrow p_2) \wedge \circ(p_2 \Rightarrow p_3)) \Rightarrow (\circ p_1 \vee \circ p_4 \vee \circ(p_4 \Rightarrow p_3) \vee \circ(p_3 \Rightarrow p_2) \vee \circ(p_2 \Rightarrow p_1))}\ \mathsf{r}_3^{\top}$$

This axiomatization should be compared with the equational presentation given in Definition 7.3, and the claimed completeness of the axiomatization will follow from Theorems 7.4 and 7.5. Definition 7.3(2) matches $\mathsf{r}_3^{\top}$. Definition 7.3(1) says that the algebra has a PP-algebra reduct: as observed in Marcelino and Rivieccio [2022, p. 3150], for an algebra that has a pseudo-complement negation (as all symmetric Heyting algebras do), it is sufficient to impose the equation ${\sim}\neg\neg x \approx \neg x$ to obtain an involutive Stone algebra (i.e., modulo the language, a PP-algebra); clearly the equation ${\sim}\neg\neg x \approx \neg x$ corresponds, via algebraizability, to $\mathsf{r}_1^{\top}$ and $\mathsf{r}_2^{\top}$.

One can also show that $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ may be obtained from the preceding axiomatization by taking as axioms all the valid formulas while dropping the contraposition rule of Moisil's logic (see, e.g., Bou et al. [2011, p.11]).

We proceed to obtain further information on the subclass of symmetric Heyting algebras that are models of $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ and $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$. Monteiro [1980] carried out an extensive study of symmetric Heyting (and related) algebras; independently, some of Monteiro's results were rediscovered and a number of new ones obtained in Sankappanavar [1987]. From these works we shall recall only the few lemmas needed for our purposes.

The following example is of special relevance to us because, as we shall see, the symmetric Heyting algebras we are mostly interested in have the shape described therein:

Example 7 (Sankappanavar [1987], p. 568)**.** *Let* $\mathbf{D}$ *be a finite De Morgan algebra and let* $\mathbf{C}_n^+$ *and* $\mathbf{C}_n^-$ *be two n-element chains, which we view as lattices. Denoting* $C_n^+ := \{c_1, \ldots, c_n\}$*, with* $c_1 < \ldots < c_n$*, let* $C_n^- := \{\sim c_1, \ldots, \sim c_n\}$*, with* $\sim c_n < \ldots < \sim c_1$*. Consider the ordinal sum of these lattices,* $\mathbf{D}^{\mathbf{C}} := \mathbf{C}_n^- \oplus \mathbf{D} \oplus \mathbf{C}_n^+$*, in which the order and the De Morgan negation are defined as follows: for all* $d \in D$ *and* $c_i \in C_n^+$*, we let* $\sim c_i < d < c_i$ *and* $\sim^{\mathbf{D}^{\mathbf{C}}} d = \sim^{\mathbf{D}} d$*. Then,* $\mathbf{D}^{\mathbf{C}}$ *is a finite De Morgan algebra, and can therefore be endowed with the Heyting implication determined by the order, turning it into a symmetric Heyting algebra.*

We now proceed to the axiomatization of $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$. First, we define an equational class $\mathbb{PP}^{\Rightarrow_{\mathsf{H}}}$, then we show that it coincides with $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$.

Definition 7.3. *Let* $\mathbf{A} := \langle A; \wedge, \vee, \Rightarrow, \sim, \circ, \bot, \top\rangle$ *be a symmetric Heyting algebra expanded with an operation* $\circ$*. We say that* $\mathbf{A}$ *is in the class* $\mathbb{PP}^{\Rightarrow_{\mathsf{H}}}$ *if the following conditions are satisfied:*

(i) The reduct $\langle A; \wedge, \vee, \circ, \sim, \bot, \top\rangle$ *is a PP-algebra.*

(ii) $\mathbf{A}$ *satisfies the following equation:*

$$\circ(x_1 \Rightarrow x_2) \wedge \circ(x_2 \Rightarrow x_3) \leq \circ x_1 \vee \circ x_4 \vee \circ(x_4 \Rightarrow x_3) \vee \circ(x_3 \Rightarrow x_2) \vee \circ(x_2 \Rightarrow x_1).$$

Our next aim is to check the following results:

Theorem 7.4. $\mathbb{PP}^{\Rightarrow_{\mathsf{H}}}$ *is the variety (and the quasi-variety) generated by* $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$.

Theorem 7.5. $\mathbb{PP}^{\Rightarrow_{\mathsf{H}}}$ *is the equivalent algebraic semantics of the algebraizable logic* $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$.

We shall prove the above by relying on a few lemmas which also have an independent interest, in that they throw some light on the structures of symmetric Heyting algebras in general and of algebras in $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$ in particular.

