THE JOURNAL OF SYMBOLIC LOGIC
Volume 6, Number 4, December 1941

A SOLUTION OF THE DECISION PROBLEM FOR THE LEWIS SYSTEMS S2 AND S4, WITH AN APPLICATION TO TOPOLOGY

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I. Introduction. In this paper I shall give a solution of the decision problem for the Lewis systems[1] S2 and S4; i.e., I shall establish a constructive method for deciding whether an arbitrary given sentence of one of these systems is provable. The method is laborious to apply, since, in order to decide by means of it whether a given sentence is provable, it is necessary to construct a (usually very large) finite matrix. The argument will perhaps be of general interest, however, because it does not seem to depend too closely on the special features of these particular systems, so that it may be possible to apply it in order to solve the decision problem for other such systems.

Section II presents the decision method for S2, and Section III for S4. In Section IV, I shall establish a certain correspondence between S4 and topology, which will provide a solution for a decision problem in topology; this correspondence also enables us to settle a previously unsolved problem with regard to the Lewis systems.

II. The system S2. In treating of this system, I shall use the notation of Lewis, with the single exception that I shall use the symbol "$\equiv$", instead of Lewis's symbol "$=$", for strict equivalence. I shall use the symbol "$=$" to denote identity. Thus "$p\equiv q$" is a formula of S2, while "$x=y$" is the statement asserting that x and y are identical. I shall refer to the rules, primitive sentences and theorems of S2 by the names and numbers used by Lewis. Whenever a theorem stated by Lewis involves the symbol "$=$", I shall of course suppose that this symbol has been replaced throughout by "$\equiv$": thus, for example, I shall take 19.82 to be "$(\Diamond p \vee \Diamond q) \equiv \Diamond(p \vee q)$", instead of "$(\Diamond p \vee \Diamond q) = \Diamond(p \vee q)$".

I shall take the primitive sentential constants of S2 to be "$\sim$", "$\Diamond$", and "$\cdot$". I shall take 11.01, 11.02, and 11.03 to be replaced respectively by the conventions: that $\alpha \vee \beta$ is an abbreviation for $\sim(\sim\alpha \cdot \sim\beta)$; that $\alpha \prec \beta$ is an abbreviation for $\sim\Diamond(\alpha \cdot \sim\beta)$; and that $\alpha \equiv \beta$ is an abbreviation for $(\alpha \prec \beta)\cdot(\beta \prec \alpha)$.

We shall henceforth to a large extent be concerned with matrices. By a *matrix* I shall mean an ordered class $(K, D, -, {}^*, \times)$ where K is a set, D is a non-empty proper subset of K, $-$ and * are unary functions defined over K and class-closing on K, and $\times$ is a binary function defined over K and class-closing on K. In discussing matrices, I shall also sometimes use the symbols "$+$", "$\rightarrow$", and "$\leftrightarrow$". These stand for binary functions defined over K as follows: $(y+z) = -(-y\times -z)$; $(y\rightarrow z) = -{}^*(y\times -z)$; $(y\leftrightarrow z) = (y\rightarrow z)\times(z\rightarrow y)$. We can think of K as the set of elements of the matrix, D as the set of "designated" elements, and $-$, *, $\times$, $+$, $\rightarrow$, and $\leftrightarrow$ as corresponding, respec-

Received August 2, 1941.

[1] See Lewis and Langford, ***Symbolic logic***, pp. 122–178, and pp. 492–502.

tively, to negation, possibility, conjunction, disjunction, strict implication, and strict equivalence.

A matrix $\mathfrak{M} = (K, D, -, *, \times)$ is said to *satisfy*[2] a sentence α of S2 if every way of evaluating α on the basis of K, using $-$, $*$, $\times$, $+$, $\rightarrow$, and $\leftrightarrow$ respectively in place of $\sim$, $\Diamond$, $\cdot$, $\vee$, $\prec$, and $\equiv$ leads to an element of D. $\mathfrak{M}$ is called an *S2-matrix* if it satisfies every provable formula of S2.

In what follows we shall be especially concerned with a more special sort of matrix:

Definition 1. A matrix $\mathfrak{M} = (K, D, -, *, \times)$ is a *normal matrix*, if and only if the following conditions are satisfied:[3]

(i) if $x \in D$ and $(x \rightarrow y) \in D$ and $y \in K$, then $y \in D$,

(ii) if $x \in D$ and $y \in D$, then $x \times y \in D$,

(iii) if $x \in K$ and $y \in K$ and $x \leftrightarrow y \in D$, then $x = y$.

We see that conditions (i), (ii), and (iii) of Definition 1 correspond respectively to Lewis's rules of inference, adjunction, and replacement. Hence a normal matrix which satisfies all the primitive sentences of S2 must also satisfy all the provable sentences, and hence be an S2-matrix.

Definition 2. If a and b are elements of a matrix $\mathfrak{M} = (K, D, -, *, \times)$ then we shall say that $a < b$ if and only if $a \rightarrow b$ is an element of D.

Theorem 1. If $\mathfrak{M} = (K, D, -, *, \times)$ is a normal S2-matrix, then K is a Boolean algebra with respect to $\times$, $+$, $-$, and $<$.[4]

Proof. I shall take Boolean algebra to be the mathematical system determined by the following set of postulates;[5] the undefined terms here are K, $\times$, $+$, $-$, and $<$:

P1. K contains at least two elements.

P2. If a and b are in K, then $a \times b$, $a + b$, and $-a$ are in K.

P3. If a and b are in K, then $a + b = b + a$.

P4. If a and b are in K, then $a \times b = b \times a$.

P5. If a, b, and c are in K, then $a + (b \times c) = ((a + b) \times (a + c))$.

P6. If a, b, and c, are in K, then $a \times (b + c) = ((a \times b) + (a \times c))$.

P7. If a and b are in K, then $a + (b \times -b) = a$.

P8. If a and b are in K, then $a \times (b + -b) = a$.

P9. If a and b are in K, then $a < b$ holds if and only if $a = a \times b$.

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[2] For a fuller explanation of this notion, see §2 of Tarski's *Der Aussagenkalkül und die Topologie*, ***Fundamenta mathematicae***, vol. 31 (1938), pp. 103–134.

[3] Throughout the paper I shall use symbols from the theory of aggregates as abbreviations for certain English expressions. Thus I write "$D \subset K$" as an abbreviation for "D is a subset of K," and "$x \in K$" as an abbreviation for "x is a member of K." I use "Λ" for the empty set, and "V" for the universal set. Later I shall use the symbol "$\cap$" to indicate the intersection of two sets, and the symbol "$\cup$" to indicate the join (logical sum) of two sets.