Adopting Monteiro's notation, we use $\Delta x := \neg{\sim}x$, where $\neg x := x \Rightarrow {\sim}(x \Rightarrow x)$ as abbreviations (recall that, on every algebra having a PP-algebra reduct (Gomes et al. [2022]), one may take $\circ x := \neg x \vee \Delta x$ and $\Delta x := x \wedge \circ x$). In $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$, we have:

$$\neg^{\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}}(a) = \begin{cases} \hat{\mathbf{t}} & \text{if } a = \hat{\mathbf{f}}, \\ \hat{\mathbf{f}} & \text{otherwise.} \end{cases} \quad \text{and} \quad \Delta^{\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}}(a) = \begin{cases} \hat{\mathbf{t}} & \text{if } a = \hat{\mathbf{t}}, \\ \hat{\mathbf{f}} & \text{otherwise.} \end{cases}$$

Given a SHA $\mathbf{A}$ and a subset $F \subseteq A$, we shall say that F is a *filter* if it is a non-empty lattice filter of the bounded lattice reduct of $\mathbf{A}$. A filter F will be called *regular* if $\Delta a \in F$ whenever $a \in F$, for all $a \in A$. For example, the algebra $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$ has only one proper regular filter, namely the singleton $\{\hat{\mathbf{t}}\}$. On every SHA, the regular filters form a closure system (hence, a complete lattice) and may be characterized in a number of alternative ways (see e.g. Monteiro [1980, Thm. 4.3, p. 75, and Thm. 4.11, p. 80]). In particular, it can be shown that regular filters coincide with the lattice filters F that further satisfy the contraposition rule, that is: ${\sim}b \Rightarrow {\sim}a \in F$ whenever $a \Rightarrow b \in F$. By the algebraizability of Moisil's logic, this observation yields the following result, which Sankappanavar proved in a more general (and purely algebraic) context:

Lemma 7.6 (Sankappanavar [1987], Thm. 3.3)**.** *The lattice of congruences of each SHA* $\mathbf{A}$ *is isomorphic to the lattice of regular filters on* $\mathbf{A}$.

We now focus on a subvariety of SHAs (to which $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$ obviously belongs) where regular filter generation admits a particularly simple description. As we will see, the following equation will play a key role:

$$\Delta\Delta x \approx \Delta x \tag{Δ-idemp}$$

Lemma 7.7 (Monteiro [1980, Thm. 4.17, p. 82])**.** *Let* $\mathbf{A}$ *be a SHA that satisfies* (Δ-idemp) *and let* $B \subseteq A$*. The regular filter* $F(B)$ *generated by* B *is given by:*

$$F(B) := \{a \in A : \Delta(a_1 \wedge \ldots \wedge a_n) \leq a \text{ for some } a_1, \ldots, a_n \in B\}.$$

Lemma 7.8 (Sankappanavar [1987, Cor. 4.8])**.** *Let* $\mathbf{A}$ *be a SHA that satisfies* (Δ-idemp)*. The following are equivalent:*

(i) $\mathbf{A}$ *is directly indecomposable.*

(ii) $\mathbf{A}$ *is subdirectly irreducible.*

(iii) $\mathbf{A}$ *is simple.*

The subvariety of SHAs defined by the equation (Δ-idemp) is dubbed $\mathbb{SDH}_1$ by Sankappanavar [1987], who observes that it is a discriminator[2] variety (see Burris and Sankappanavar [2011, Def. IV.9.3]). This entails that $\mathbb{PP}^{\Rightarrow_{\mathsf{H}}}$ is also a discriminator variety. The discriminator term for $\mathbb{PP}^{\Rightarrow_{\mathsf{H}}}$ is the following:

$$t(x, y, z) := (\Delta(x \Leftrightarrow y) \wedge z) \vee ({\sim}\Delta(x \Leftrightarrow y) \wedge x)$$

where $x \Leftrightarrow y := (x \Rightarrow y) \wedge (y \Rightarrow x)$. To see this, consider Theorem 7.4 and note that on $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$ we have, for all $c, d \in \mathcal{V}_6$,

$$\Delta^{\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}}(c \Leftrightarrow d) = \begin{cases} \hat{\mathbf{t}} & \text{if } c = d, \\ \hat{\mathbf{f}} & \text{otherwise.} \end{cases}$$

Recall that every variety of algebras is generated by its subdirectly irreducible members, which in our case (since $\mathbb{PP}^{\Rightarrow_{\mathsf{H}}} \subseteq \mathbb{SDH}_1$) are simple, by Lemma 7.8. By Lemma 7.6, this means that every such algebra $\mathbf{A}$ with a top element $\top$ has (as we have seen of $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$) a unique non-trivial regular filter, namely the singleton $\{\top\}$. The following is now the main lemma we need:

Lemma 7.9. *Every simple algebra* $\mathbf{A} \in \mathbb{PP}^{\Rightarrow_{\mathsf{H}}}$ *is (isomorphic to) a subalgebra of* $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$.

Proof. In the proof we shall use the equations $\Delta x \leq x$ and $\Delta(x \vee y) = \Delta x \vee \Delta y$, which can be easily verified in $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$. If $\mathbf{A}$ is simple, then $\mathbf{A}$ has only one non-trivial congruence, corresponding to the regular filter $\{\top\}$ — which in this case must be prime, as we now argue. In fact, since $\mathbf{A}$ satisfies $\Delta\Delta x \approx \Delta x$, we have $\Delta a = \bot$ whenever $a \neq \top$. For, otherwise, the regular filter $F(a) = \{b \in A : \Delta a \leq b\}$ would be proper (Lemma 7.7). Now, assume $a_1 \vee a_2 = \top$ for some $a_1, a_2 \in A$. We have $\Delta(a_1 \vee a_2) = \Delta a_1 \vee \Delta a_2 = \top = \Delta\top$. Thus, if $a_1 \neq \top$, then $\Delta a_1 = \bot$ and $a_2 \geq \Delta a_2 = \bot \vee \Delta a_2 = \Delta a_1 \vee \Delta a_2 = \top$, so $a_2 = \top$. Thus $\{\top\}$ is a prime filter (entailing, since $\mathbf{A}$ has a De Morgan algebra reduct, that $\{\bot\}$ is a prime ideal); hence the last elements of $\mathbf{A}$ form a chain C^+, and the first elements of $\mathbf{A}$ form a chain C^-. This means that $\mathbf{A}$ has the shape described in Example 7, except that it need not be finite.

Let $D := A - \{\bot, \top\}$. We claim that any chain of elements in D must have length at most 3. To see this, notice that, for all $a \in A$, we have $a \in D$ if, and only if, $\circ a = \bot$. In fact, for $a \in D$, we have $\neg a = \bot = \Delta a$, so $\circ a = \neg a \vee \Delta a = \bot$. Notice also that, for $a_1, a_2 \in D$, we have $\circ(a_1 \Rightarrow a_2) = \bot$ if (and only if) $a_1 \nleq a_2$.

Now assume, by way of contradiction, that there are elements $a_1, a_2, a_3, a_4 \in D$ such that $a_1 < a_2 < a_3 < a_4$, forming a four-element chain. Then the inequality in the second item of Definition 7.3 would fail, for we would have:

$$\circ(a_1 \Rightarrow a_2) \wedge \circ(a_2 \Rightarrow a_3) = \circ\top \wedge \circ\top = \top \nleq \bot = \circ a_1 \vee \circ a_4 \vee \circ(a_4 \Rightarrow a_3) \vee \circ(a_3 \Rightarrow a_2) \vee \circ(a_2 \Rightarrow a_1).$$

Thus, all chains in D have at most three elements. Hence, as $\mathbf{A}$ is a distributive lattice, it is easy to verify that $\mathbf{A}$ must be finite, with a coatom (call it $\mathbf{t}$) and an atom $\mathbf{f}$. This easily entails that $\mathbf{A}$ must be isomorphic to one of the subalgebras of $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$. □

Theorem 7.4 is now an immediate corollary of Lemma 7.9: $\mathbb{PP}^{\Rightarrow_{\mathsf{H}}}$ is both the variety and the quasivariety generated by $\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}$ (for the latter statement, see e.g. Clark and Davey [1998, Thm. 1.3.6.ii]). We are now ready to prove the following:

Theorem 7.10. *The variety* $\mathbb{PP}^{\Rightarrow_{\mathsf{H}}}$ *is the equivalent algebraic semantics of the algebraizable logic* $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$.

[2] Note that this notion of discriminator variety is unrelated to the above-mentioned discriminator for a monadic matrix (see Theorem 4.3).

Proof. We know from the algebraizability of Moisil's logic (with respect to all SHAs) that $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ is algebraizable, with the same translations, with respect to a sub(quasi)variety of SHAs which is axiomatized by the equations that translate the new axioms. Recalling that the equation $\neg x \approx {\sim}\neg\neg x$ guarantees that a Heyting algebra has an involutive Stone algebra reduct (see Cignoli and Sagastume [1983, Remark 2.2]), is easy to verify that the translations of the new axioms are indeed equivalent to the equations introduced in Definition 7.3. $\square$

Another consequence of Lemma 7.9 which has a logical impact is the following. Recall that the universes of subalgebras of $\mathbf{PP_6}^{\Rightarrow_{\mathsf{H}}}$ are the same as those of $\mathbf{PP_6}$, minus the five-element chain. Let us denote the corresponding algebras by $\mathbf{PP_4}^{\Rightarrow_{\mathsf{H}}}, \mathbf{PP_3}^{\Rightarrow_{\mathsf{H}}}$ and $\mathbf{PP_2}^{\Rightarrow_{\mathsf{H}}}$.

Corollary 7.11. *The (proper, non-trivial) subvarieties of $\mathbb{PP}^{\Rightarrow_{\mathsf{H}}}$ are precisely the following (the axiomatizations are obtained by adding the mentioned equations to the axiomatization of $\mathbb{PP}^{\Rightarrow_{\mathsf{H}}}$):*

1. $\mathbb{V}(\mathbf{PP_4}^{\Rightarrow_{\mathsf{H}}})$*, axiomatized by the equation* $x \wedge {\sim}x \leq y \vee {\sim}y$*.*
2. $\mathbb{V}(\mathbf{PP_3}^{\Rightarrow_{\mathsf{H}}})$*, axiomatized by* $x \vee (x \Rightarrow (y \vee \neg y)) \approx \top$*.*
3. $\mathbb{V}(\mathbf{PP_2}^{\Rightarrow_{\mathsf{H}}})$*, axiomatized by* $x \vee {\sim}x \approx \top$*.*