[4] Essentially the same result as that formulated in this theorem will be found in E. V. Huntington's *Postulates for assertion, conjunction, negation, and equality*, ***Proceedings of the American Academy of Arts and Sciences***, vol. 72 (1937), pp. 1–44.

[5] These postulates can easily be shown to be equivalent to Huntington's first set of postulates for Boolean algebra (***Transactions of the American Mathematical Society***, vol. 5 (1904), pp. 288–309), from which they were derived by some trivial modifications.

I shall now show that these postulates are all satisfied by a normal S2-matrix.

It is clear, from the definition of a matrix, that P1 is satisfied; for, according to this definition, K contains a non-empty proper sub-class D, and hence must itself contain at least two elements. It is also seen immediately that P2 is satisfied.

By 13.11, $(p \vee q) \equiv (q \vee p)$ is provable in S2. Hence since $\mathfrak{M}$ is an S2-matrix, we see that, for every a and b in K, $(a+b) \leftrightarrow (b+a) \in D$. Hence, since $\mathfrak{M}$ is a normal matrix, $a+b = b+a$. Hence P3 is satisfied. P4 can similarly be shown to be satisfied, making use of 12.15, according to which $(p \cdot q) \equiv (q \cdot p)$ is provable is S2. P5, P6, P7, and P8 are seen to be satisfied by means of 16.73, 16.72, 19.58, and 18.92, respectively.

It remains to show that P9 is satisfied.

Suppose first that $a < b$. Then, by Definition 2, $a \rightarrow b \in D$. By 12.1 we see that $a \rightarrow a \in D$. Hence, by Definition 1(ii), $(a \rightarrow a) \times (a \rightarrow b) \in D$. By 19.61 we see that $[(a \rightarrow a) \times (a \rightarrow b)] \rightarrow (a \rightarrow a \times b) \in D$. Hence, by Definition 1(i), $a \rightarrow a \times b \in D$. By 11.2 we see that $a \times b \rightarrow a \in D$. Hence, by Definition 1(ii), $(a \rightarrow a \times b) \times (a \times b \rightarrow a) \in D$. Thus, by the convention defining "$\leftrightarrow$", we see that $a \leftrightarrow a \times b \in D$. Hence, by Definition 1(iii), $a = a \times b$, as was to be shown.

Suppose, on the other hand, that $a = a \times b$. Since, by 12.1, we have $a \rightarrow a \in D$, we then see that $a \rightarrow a \times b \in D$. By 12.17 we see that $a \times b \rightarrow b \in D$. Hence by Definition 1(ii), $(a \rightarrow a \times b) \times (a \times b \rightarrow b) \in D$. By 11.6 we see that $[(a \rightarrow a \times b) \times (a \times b \rightarrow b)] \rightarrow (a \rightarrow b) \in D$. Hence, by Definition 1(i), $a \rightarrow b \in D$. Hence, by Definition 2, $a < b$, as was to be shown.

In view of Theorem 1, I shall henceforth use the ordinary symbolism and terminology of Boolean algebra in dealing with normal S2-matrices. In particular, I shall use the symbol "0" to denote the element of K such that $x+0 = x$ is true for every x in K, and "1" to denote the element of K such that $x \times 1 = x$ is true for every x in K. If a and b are elements of such a matrix, I call $a \times b$, $a+b$, and $-a$, the "product of an a and b," the "sum of a and b," and the "negation of a," respectively.

The following theorem exhibits some of the more closely characteristic properties of normal S2-matrices.

THEOREM 2. If $\mathfrak{M} = (K, D, -, *, \times)$ is a normal S2-matrix, then:

.1 $-{*0} \in D$,

.2 if $-{*x} \in D$, then $x=0$,

.3 $x < {*x}$,

.4 $*(y+z) = {*y}+{*z}$,

.5 if $y<z$, then $*y<*z$,

.6 if $x \in D$ and $x<y$, then $y \in D$,

.7 $0 \notin D$,

.8 $1 \in D$,

.9 if $y<z_1, \cdots, y<z_n$, then $*y < {*z_1} \times \cdots \times {*z_n}$, and $y < {*z_1} \times \cdots \times {*z_n}$.

.11 $*1=1$.

Proof. To prove .1, we see by 18.8 that $\sim\Diamond(p \cdot \sim p)$ is provable in S2. Hence, since $\mathfrak{M}$ is an S2-matrix, $-{*(0 \times -0)} \in D$. But $0 \times -0=0$, from Boolean algebra. Hence $-{*0} \in D$, as was to be shown.

To prove .2, suppose that $-{}^*x \in D$. By 19.74 we see that $-{}^*x \rightarrow (x \rightarrow 0) \in D$. Hence $x \rightarrow 0 \in D$, or $x < 0$; and hence $x = 0$, as was to be shown.

To prove .3, we notice that, by 18.4, $p \prec \Diamond p$ is provable in S2. Hence, if x is any element of K, we have $x \rightarrow {}^*x \in D$, or $x < {}^*x$.

.4 is seen to follow immediately from 19.82, together with the fact that $\mathfrak{M}$ is a normal S2-matrix.

To prove .5, suppose that $y < z$. Then, by Theorem 1, $y = y \times z$; hence, ${}^*y = {}^*(y \times z)$. By 19.13, we see that $\Diamond(p \cdot q) \prec \Diamond q$ is provable in S2. Hence ${}^*(y \times z) \rightarrow {}^*(z) \in D$, or ${}^*(y \times z) < {}^*z$. Hence ${}^*y < {}^*z$, as was to be shown.

To prove .6, suppose that $x \in D$ and $x < y$. Then $x \in D$ and $x \rightarrow y \in D$, and hence $y \in D$.

To prove .7, suppose that $0 \in D$. From Boolean algebra, we see that $0 < x$ is true for every x in K; hence, by .6, $x \in D$. Thus $D = K$, contrary to the definition of a matrix.

To prove .8, we notice that, by 13.5, $p \vee \sim p$ is provable in S2. Hence $0 + -0 \in D$, or $1 \in D$, as was to be shown.

To prove .9, suppose that $y < z_1, y < z_2, \cdots, y < z_n$. Then by .6 we see that ${}^*y < {}^*z_1, {}^*y < {}^*z_2, \cdots, {}^*y < {}^*z_n$. Hence ${}^*y < {}^*z_1 \times {}^*z_2 \times \cdots \times {}^*z_n$. By .3 we have $y < {}^*y$. Hence $y < {}^*z_1 \times {}^*z_2 \times \cdots \times {}^*z_n$.

To prove .11, we see that $1 < {}^*1$ by .3; and that ${}^*1 < 1$ is true from Boolean algebra.