Note that $\mathbb{V}(\mathbf{PP_4}^{\Rightarrow_{\mathsf{H}}})$ *and* $\mathbb{V}(\mathbf{PP_3}^{\Rightarrow_{\mathsf{H}}})$ *are incomparable, while* $\mathbb{V}(\mathbf{PP_2}^{\Rightarrow_{\mathsf{H}}})$*, which is (up a choice of language) just the variety of Boolean algebras, is included in both of them.*

Proof. All claims are established by easy computations. The main observation we need is that, by Jónsson's Lemma (see Burris and Sankappanavar [2011, Cor. IV.6.10]), for $\mathbf{A} \in \{\mathbf{PP_4}^{\Rightarrow_{\mathsf{H}}}, \mathbf{PP_3}^{\Rightarrow_{\mathsf{H}}}, \mathbf{PP_2}^{\Rightarrow_{\mathsf{H}}}\}$, the subdirectly irreducible (here meaning simple) algebras in each $\mathbb{V}(\mathbf{A})$ are in $\mathbb{HS}(\mathbf{A})$, and $\mathbb{HS}(\mathbf{A}) = \mathbb{S}(\mathbf{A})$. $\square$

By Theorem 7.10, the logic $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ is complete with respect to the class of all matrices $\langle \mathbf{A}, \{\top^{\mathbf{A}}\}\rangle$ such that $\mathbf{A} \in \mathbb{V}(\mathbf{PP_6}^{\Rightarrow_{\mathsf{H}}})$. But, by Theorem 7.4, we know that the single matrix $\langle \mathbf{PP_6}^{\Rightarrow_{\mathsf{H}}}, \{\hat{\mathbf{t}}\}\rangle$ suffices. Thus:

Proposition 7.12. $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ *is determined by* $\langle \mathbf{PP_6}^{\Rightarrow_{\mathsf{H}}}, \{\hat{\mathbf{t}}\}\rangle$*.*

This observation may be used to verify the following equivalence, which holds for all $\Phi \cup \{\varphi\} \subseteq L_{\Sigma}(P)$:

$$\Phi \vdash_{\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}} \varphi \quad \text{if, and only if,} \quad \Delta\Phi \vdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} \varphi$$

where $\Delta\Phi := \{\Delta\psi : \psi \in \Phi\}$. From the latter, relying on the DDT for $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$, we can obtain the DDT for $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$: for all $\Phi, \{\varphi, \psi\} \subseteq L_{\Sigma}(P)$,

$$\Phi, \varphi \vdash_{\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}} \psi \quad \text{iff} \quad \Phi \vdash_{\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}} \Delta\varphi \Rightarrow \psi.$$

The class of algebraic reducts of reduced matrices for $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ is also $\mathbb{V}(\mathbf{PP_6}^{\Rightarrow_{\mathsf{H}}})$. In fact, it can be shown that $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ is complete with respect to the class of all matrices $\langle \mathbf{A}, D\rangle$ with $\mathbf{A} \in \mathbb{V}(\mathbf{PP_6}^{\Rightarrow_{\mathsf{H}}})$ and D a non-empty lattice filter of $\mathbf{A}$. The reduced models of $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ may be characterized as follows:

Proposition 7.13. *Given* $\mathbf{A} \in \mathbb{V}(\mathbf{PP_6}^{\Rightarrow_{\mathsf{H}}})$*, we have that a matrix* $\langle \mathbf{A}, D\rangle$ *is a reduced model of* $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ *if, and only if,* D *is a lattice filter that contains exactly one regular filter (namely* $\{\top^{\mathbf{A}}\}$*).*

Proof. We shall prove both implications by contraposition. Assume first that $\langle \mathbf{A}, D\rangle$ is not reduced. Then, by the characterization of the Leibniz congruence given in Proposition 5.8, there are elements $a, b \in A$ such that $a \neq b$ and

$$\Delta(a \Rightarrow b), \Delta(b \Rightarrow a) \in D.$$

Assuming $a \nleq b$, we have $a \Rightarrow b \neq \top$. Recall that the regular filter generated by the element $a \Rightarrow b$ is

$$F(\{a \Rightarrow b\}) = \{c \in A : \Delta(a \Rightarrow b) \leq c\}.$$

From the assumption $\Delta(a \Rightarrow b) \in D$, we have $F(\{a \Rightarrow b\}) \subseteq D$. So D contains a regular filter distinct from $\{\top\}$.

Conversely, assume D contains a regular filter $F \neq \{\top\}$. Then there is $a \in F$ such that $a < \top$. This means that $\Delta(a \Rightarrow \top) = \Delta\top = \top \in F$ and (since F is regular) $\Delta(\top \Rightarrow a) = \Delta a \in F$. Then, by the characterization of Proposition 5.8, the pair $(a, \top)$ is identified by the Leibniz congruence of D, which would make $\langle \mathbf{A}, D\rangle$ not reduced. $\square$

8 On interpolation for $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ and $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$, and amalgamation for $\mathbb{PP}^{\Rightarrow_{\mathsf{H}}}$

We begin by defining three basic notions of interpolation according to the terminology of Czelakowski and Pigozzi [1998, Def. 3.1]. In this section we will focus only in SET-FMLA logics, since this is the framework in which the properties of interpolation are commonly formulated and investigated in the literature.