THEOREM 3. A necessary and sufficient condition that a matrix $\mathfrak{M} = (K, D, -, {}^*, \times)$ be a normal S2-matrix is that the following statements be true:

(1) K is a Boolean algebra with respect to $-$ and $\times$,

(2) D is an additive ideal of K (i.e., D is a non-empty proper subset of K such that, if x and y are in D then $x \times y$ is in D; and if x is in D and y is in K, then $x+y$ is in D),

(3) $-{}^*0$ is in D,

(4) if $-{}^*x$ is in D, then $x = 0$,

(5) if x is in K, then $x < {}^*x$,

(6) if x and y are in K, then ${}^*(x+y) = {}^*x + {}^*y$.

Proof. To see that the condition is necessary, let $\mathfrak{M}$ be a normal S2-matrix.

By Theorem 1, we see that condition (1) is satisfied.

By the definition of a matrix, we see that D is a non-empty proper subset of K. By Definition 1(ii), we see that if x is in D and y is in D, then $x \times y$ is in D. Since $x < x+y$, we see by Theorem 2.6 that if x is in D and y is in K then $x+y$ is in D. Hence condition (2) is satisfied.

By Theorems 2.1, 2.2, 2.3 and 2.4, respectively, we see that conditions (3), (4), (5), and (6) are satisfied.

Hence the condition is necessary.

To show that the condition is also sufficient, suppose that $\mathfrak{M} = (K, D, -, {}^*, \times)$ is a matrix satisfying (1)–(6). We are to show that $\mathfrak{M}$ is a normal S2-matrix.

We first notice that, under the given hypothesis, if $x \in D$ and $y \in D$ then $x \times y \in D$, so that part (ii) of Definition 1 is satisfied. Suppose, conversely, that $x \times y \in D$; then, by (2), we see that $(x \times y)+x \in D$ and $(x \times y)+y \in D$; but $(x \times y)+x = x$, and $(x \times y)+y = y$; and hence $x \in D$ and $y \in D$.

I next show that $x \rightarrow y \in D$ holds if and only if $x < y$. If $x \rightarrow y \in D$, then $-{}^*(x \times -y) \in D$, so that $x \times -y = 0$, by part (4) of the hypothesis; but this is equivalent to $x < y$. If $x < y$, of the other hand, then $x \times -y = 0$, and hence $-{}^*(x \times -y) \in D$, or $x \rightarrow y \in D$, by part (3) of the hypothesis.

I next show that part (i) of Definition 1 is satisfied. Suppose that $x \in D$ and $x \rightarrow y \in D$. Then $x < y$, by what has just been shown, so that $y = x+y$, and hence $y \in D$ by part (2) of the hypothesis.

I next show that part (iii) of Definition 1 is satisfied. Suppose that $x \leftrightarrow y \in D$. Then $(x \rightarrow y) \times (y \rightarrow x) \in D$. Hence, by what was shown in the first paragraph, $x \rightarrow y \in D$ and $y \rightarrow x \in D$. Hence, by what was shown in the second paragraph $x < y$ and $y < x$. Hence $x = y$, as was to be shown.

This completes the proof that $\mathfrak{M}$ is a normal matrix. In order to show that $\mathfrak{M}$ is an S2-matrix, it now suffices to show that the primitive sentences B1-B8 are satisfied.

I shall first show, however, that if $x < y$ then ${}^*x < {}^*y$. For if $x < y$, then $y = x+y$, and hence ${}^*y = {}^*(x+y)$. Hence, by part (6) of the hypothesis, ${}^*y = {}^*x + {}^*y$, or ${}^*x < {}^*y$.

To see that B1 is satisfied, we notice that $[(x \times y) \rightarrow (y \times x)] = -{}^*[(x \times y) \times -(y \times x)] = -{}^*0$, and that $-{}^*0 \in D$ by part (3) of the hypothesis. The proofs, that B2, B3, B4, and B5 are satisfied, are similar.

To see that B6 is satisfied, we first notice that $(x \times -z) < [(x \times -y)+(y \times -z)]$. Hence ${}^*(x \times -z) < {}^*[(x \times -y)+(y \times -z)]$. Hence, by part (6) of the hypothesis, ${}^*(x \times -z) < {}^*(x \times -y) + {}^*(y \times -z)$, and therefore $-[{}^*(x \times -y) + {}^*(y \times -z)] < -{}^*(x \times -z)$, or $-{}^*(x \times -y) \times -{}^*(y \times -z) < -{}^*(x \times -z)$. Hence $[(x \rightarrow y) \times (y \rightarrow z)] < (x \rightarrow z)$, and therefore by the second paragraph $\{[(x \rightarrow y) \times (y \rightarrow z)] \rightarrow (x \rightarrow z)\} \in D$, as was to be shown.

To show that B7 is satisfied, we notice that $-y < -x + (x \times -y)$. Also, by part (5) of the hypothesis, $x \times -y < {}^*(x \times -y)$. Hence $-y < -x + {}^*(x \times -y)$. Hence $-[-x + {}^*(x \times -y)] < y$, or $x \times -{}^*(x \times -y) < y$. Thus $x \times (x \rightarrow y) < y$, or, by the second paragraph, $\{[x \times (x \rightarrow y)] \rightarrow y\} \in D$, as was to be shown.

To see that B8 is satisfied, we notice that $x \times y < x$, and hence ${}^*(x \times y) < {}^*x$; hence, by the second paragraph, $[{}^*(x \times y) \rightarrow {}^*x] \in D$, as was to be shown.

Remark. As an immediate consequence of Theorem 3, we are provided with a simple "recipe" for constructing all possible finite normal S2-matrices. We take a finite Boolean algebra $(K, -, \times)$, pick an element $d \neq 0$ of K, and let D be the set of all elements x of K such that $d < x$. We choose an element a of D, and set ${}^*0 = -a$. If, then, p is any atom in K (i.e., an element such that $x < p$ implies that either $x = 0$ or $x = p$) we define ${}^*p$ arbitrarily, but in such a way that $p < {}^*p$, ${}^*0 < {}^*p$, and $d \times {}^*p \neq 0$. If finally, z is any element of K, such that $z = p_1 + p_2 + \cdots + p_r$, where $p_1, p_2, \cdots, p_r$, are atoms, then we set ${}^*z = {}^*p_1 + {}^*p_2 + \cdots + {}^*p_r$. It follows easily from Theorem 3, that the matrix constructed in this way is a normal S2-matrix, and furthermore that every finite normal S2-matrix can be constructed in this way.

Definition 3. By an *S2-characteristic matrix* is meant a matrix which satisfies every provable sentence of S2 (so that it is an S2-matrix) and which is such, conversely, that every sentence which is satisfied by it is provable in S2. Thus the class of sentences which are satisfied by an S2-characteristic matrix coincides with the class of provable sentences of S2.

If we were able to find a finite S2-characteristic matrix, we should of course thereby be also provided with a decision-method for S2; but Dugundji[6] has shown that there does not exist a finite S2-characteristic matrix. The following theorem, however, assures us that there is an infinite S2-characteristic matrix.

Theorem 4. There exists a normal S2-characteristic matrix $\mathfrak{M} = (K, D, -, *, \times)$.

Proof. I first show, by means of an unpublished method of Lindenbaum,[7] that there is a matrix $\mathfrak{M}_1 = (K_1, D_1, -_1, *_1, \times_1)$ which is S2-characteristic, though not normal. Later I shall show how a normal S2-characteristic matrix can be constructed from $\mathfrak{M}_1$.

Let K_1 consist of all the sentences of S2, and let D_1 consist of all the provable sentences of S2. If α is an element of K_1, then let $-_1\, \alpha$ be the sentence which arises by prefixing a negation-sign to the formula α; i.e., $-_1\, \alpha = \sim\alpha$. Similarly, let $*_1\, \alpha = \Diamond\alpha$. If α and β are elements of K, then let $\alpha \times_1 \beta = \alpha\cdot\beta$.

It is easily shown that $\mathfrak{M}_1$ is S2-characteristic. For suppose first that α is a provable sentence of S2, and let β be any result of substituting elements of K_1 for variables in α; then β is also a provable sentence of S2, since S2 had a rule of substitution for sentential variables; so that $\beta \in D_1$ by the definition of D_1; hence α is satisfied by $\mathfrak{M}_1$. Now suppose, on the other hand, that α is a formula which is satisfied by $\mathfrak{M}_1$; then every result of substituting elements of K_1 for the variables in α, is in D_1. Since all sentential variables are sentences, and hence elements of K_1, we see, in particular, that the sentence β which results from α by replacing every variable in α by itself, must be in D_1. But clearly $\alpha = \beta$, so that $\alpha \in D_1$; thus α is provable in S2, as was to be shown.

It will be observed that $\mathfrak{M}_1$ satisfies parts (i) and (ii) of Definition 1. For if $\alpha \in D_1$ and $\beta \in D_1$, then α and β are provable, and hence (by the rule of adjunction) $\alpha\cdot\beta$ is provable, so that $\alpha \times_1 \beta \in D_1$. Similarly, if $\alpha \in D_1$ and $\alpha \rightarrow_1 \beta \in D_1$ then α is provable and $\alpha \prec \beta$ is provable, and hence (by the rule of inference) β is provable, so that $\beta \in D_1$.

On the other hand, condition (iii) of Definition 1 is not satisfied by $\mathfrak{M}_1$. For by 12.3, $p \equiv \sim\sim p$ is provable, so that $p \leftrightarrow_1 -_1 -_1 p \in D_1$, while "$p$" is not identical with "$-_1 -_1 p$".

I shall now construct from $\mathfrak{M}_1$ a new matrix $\mathfrak{M} = (K, D, -, *, \times)$ which will also satisfy the condition (iii) of Definition 1.

By saying that $\alpha \approx \beta$, I shall mean that $\alpha \leftrightarrow_1 \beta \in D_1$. I shall now show that

[6] In his *Note on a property of matrices for Lewis and Langford's calculi of propositions*, this Journal, vol. 5 (1940), pp. 150–151.

[7] This method is very general, and applies to any sentential calculus which has a rule of substitution for sentential variables. The method was explained to me by Professor Tarski, to whom I am also indebted for many other suggestions in connection with the present paper.

the relation $\approx$ is a congruence relation (in the sense of modern algebra) over K_1. That is to say, I shall prove the following: (1) $\alpha \approx \alpha$, for every α in K_1; (2) if $\alpha \approx \beta$, then $\beta \approx \alpha$; (3) if $\alpha \approx \beta$ and $\beta \approx \gamma$, then $\alpha \approx \gamma$; (4) if $\alpha \approx \beta$, then $-_1 \alpha \approx -_1 \beta$; (5) if $\alpha \approx \beta$, then $*_1 \alpha \approx *_1 \beta$; (6) if $\alpha \approx \beta$ and $\gamma \approx \delta$, then $\alpha \times_1 \gamma \approx \beta \times_1 \delta$.

To prove (1) we notice that, by 12.11 and substitution, $\alpha \equiv \alpha$ is provable in S2, so that $\alpha \leftrightarrow_1 \alpha \in D_1$ (or $\alpha \approx \alpha$), since $\mathfrak{M}_1$ is an S2-matrix. To prove (2), suppose that $\alpha \approx \beta$. Then $\alpha \equiv \beta$ is provable in S2. By 12.11 and substitution, we see that $\alpha \equiv \alpha$ is provable in S2. Hence, by part (a) of the rule of substitution, $\beta \equiv \alpha$ is provable in S2, so that $\beta \approx \alpha$, as was to be shown. The proofs of (3), (4), (5), and (6) can be carried through in a similar way, using 12.11 and part (a) of the rule of substitution.

In view of the fact that $\approx$ is a congruence relation, we are justified in giving the following definitions. If α is any element of K_1, then by $E(\alpha)$ we mean the set of all elements β of K_1 such that $\alpha \approx \beta$. We take K to be the class of all sets $E(\alpha)$ such that $\alpha \in K_1$. We take D to be the class of all sets $E(\alpha)$ such that $\alpha \in D_1$. We then define $-E(\alpha) = E(-_1 \alpha)$, $*E(\alpha) = E(*_1(\alpha))$, and $E(\alpha) \times E(\beta) = E(\alpha \times_1 \beta)$.

It is now easily seen that the matrix $\mathfrak{M}$ satisfies exactly the same formulas as does the matrix $\mathfrak{M}_1$, and hence is again an S2-characteristic matrix.

Moreover $\mathfrak{M}$ satisfies the three conditions of Definition 1, and hence is a normal matrix.

To see that condition (ii) of Definition 1 is satisfied, let $E(\alpha) \in D$ and $E(\beta) \in D$. Then $\alpha \in D_1$ and $B \in D_1$. Hence, by the earlier part of the proof, $\alpha \times_1 \beta \in D_1$. Hence $E(\alpha \times_1 \beta) \in D$, which by definition is equivalent to saying that $E(\alpha) \times E(\beta) \in D$.