Definition 8.1. *A* SET-FMLA *Σ-logic $\vdash$ has the*

1. extension interpolation property *(EIP) if having $\Phi, \Psi \vdash \varphi$ implies that there is $\Pi \subseteq L_\Sigma(\mathsf{props}(\Psi \cup \{\varphi\}))$ such that $\Phi \vdash \psi$ for all $\psi \in \Pi$ and $\Pi, \Psi \vdash \varphi$.*
2. Craig interpolation property *(CIP) if, whenever $\mathsf{props}(\Phi) \cap \mathsf{props}(\varphi) \neq \varnothing$, having $\Phi \vdash \varphi$ implies that there is $\Pi \subseteq L_\Sigma(\mathsf{props}(\Phi) \cap \mathsf{props}(\varphi))$ such that $\Phi \vdash \psi$ for all $\psi \in \Pi$ and $\Pi \vdash \varphi$.*
3. Maehara interpolation property *(MIP) if, whenever $\mathsf{props}(\Phi) \cap \mathsf{props}(\Psi \cup \{\varphi\}) \neq \varnothing$, having $\Phi, \Psi \vdash \varphi$ implies that there is $\Pi \subseteq L_\Sigma(\mathsf{props}(\Phi) \cap \mathsf{props}(\Psi \cup \{\varphi\}))$ such that $\Phi \vdash \psi$ for all $\psi \in \Pi$ and $\Pi, \Psi \vdash \varphi$.*

Note that the (MIP) implies the (CIP) — just take $\Psi = \varnothing$. It also implies the (EIP) when the logic has theses (that is, formulas φ such that $\varnothing \vdash \varphi$) on a single variable and every formula without variables is logically equivalent to some constant in the signature. Note that for the logics $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ and $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ both conditions hold good.

Let us see now which of these interpolation properties the logic $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ satisfies. For what follows, we recall that this logic satisfies the *deduction-detachment theorem* (DDT) with respect to the connective $\Rightarrow$, that is, we have $\Phi, \varphi \vdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} \psi$ if, and only if, $\Phi \vdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} \varphi \Rightarrow \psi$.

Theorem 8.2. *The logic $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ has the (EIP) but does not have the (CIP) nor the (MIP).*

Proof. Since $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ is finitary, we may consider only finite candidates for Ψ and Π. To see that $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ has (EIP), assume that $\Phi, \Psi \vdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} \varphi$. The property is satisfied if we choose $\Pi := \{\bigwedge \Psi \Rightarrow \varphi\}$, since $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ has the DDT.

Given that (MIP) implies (CIP), it is enough to show that the latter fails. First of all, note that $(p \wedge {\sim}p \wedge q \wedge {\sim}q \wedge {\sim}{\circ}(p \Rightarrow q)) \vee s \vdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} (r \vee {\sim}r) \vee s$ (recall the proof of Theorem 5.6). Based on the definition of (CIP), let $\Phi := \{(p \wedge {\sim}p \wedge q \wedge {\sim}q \wedge {\sim}{\circ}(p \Rightarrow q)) \vee s\}$ and $\varphi := (r \vee {\sim}r) \vee s$. Note that $\mathsf{props}(\Phi) \cap \mathsf{props}(\varphi) = \{s\}$. Assume there is such Π, then $\psi(s) := \bigwedge \Pi$ is a formula with a single variable s. We will see now that we cannot have both (i) $\Phi \vdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} \psi(s)$ and (ii) $\psi(s) \vdash_{\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}} \varphi$. Let us fix $v(p) := \mathbf{b}$, $v(q) := \mathbf{n}$, $v(r) := \mathbf{b}$ and $v(s) := \hat{\mathbf{f}}$ thus making $v(\Phi) = \{\mathbf{f}\}$ and $v(\varphi) = \mathbf{b}$. Note that $v(\psi) \in \{\hat{\mathbf{f}}, \hat{\mathbf{t}}\}$. Hence, if $v(\psi) = \hat{\mathbf{f}}$, then (i) fails, and, if $v(\psi) = \hat{\mathbf{t}}$, (ii) fails. $\square$

Let us now take a look at the situation for $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$. Recall from Proposition 7.12 that this logic is determined by the single matrix $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, \{\hat{\mathbf{t}}\}\rangle$.

Theorem 8.3. *The logic $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ has the (EIP), the (CIP) and the (MIP).*

Proof. It is enough to show it has the (MIP). Since $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ is finitary, it suffices to consider finite sets. Given $\Phi, \Psi \vdash_{\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}} \varphi$, let $\mathsf{props}(\Phi) \cap \mathsf{props}(\Psi \cup \{\varphi\}) = \{p_1, \dots, p_k\} \neq \varnothing$. Consider $\mathcal{U} := \{v \in \mathsf{Hom}(L_\Sigma(\{p_1, \dots, p_k\}), \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}) \mid v(\Phi) = \hat{\mathbf{t}}\}$. For each $v \in \mathcal{U}$ and $1 \leq i \leq k$, let