To see that (i) is satisfied, let $E(\alpha) \in D$ and $E(\alpha) \rightarrow E(\beta) \in D$. Then $E(\alpha) \in D$ and $E(\alpha \rightarrow_1 \beta) \in D$, so that $\alpha \in D_1$ and $\alpha \rightarrow_1 \beta \in D_1$. Hence $\beta \in D_1$, by what was proved earlier, so that $E(\beta) \in D$.

Finally, to see that (iii) is satisfied, suppose that $E(\alpha) \leftrightarrow E(\beta) \in D$. Then $E(\alpha \leftrightarrow_1 \beta) \in D$, so that $\alpha \leftrightarrow_1 \beta \in D_1$, or $\alpha \approx \beta$. Now if γ is any element of $E(\alpha)$ we have $\gamma \approx \alpha$, and hence $\gamma \approx \beta$, and hence $\gamma \in E(\beta)$; whence $E(\alpha)$ is a subset of $E(\beta)$. In a similar way we see that $E(\beta)$ is a subset of $E(\alpha)$. Hence $E(\alpha) = E(\beta)$, as was to be shown.

The following is the fundamental theorem[8] of this paper.

THEOREM 5. Let $\mathfrak{M} = (K, D, -, *, \times)$ be a normal S2-matrix, and let $a_1, a_2, \cdots, a_r$ be a finite sequence of elements of K. Then there exists a finite normal S2-matrix $\mathfrak{M}_1 = (K_1, D_1, -_1, *_1, \times_1)$, with at the most $2^{2^{r+1}}$ elements, such that K_1 contains a sequence $b_1, b_2, \cdots, b_r$ of elements satisfying the following conditions:

(1) $a_i \in D$ holds, if and only if $b_i \in D_1$ holds,
(2) $-a_i = a_j$ holds, if and only if $-_1 b_i = b_j$ holds,
(3) $a_i \times a_j = a_k$ holds, if and only if $b_i \times_1 b_j = b_k$ holds,
(4) if $*a_i = a_j$, then $*_1 b_i = b_j$.

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[8] I am indebted to the referee for a suggestion leading to a considerable simplification of the proof of this theorem, as well as for other helpful criticisms.

Proof. Let K_1 be that Boolean subalgebra of K which is generated by $*0, a_1, \cdots, a_r$; i.e., let K_1 consist of all elements of K which can be obtained from the elements $*0, a_1, \cdots, a_r$ by any finite number of applications of the operations $-$ and $\times$.

Let the operations $-_1$ and $\times_1$ be the same as the operations $-$ and $\times$, except for being defined merely over K_1 instead of over K.

Let D_1 be the intersection of D and K_1.

We say that an element x of K_1 is *covered* by an element y of K_1, if $x<y$ and $*y \in K_1$. We notice that every element x of K_1 is covered by some element of K_1, since $1 = -(x\times -x) \in K_1$, $x<1$, and $*1=1$, by Theorem 2.11.

If x is an element of K_1, which is covered by $x_1, x_2, \cdots, x_n$, then we set $*_1 x = *x_1 \times *x_2 \times \cdots \times *x_n$.

It is easily seen from our general knowledge of Boolean algebra, that K_1 contains at most $2^{2^{r+1}}$ elements. For every element of K_1 is equal to a sum of the products $*0 \times a_1 \times \cdots \times a_r$, $*0 \times a_1 \times \cdots \times a_{r-1} \times -a_r, \cdots,$ $-*0 \times -a_1 \times \cdots \times -a_r$; and there are at most 2^{r+1} distinct products of this sort, and hence at most $2^{2^{r+1}}$ sums of such products.

I shall now show that, if x is any element of K_1, then $*x < *_1 x$. Let x be covered by $x_1, \cdots, x_n$. Then $*_1 x = *x_1 \times \cdots \times *x_n$. Since $x<x_1, \cdots, x<x_n$, we see by Theorem 2.9 that $*x < *x_1 \times \cdots \times *x_n$, or $*x < *_1 x$.

I shall now show that the matrix $\mathfrak{M}_1 = (K_1, D_1, -_1, *_1, \times_1)$ which has been defined satisfies conditions (1)–(4) of our theorem, if we choose $b_1, \cdots, b_r$ to be respectively equal to $a_1, \cdots, a_r$. Condition (1) is immediately seen to be satisfied in view of the way in which D_1 was defined; conditions (2) and (3), similarly, are clearly satisfied from the definitions of $-_1$ and $\times_1$. To show that (4) is satisfied, it is sufficient to prove that if $*a_i = a_j$ then $*_1 a_i = a_j$. I shall show, more generally, that if x and y are any elements of K_1 such that $*x=y$, then $*_1 x = y$. Since $*x = y$, we see that x is covered by x. Suppose that x is also covered by $x_1, \cdots, x_n$. Then, by the definition of $*_1$, we see that

$$*_1 x = *x \times *x_1 \times \cdots \times *x_n = y \times *x_1 \times \cdots \times *x_n, \quad \text{so that} \quad *_1 x < y.$$

On the other hand, we have $*x < *_1 x$, or (since by hypothesis $*x=y$) $y < *_1 x$. Thus $*_1 x = y$, as was to be shown.

To complete the proof of our theorem it is now necessary to show that $\mathfrak{M}_1$ is a normal S2-matrix. In order to do this, I shall prove that $\mathfrak{M}_1$ satisfies the six conditions of Theorem 3.

From the definition of K_1, $-_1$, and $\times_1$, it is immediately evident that K_1 is a Boolean algebra with respect to $-_1$ and $\times_1$.

To see that D_1 is a non-empty proper subset of K_1, we notice that $0 \in K_1$ and $1 \in K_1$; and that, by Theorem 2.8 and 2.7 respectively, $1 \in D$ and $0 \notin D$, and hence that $1 \in D_1 = K_1 \cap D$ and $0 \notin D_1 = K_1 \cap D$.

If $x \in D_1$ and $y \in D_1$, then $x \in D$, $x \in K_1$, $y \in D$, and $y \in K_1$. Then, since $\mathfrak{M}$ is a normal matrix, $x \times y \in D$; moreover $x \times y \in K_1$. Hence $x \times y \in D_1$.

If $x \in D_1$ and $y \in K_1$, then $x \in D$ and $y \in K$, and hence $x+y \in D$, by Theorem 3. But clearly $x+y \in K_1$, and hence $x+y \in D_1$.

From the last three paragraphs we see that D_1 is an additive ideal of K_1, so that condition (2) of Theorem 3 is satisfied.

I shall now show that condition (3) of Theorem 3 is satisfied; i.e., that $-_1 *_1 0 \epsilon D_1$. Since $0 \epsilon K_1$, $*0 \epsilon K_1$, and $*0 = *0$, we see by what was proved above that $*_1 0 = *0$, and hence that $-_1 *_1 0 = -*0$. By Theorem 2.1 we see that $-*0 \epsilon D$, and hence that $-_1 *_1 0 \epsilon D$. Thus we have $-_1 *_1 0 \epsilon K$ and $-_1 *_1 0 \epsilon D$, and hence $-_1 *_1 0 \epsilon D_1$, by the definition of D_1.