$$\varphi_i^v(p) := \begin{cases} p \wedge {\circ}p & \text{if } v(p_i) = \hat{\mathbf{t}} \\ {\sim}{\downarrow}p & \text{if } v(p_i) = \mathbf{t} \\ {\uparrow}p \wedge {\downarrow}p \wedge {\sim}{\circ}p & \text{if } v(p_i) \in \{\mathbf{b}, \mathbf{n}\} \\ {\sim}{\uparrow}p & \text{if } v(p_i) = \mathbf{f} \\ {\sim}p \wedge {\circ}p & \text{if } v(p_i) = \hat{\mathbf{f}} \end{cases}$$

$$I_{\mathbf{b}}^v := \{i \,:\, v(p_i) = \mathbf{b}\}$$

$$J_{\mathbf{n}}^v := \{j \,:\, v(p_j) = \mathbf{n}\}$$

and

$$\psi_v := \bigwedge_{1\leq i\leq k} \varphi_i^v(p_i) \wedge \bigwedge_{i\in I^v_{\mathbf{b}}, j\in J^v_{\mathbf{n}}} {\sim}\circ(p_i \Rightarrow p_j)$$
$$\phi := \bigvee_{v\in\mathcal{U}} \psi_v$$

If we set $\equiv\, \subseteq \mathcal{V}_6 \times \mathcal{V}_6$ such that $a \equiv b$ iff $a = b$ or $\{a, b\} = \{\mathbf{b}, \mathbf{n}\}$ we obtain that, for all $v' \in \mathsf{Hom}(\mathbf{L}_\Sigma(P), \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_\mathsf{H}})$,

$$v'(\varphi_i^v(p)) = \begin{cases} \hat{\mathbf{t}} & \text{if } v'(p) \equiv v(p_i) \\ \hat{\mathbf{f}} & \text{otherwise.} \end{cases}$$

Thus, if $v'(\psi_v) = \hat{\mathbf{t}}$ then $v(p_i) \equiv v'(p_i)$ for every $1 \leq i \leq k$.

Let us show that $\phi \vdash_{\mathcal{PP}_\mathsf{T}^{\Rightarrow_\mathsf{H}}} \varphi$. For every $v' \in \mathsf{Hom}(\mathbf{L}_\Sigma(P), \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_\mathsf{H}})$ such that $v'(\phi) = \hat{\mathbf{t}}$, there is $v \in \mathcal{U}$ such that $v(p_i) \equiv v'(p_i)$ for every $1 \leq i \leq k$. Also, $\{\mathbf{b}, \mathbf{n}\} = \{v(p_i), v(p_j)\}$ if, and only if, $\{\mathbf{b}, \mathbf{n}\} = \{v'(p_i), v'(p_j)\}$, and therefore $v(\psi) = v'(\psi)$ for every ψ such that $\mathsf{props}(\psi) \subseteq \{p_1, \dots, p_k\}$. Thus, without loss of generality, we proceed considering that $v(p_i) = v'(p_i)$ for $1 \leq i \leq k$.

Considering $v'' \in \mathsf{Hom}(\mathbf{L}_\Sigma(P), \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_\mathsf{H}})$ such that

$$v''(p) := \begin{cases} v(p) & p \in P \backslash \mathsf{props}(\Phi) \\ v'(p) & p \in \mathsf{props}(\Phi) \backslash \{p_1, \dots, p_k\} \end{cases}$$

we have that $v''(\Phi) = v''(\Psi) = \hat{\mathbf{t}}$, thus from $\Phi, \Psi \vdash_{\mathcal{PP}_\mathsf{T}^{\Rightarrow_\mathsf{H}}} \varphi$ we conclude that $v''(\varphi) = v'(\psi) = \hat{\mathbf{t}}$ and therefore $\phi \vdash_{\mathcal{PP}_\mathsf{T}^{\Rightarrow_\mathsf{H}}} \varphi$. Finally, to see that $\Phi \vdash_{\mathcal{PP}_\mathsf{T}^{\Rightarrow_\mathsf{H}}} \phi$, note that if $v(\Phi) = \{\hat{\mathbf{t}}\}$ then, by definition, $v \in \mathcal{U}$ and therefore $v(\psi_v) = v(\phi) = \hat{\mathbf{t}}$. □

Remark 8.4. *Recall that, in the previous section, we presented the DDT for $\mathcal{PP}_\mathsf{T}^{\Rightarrow_\mathsf{H}}$, which demands a different (derived) connective to play the role of $\Rightarrow$. This could also have been used to prove the (EIP) for this logic.*

When a SET-FMLA logic satisfies some of the above interpolation properties and we know it is algebraizable, the class of algebras corresponding to the equivalent algebraic semantics satisfies so-called 'amalgamation properties'. We will focus here on two of such properties, the *Maehara amalgamation property* (MAP) and the *flat amalgamation property* (FAP), as they are the ones corresponding respectively to the interpolation properties (MIP) and (CIP) entertained above.

We need some preliminary concepts before formulating such properties. Let K be a class of Σ-algebras and $\mathbf{A}, \mathbf{B} \in \mathsf{K}$. If $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $S \subseteq B$, $\mathbf{B}$ is said to be a K-*free extension of* $\mathbf{A}$ *over* S if for every $\mathbf{C} \in \mathsf{K}$, every homomorphism $h : \mathbf{A} \to \mathbf{C}$ and every $f : S \to C$, there is a unique homomorphism $g : \mathbf{B} \to \mathbf{C}$ such that $g \upharpoonright A = h$ and $g \upharpoonright S = f$, where $\upharpoonright$ denotes domain restriction. Write $f : \mathbf{A} \rightarrowtail \mathbf{B}$ to denote that f is an injective homomorphism, and $f : \mathbf{A} \hookrightarrow \mathbf{B}$ to denote that f is a *free injection over* K, meaning that f is an injection and $f(\mathbf{B})$ is a K-*free extension* of $\mathbf{B}$ over some set S of elements.