I shall now show that condition (4) of Theorem 3 is satisfied; i.e., if $-_1 *_1 x \epsilon D_1$ then $x=0$. Suppose that $-_1 *_1 x \epsilon D_1$; then $-_1 *_1 x \epsilon D$. Since $*x < *_1 x$, we see that $-*_1 x < -*x$, or $-_1 *_1 x < -*x$. Then, by Theorem 2.6, $-*x \epsilon D$. Hence, by Theorem 2.2, $x=0$, as was to be shown.

I shall now show that condition (5) of Theorem 3 is satisfied; i.e., if x is any element of K_1, then $x < *_1 x$. By Theorem 2.3, we see that $x < *x$. But we have shown previously that $*x < *_1 x$. Hence $x < *_1 x$, as was to be shown.

I shall now show that condition (6) of Theorem 3 is satisfied; i.e., if x and y are any elements of K_1, then $*_1(x+y) = *_1 x + *_1 y$. We notice that the elements covering $x+y$ consist of all sums of pairs of elements such that the first covers x and the second covers y; for if x_1 covers x and y_1 covers y, then $x<x_1$, $*x_1 \epsilon K_1$, $y<y_1$ and $*y_1 \epsilon K_1$, and hence $x+y < x_1+y_1$ and $*(x_1+y_1) = *x_1+*y_1 \epsilon K_1$, so that x_1+y_1 covers $x+y$; and if z_1 covers $x+y$, then $x+y < z_1$ and hence $x<z_1$ and $y<z_1$, so that z_1 covers x and y, and $z_1 = z_1+z_1$. Suppose now, that x is covered by $x_1, \cdots, x_m$, that y is covered by $y_1, \cdots, y_n$, and that $x+y$ is covered by $z_1, \cdots, z_s$. Then $*_1 x + *_1 y = (*x_1 \times *x_2 \times \cdots \times *x_m) + (*y_1 \times *y_2 \times \cdots \times *y_n) = (*x_1 + *y_1) \times (*x_1 + *y_2) \times \cdots \times (*x_m + *y_n) = *(x_1+y_1) \times *(x_1+y_2) \times \cdots \times *(x_m+y_n) = *z_1 \times *z_2 \times \cdots \times *z_s = *_1(x+y)$, as was to be shown.

This completes the proof of our theorem.

By a *sub-sentence* of a sentence α, is meant a sentence β which occurs as a part (proper or improper) of α. Thus there are five sub-sentences of the sentence "$(p \cdot q) \cdot \Diamond p$": namely, "$p$", "$q$", "$\Diamond p$", "$(p \cdot q)$", and "$(p \cdot q) \cdot \Diamond p$".

THEOREM 6. Let γ be a sentence[9] of S2 which contains just r sub-sentences. Then γ is provable if and only if it is satisfied by every normal S2-matrix with not more than $2^{2^{r+1}}$ elements.

Proof. Suppose that γ is not provable. Then γ is not satisfied by the normal S2-characteristic matrix $\mathfrak{M} = (K, D, -, *, \times)$, which exists by Theorem 4. Let the sentential variables in γ be $\gamma_1, \gamma_2, \cdots, \gamma_n$, and let $a_1, a_2, \cdots, a_n$ be elements of K such that, when $\gamma_1, \gamma_2, \cdots, \gamma_n$ are replaced in γ by $a_1, a_2, \cdots, a_n$, respectively, an element is obtained which is not in D. Let the sub-sentences of γ, other than the sentential variables, be $\gamma_{n+1}, \cdots, \gamma_r$, and suppose that the given substitution carries $\gamma_{n+1}, \cdots, \gamma_r$, into $a_{n+1}, \cdots, a_r$ respectively. Without loss of generality we can suppose that $\gamma_r = \gamma$, and hence that $a_r \notin D$.

Then we see, by Theorem 5, that there exists a finite normal S2-matrix

[9] γ is supposed to contain no sentential constants except "$\sim$", "$\Diamond$", and "$\cdot$"; if the original sentence contains "$-\!\!3$", for example, this is first to be eliminated by means of the convention that $\alpha -\!\!3 \beta$ is an abbreviation for $\sim\Diamond(\alpha \cdot \sim\beta)$.

$\mathfrak{M}_1 = (K_1, D_1, -_1, *_1, \times_1)$ with at most $2^{2^{r+1}}$ elements such that K_1 contains a sequence $b_1, b_2, \cdots, b_r$ of elements satisfying the following conditions:

(1) if $a_i \notin D$, then $b_i \notin D_1$,

(2) if $-a_i = a_j$, then $-_1 b_i = b_j$,

(3) if $a_i \times a_j = a_k$, then $b_i \times_1 b_j = b_k$,

(4) if $*a_i = a_j$, then $*_1 b_i = b_j$.

Suppose now that we replace the sentential variables $\gamma_1, \gamma_2, \cdots, \gamma_n$ in γ by $b_1, b_2, \cdots, b_n$ respectively. It is seen from the last three of the above conditions that this substitution will also take the sub-sentences $\gamma_{n+1}, \cdots, \gamma_r$ into $b_{n+1}, \cdots, b_r$. Thus γ_r, which is the same as γ, is transformed into b_r, which, by (1), is not in D_1. Hence γ is not satisfied by the normal S2-matrix $\mathfrak{M}_1$, which contains at most $2^{2^{r+1}}$ elements.

Theorem 6 provides us with a decision-method for S2. For if γ is a given sentence, then the number, r, of the sub-sentences of γ can be found in a constructive way. Then all normal S2-matrices with not more than $2^{2^{r+1}}$ elements can be constructed. And then we can determine, again by a constructive method, whether γ is satisfied by all these matrices. If it is satisfied by all of them, then, by Theorem 6, it is provable; and otherwise it is not provable.

I shall conclude this section by pointing out that it is also possible to define a sequence of finite matrices $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$, such that a sentence γ, which contains just r sub-sentences is provable in S2 if and only if it is satisfied by $\mathfrak{M}_r$. We first construct all normal S2-matrices $\mathfrak{M}$, whose elements consist of the first n integers, for $n \leqq 2^{2^{r+1}}$; it is clear that every normal S2-matrix is isomorphic to one of these matrices. We then take $\mathfrak{M}_r$ to be the direct product of all these matrices. Since γ is satisfied by $\mathfrak{M}_r$ if and only if it is satisfied by every one of the matrices of which $\mathfrak{M}_r$ is the direct product, we see, by Theorem 6, that γ is provable in S2 if and only if it is satisfied by $\mathfrak{M}_r$.