Definition 8.5 (Czelakowski and Pigozzi [1998], Def. 5.2)**.** *A class of algebras* K *has the*

1. Maehara amalgamation property *(MAP) when, for all* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{K}$*, and all homomorphisms* $f : \mathbf{C} \rightarrowtail \mathbf{A}$ *and* $g : \mathbf{C} \to \mathbf{B}$*, there exists* $\mathbf{D} \in \mathsf{K}$ *and homomorphisms* $h : \mathbf{A} \to \mathbf{D}$ *and* $k : \mathbf{B} \rightarrowtail \mathbf{D}$ *such that* $hf = kg$*.*
2. flat amalgamation property *(FAP) when for all* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{K}$*, and all homomorphisms* $f : \mathbf{C} \rightarrowtail \mathbf{A}$ *and* $g : \mathbf{C} \hookrightarrow \mathbf{B}$*, there exists* $\mathbf{D} \in \mathsf{K}$ *and homomorphisms* $h : \mathbf{A} \hookrightarrow \mathbf{D}$ *and* $k : \mathbf{B} \rightarrowtail \mathbf{D}$ *such that* $hf = kg$*.*

Theorem 8.6. $\mathbb{PP}^{\Rightarrow_\mathsf{H}}$ *has the (MAP) and the (FAP).*

Proof. By Theorem 7.5, we know that $\mathbb{PP}^{\Rightarrow_\mathsf{H}}$ is the equivalent algebraic semantics of $\mathcal{PP}_\mathsf{T}^{\Rightarrow_\mathsf{H}}$. We just saw that this logic satisfies (MIP) and (CIP), thus, by [Czelakowski and Pigozzi, 1998, Cor. 5.27 and Cor. 5.29], $\mathbb{PP}^{\Rightarrow_\mathsf{H}}$ satisfies (MAP) and (FAP). □

9 Conclusions and future work

In the present paper we have initiated the study of implicative expansions of logics of perfect paradefinite algebras by considering classic-like and Heyting implications, both in the SET-SET and in the SET-FMLA framework. We investigated semantical characterizations (via classes of algebras and logical matrices) as well as proof-theoretical ones (via SET-SET and SET-FMLA Hilbert-style calculi) of these extensions. For the extensions with a Heyting implication, in particular, the new connective introduced many challenges for not allowing the logic to be characterized by any single logical matrix; we proved, however, that it can be characterized by a single finite PNmatrix. Over such extensions we also studied properties of interpolation and amalgamation for the corresponding algebraic models. We indicate below a few directions that we believe could prove worthwhile pursuing in future research.

Alternative expansions of the logics of PP-algebras. In Section 4 we have briefly considered other logics that may be obtained by conservatively adding an implication to the logics of PP-algebras $\mathcal{PP}_{\leq}^{\triangleright}$ and $\mathcal{PP}_{\leq}$. We did not explore these alternatives much further, preferring instead (from Section 5 onward) to focus our attention on logics having a more straightforward connection to existing frameworks (Moisil's logic and symmetric Heyting algebras). Nevertheless, we feel that such alternative systems may deserve further study, and the connection we noted with the recent work by Coniglio and Rodrigues [2023] provides further motivation for this project. Additionally, it would be interesting to systematically investigate the logics determined by refinements of the PNmatrix $\mathfrak{M}_{\mathsf{up}}$, providing axiomatizations, classifying them within the hierarchies of algebraic logic (e.g., which among them are algebraizable?) and studying the corresponding classes of algebras as done in Section 7 for $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$. Lastly, further enlarging the scope, one might ask whether condition (A1) could be relaxed, perhaps requiring the new implication to be axiomatized by the usual inference rules for Heyting implications (rather than classic-like implications, which correspond to (A1), as we have seen). In this way we could circumvent the limitation of Theorem 4.1 exploring alternative self-extensional expansions of the logics of PP-algebras.

Extensions of $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$. Our preliminary investigations suggest that the landscape of extensions of the base logics considered in the present paper is quite interesting and complex. Concerning the *finitary* SET-FMLA extensions of $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ we may affirm the following:

- By [Jansana, 2006, Thm. 3.7], the finitary SET-FMLA self-extensional extensions of $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ are in one-to-one correspondence with the subvarieties of $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$. Thus, by Corollary 7.11, there are only three of them, all of them axiomatic (none being conservative expansions over $\mathcal{PP}_{\leq}$). These logics may be axiomatized, relatively to $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$, by adding the axioms corresponding to the equations in Corollary 7.11, to wit:
  1. the logic of $\mathbb{V}(\mathbf{PP}_{\mathbf{4}}^{\Rightarrow_{\mathsf{H}}})$ is axiomatized by $(p \wedge {\sim}p) \Rightarrow (q \vee {\sim}q)$;
  2. the logic of $\mathbb{V}(\mathbf{PP}_{\mathbf{3}}^{\Rightarrow_{\mathsf{H}}})$ by $p \vee (p \Rightarrow (q \vee \neg q))$; and
  3. the logic of $\mathbb{V}(\mathbf{PP}_{\mathbf{2}}^{\Rightarrow_{\mathsf{H}}})$, which is (up to the choice of language) just classical logic, by $p \vee {\sim}p$.
- The number of all the axiomatic SET-FMLA extensions of $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ (which are obviously finitary) is larger but also finite, for each axiomatic extension may be be characterized by the submatrices of the original matrices that satisfy the axioms[3].
- As we have seen, the assertional logic $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ is itself a (non-axiomatic, non-self-extensional) extension of $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$. Algebraizability of $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ (Theorem 7.5) entails that its axiomatic SET-FMLA extensions are in one-to-one correspondence with the subvarieties of $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$, so again we can conclude that there are only three of them.
- In contrast to the preceeding results, we conjecture that it may be possible to construct countably many distinct (non-axiomatic) SET-FMLA finitary extensions of $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$. By algebraizability, this would tell us that the variety $\mathbb{PP}^{\Rightarrow_{\mathsf{H}}}$ has at least countably many sub-quasivarieties.

Now considering the SET-SET extensions of $\mathcal{PP}_{\mathsf{up}}^{\triangleright,\Rightarrow_{\mathsf{H}}}$, we can say the following:

[3]This result can be easily established using [Caleiro and Marcelino, 2021, Prop. 3.1] together with the observation that the total components of the PNmatrix $\mathfrak{M}_{\mathsf{up}}$ are deterministic; thus, the submatrices of $\mathfrak{M}_{\mathsf{up}}$ which are images of valuations that satisfy every instance of an axiom are sound with respect to that particular axiom.

- Similarly to the SET-FMLA setting, all *axiomatic* SET-SET extensions of $\mathcal{PP}_{\mathsf{up}}^{\triangleright,\Rightarrow_{\mathsf{H}}}$ are characterized by the sets of (sub)matrices in $\{\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, {\uparrow}a\rangle \mid a \in \mathcal{V}\}$ that satisfy the corresponding axioms, hence their cardinality is also finite. Their SET-FMLA companions are obtained by adding the same axioms to $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$.
- In consequence, there are *at least* three self-extensional SET-SET extensions of $\mathcal{PP}_{\mathsf{up}}^{\triangleright,\Rightarrow_{\mathsf{H}}}$, namely, the ones determined by the matrices which satisfy the corresponding axioms. (There may be more, but each of them will have as SET-FMLA fragment one of the three self-extensional SET-FMLA extensions of $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ mentioned earlier.)
- The methods used in the previous sections may be used to obtain analytic SET-SET axiomatizations for all the logics determined by sets of (sub)matrices in $\{\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, {\uparrow}a\rangle \mid a \in \mathcal{V}\}$.

The preceding considerations suggest that a complete description of the lattice of all extensions of our base logics is well beyond the scope of the present work, and will have to be pursued in further research. We summarize such research path in terms of the following couple of problems:

Problem 1. *Describe the lattice of all (*SET-FMLA*) extensions of* $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ *and the lattice of all (*SET-SET*) extensions of* $\mathcal{PP}_{\mathsf{up}}^{\triangleright,\Rightarrow_{\mathsf{H}}}$.

Problem 2. *Same as the preceding one, but restricting one's attention to the sublattices consisting of all the* SET-FMLA *(*SET-SET*) extensions of* $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$ *(of its* SET-SET *companion determined by the matrix* $\langle \mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}}, {\uparrow}\hat{\mathbf{t}}\rangle$*). Due to the algebraizability of* $\mathcal{PP}_{\top}^{\Rightarrow_{\mathsf{H}}}$*, for finitary* SET-FMLA *logics the problem may be rephrased as: describe the lattice of all subquasivarieties of* $\mathbb{V}(\mathbf{PP}_{\mathbf{6}}^{\Rightarrow_{\mathsf{H}}})$.

Fragments of the language of $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$. A close inspection of the methods employed in the previous sections to axiomatize $\mathcal{PP}_{\leq}^{\Rightarrow_{\mathsf{H}}}$ and its SET-SET companions suggests that these may also be applied so as to obtain analytic axiomatizations for the logics corresponding to those fragments of the language over the connectives in $\{\wedge, \vee, \Rightarrow, {\sim}, \circ, \bot, \top\}$ that are sufficiently rich to express an appropriate set of separators. Some of these, we believe, have intrinsic logical and algebraic interest, and may deserve further study. Let us single out, for instance, the fragments corresponding to the connectives $\{\Rightarrow, \circ\}$, $\{\Rightarrow, \Delta\}$ (recall that $\Delta x := {\sim}x \Rightarrow {\sim}(x \Rightarrow x)$) and $\{\Rightarrow, {\sim}\}$. The first of them may be of interest as a minimal Logic of Formal Inconsistency, while the second could be studied in the setting of implication fragments of modal systems. The study of the third could lead to an interesting generalization of Monteiro's results on symmetric Heyting algebras and their logic.

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