III. The system S4. In treating of the system S4, I shall again use the symbol "$\equiv$" instead of "$=$" for strict equivalence. The system arises from S2 by the addition of the primitive sentence:[10]

C10.1 $$\Diamond\Diamond p \equiv \Diamond p.$$

In dealing with this system, we shall have need of the theorem:

THEOREM 7. The following is a provable sentence of S4:

$$(p \cdot \sim p) \equiv \Diamond(p \cdot \sim p).$$

Proof.

(1) $\sim\Diamond\sim(\sim\Diamond(p\cdot\sim p)) \prec [\sim(p\cdot\sim p) \prec \sim\Diamond(p\cdot\sim p)]$ by 19.75.

(2) $\sim\Diamond\Diamond(p\cdot\sim p) \prec [\sim(p\cdot\sim p) \prec \sim\Diamond(p\cdot\sim p)]$ by (1) and 12.3.

(3) $\sim\Diamond(p\cdot\sim p) \prec [\sim(p\cdot\sim p) \prec \sim\Diamond(p\cdot\sim p)]$ by (2) and C10.1.

(4) $\sim\Diamond(p\cdot\sim p)$ by 8.8.

(5) $\sim(p\cdot\sim p) \prec \sim\Diamond(p\cdot\sim p)$ by (3) and (4).

[10] See Lewis and Langford, page 497 and page 501.

(6) $\Diamond(p \cdot \sim p) \prec (p \cdot \sim p)$ by (5) and 12.44.

(7) $(p \cdot \sim p) \prec \Diamond(p \cdot \sim p)$ by 18.4.

(8) $(p \cdot \sim p) \equiv \Diamond(p \cdot \sim p)$ by (6) and (7).

I shall now show how the decision-method given in Section II can be modified so as to cover S4. The proofs of the theorems here will not be given in detail, but it will be indicated merely wherein they differ from the proofs of the corresponding theorems of S2. I shall also formulate the necessary new definitions for S4.

A matrix $\mathfrak{M}$ is called an *S4-matrix* if it satisfies every provable formula of S4.

THEOREM 8. If $\mathfrak{M} = (K, D, -, *, \times)$ is a normal S4-matrix, then K is a Boolean algebra with respect to $\times$, $+$, $-$, and $<$.

Proof. This is an immediate consequence of Theorem 1, since every S4-matrix is also an S2-matrix.

Similarly we see that Theorem 2 is likewise true of normal S4-matrices. In addition to these properties, we have for normal S4-matrices the following:

THEOREM 9. If $\mathfrak{M} = (K, D, -, *, \times)$ is a normal S4-matrix, and if x is any element of K, then:

.1 $**x = *x$

.2 $*0 = 0$

Proof. By C10.1, we see that $\Diamond\Diamond p \equiv p$ is provable in S4. Hence, if x is any element of K we have $**x \leftrightarrow *x \in D$, and hence, since $\mathfrak{M}$ is a normal matrix, $**x = *x$, as was to be shown.

The fact that $*0 = 0$ follows from Theorem 7.

THEOREM 10. A necessary and sufficient condition that a matrix $\mathfrak{M} = (K, D, -, *, \times)$ be a normal S4-matrix, is that it satisfy the six conditions of Theorem 3, and in addition the following: if x is any element of K, then $**x = *x$.

Proof. Since an S4-matrix is also an S2-matrix, we see by Theorem 3 that a normal S4-matrix satisfies the six conditions of Theorem 3. By Theorem 9, a normal S4-matrix satisfies the condition that $**x = *x$. Hence the condition is necessary.

Suppose now, on the other hand, that $\mathfrak{M} = (K, D, -, *, \times)$ is a matrix satisfying the six conditions of Theorem 3, and in addition the condition that $**x = *x$. Then we see by Theorem 3 that $\mathfrak{M}$ is a normal S2-matrix. To prove that $\mathfrak{M}$ is a normal S4-matrix, it remains therefore only to show that it satisfies C10.1. But this is immediately evident, since, in view of the fact that $\mathfrak{M}$ is an S2-matrix we have $*x \leftrightarrow *x \in D$, and hence (since $**x = *x$) $**x \leftrightarrow *x \in D$. Thus the condition is also sufficient.

By an *S4-characteristic matrix* is meant a matrix which satisfies every provable formula of S4, and which is such, conversely, that every formula which is satisfied by it is provable in S4.

THEOREM 11. There exists a normal S4-characteristic matrix $\mathfrak{M}$.

Proof. The proof of this is exactly analogous to the proof of Theorem 4.

We are able to prove for S4 a theorem which is a slight improvement on Theorem 5, in that we can replace "$2^{2^{r+1}}$" by "2^{2^r}".

THEOREM 12. Let $\mathfrak{M} = (K, D, -, *, \times)$ be a normal S4-matrix, and let $a_1, a_2, \cdots, a_r$ be a finite sequence of elements of K. Then there exists a finite normal S4-matrix $\mathfrak{M}_1 = (K_1, D_1, -_1, *_1, \times_1)$, with at most 2^{2^r} elements, such that K_1 contains a sequence $b_1, b_2, \cdots, b_r$ of elements satisfying the following conditions:

(1) $a_i \epsilon D$ holds, if and only if $b_i \epsilon D_1$ holds,
(2) $-a_i = a_j$ holds, if and only if $-_1 b_i = b_j$ holds,
(3) $a_i \times a_j = a_k$ holds, if and only if $b_i \times_1 b_j = b_k$ holds,
(4) if $*a_i = a_j$, then $*_1 b_i = b_j$.

Proof. We define $\mathfrak{M}_1$ as in the proof of Theorem 5, with the exception that we take K_1 to be the Boolean algebra generated simply from the elements $a_1, \cdots, a_r$ (it is not necessary to add $*0$ to this list, since, by Theorem 9, $*0=0$).

The existence of a sequence having the asserted properties is proved as the proof of Theorem 5. In a similar way, we prove that $\mathfrak{M}_1$ is a normal S2-matrix. It remains therefore, by Theorem 10, merely to show, that if x is any element of K_1, then $*_1 *_1 x = *_1 x$.

As in the proof of Theorem 5, we see that, if x is any element of K_1, we have $x < *_1 x$. Hence, in particular, $*_1 x < *_1 *_1 x$. It remains therefore, only to show that $*_1 *_1 x < *_1 x$.

Suppose that x is covered by $x_1, \cdots, x_m$, so that $*_1 x = *x_1 \times \cdots \times *x_m$. Then $*_1 x < *x_1, *_1 x < *x_2, \cdots, *_1 x < *x_m$. Since, moreover, by Theorem 9.1, $**x_1 = *x_1, \cdots, **x_m = *x_m$, we see that $**x_1, \cdots, **x_m$ are all elements of K_1. Hence $*_1 x$ is covered by $*x_1, \cdots, *x_m$; suppose that $*_1 x$ is also covered by $x_{m+1}, \cdots, x_n$. Then $*_1 *_1 x = **x_1 \times \cdots \times **x_m \times *x_{m+1} \times \cdots \times *x_n = *x_1 \times \cdots \times *x_m \times *x_{m+1} \times \cdots \times *x_n = *_1 x \times *x_{m+1} \times \cdots \times *x_n$, and hence $*_1 *_1 x < *_1 x$, as was to be shown.

THEOREM 13. Let γ be a sentence of S4 which contains just r sub-sentences. Then γ is provable if and only if it is satisfied by every normal S4-matrix with not more than 2^{2^r} elements.

Proof. Analogous to the proof of Theorem 6.

Just as Theorem 6 provides us with a decision-method for S2, so Theorem 13 provides us with a decision-method for S4.

It is clear that, by the sort of construction described at the end of Section II, we can also give a sequence of finite matrices $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$, such that a sentence which involves just r sub-sentences will be provable in S4 if and only if it is satisfied by $\mathfrak{M}_r$.

IV. An application to topology. In this section,[11] I shall show how the method employed in Section II provides us also with a solution for a certain methodological problem regarding topology: namely, it gives us a constructive method for deciding whether an arbitrary given equation (involving symbols for the Boolean operations and for the topological operation of closure) is true in every topological space. This result can be regarded as a special case of what

[11] This extension of my results to topology was first proposed to me by Professor Tarski, who has also suggested several changes in my first draft of this section. I am also indebted to Dr. Morris Kline, who has supplied me with some information regarding topology.

should be called Jaśkowski's theorem, which provides a solution for a more general decision problem for these spaces.[12] The first steps in the direction of a study of the relationship between topology and the Lewis calculus were made by Tang Tsao-Chen.[13]

I shall take a *topological space in the wider sense*[14] to be any realization of the following postulates (where the undefined terms are S and $*$):

T1 S is not an empty set.

T2 If α is any subset of S, then $*\alpha$ is a subset of S.

T3 If α is any subset of S, then $\alpha \subset *\alpha$.

T4 If α and β are any subsets of S, then $*(\alpha \cup \beta) = *\alpha \cup *\beta$.

T5 If α is any subset of S, then $**\alpha = *\alpha$.

T6 If Λ is the null-set, then $*\Lambda = \Lambda$.

Where there is no danger of ambiguity, I shall say simply "topological space," instead of "topological space in the wider sense."

If A is a set, I denote the complement of A by "$-A$". If A and B are sets, I consider "$A \rightarrow B$" as an abbreviation for "$-*(A \cap -B)$", and "$A \leftrightarrow B$" as an abbreviation for "$(A \rightarrow B) \cap (B \rightarrow A)$".

By a *topological expression,* I shall mean any meaningful expression built up from set-variables, together with the symbols "Λ" and "V", by a finite number of applications of the operations, $*$, $-$, $\cup$, $\cap$, $\rightarrow$, and $\leftrightarrow$.

By a *topological equation,* I shall mean an equation both of whose members are topological expressions. A *topological formula* is a topological equation whose right member is the symbol "V" and whose left member does not contain the constants "V" and "Λ".

It is well known from Boolean algebra that an equation "$A = B$" is true in a given space if and only if "$(A \cup -B) \cap (B \cup -A) = V$" is true. Also if "$\Lambda$", or "V", occurs in an equation, they may be replaced by "$x \cap -x$", or "$x \cup -x$", respectively, where x is any variable. Hence the decision problem for topological equations reduces to the decision problem for topological formulas.

I shall say that a topological expression A *corresponds* to a sentence α of S4, if α results from A by replacing the distinct set-variables by distinct sentential variables, and the symbols $-$, $*$, $\cap$, $\cup$, $\rightarrow$, and $\leftrightarrow$ by the symbols $\sim$, $\Diamond$, $\cdot$, $\vee$, $\prec$, and $\equiv$, respectively.

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[12] Jaśkowski's theorem provides a decision procedure for sentential functions (built up by means of "not", "and", etc.) of equations involving symbols for the Boolean operations and for the topological operation of derivative. Since closure is definable in terms of derivative by means of the equation $*A = A + \acute{A}$ (where $\acute{A}$ is the derivative of A), it is clear that this result contains mine as a special case. I am informed by Professor Tarski, that this result was obtained by Jaśkowski in 1939, but that it was not published on account of the war.

[13] See his paper, *Algebraic postulates and a geometric interpretation for the Lewis calculus of strict implication*, ***Bulletin of the American Mathematical Society,*** vol. 44 (1938), pp. 737–744. A proof is given here that every provable sentence of the Lewis calculus corresponds to a true equation of topology, although no proof is given for the fact that every true equation of the given topological system corresponds to a provable sentence of the Lewis calculus.

[14] See C. Kuratowski, *Sur l'operation $\bar{A}$ de l'analysis situs*, ***Fundamenta mathematicae,*** vol. 3 (1922), pp. 182–199.

I shall show that a topological formula $A = \text{V}$ is true for every topological space, if and only if the sentence α is provable in S4, where α corresponds to A. In view of Theorem 13, this will provide us with a method of deciding whether a given topological formula is true in every topological space.

THEOREM 14. The following topological formulas are true for every topological space in the wider sense:

.1 $[(X \cap Y) \to (Y \cap X)] = \text{V}$.

.2 $[(X \cap Y) \to X] = \text{V}$.

.3 $[X \to (X \cap X)] = \text{V}$.

.4 $\{[(X \cap Y) \cap Z] \to [X \cap (Y \cap Z)]\} = \text{V}$.

.5 $[X \to -(-X)] = \text{V}$.

.6 $\{[(X \to Y) \cap (Y \to Z)] \to (X \to Z)\} = \text{V}$.

.7 $\{[X \cap (X \to Y)] \to Y\} = \text{V}$.

.8 $[{}^*(X \cap Y) \to {}^*X] = \text{V}$.

.9 $[{}^{**}X \leftrightarrow {}^*X] = \text{V}$.

Proof. We first notice that in any topological space we have $(X \to X) = \text{V}$. For $(X \to X) = -{}^*(X \cap -X) = -{}^*\Lambda = -\Lambda = \text{V}$.

From this it follows that parts .1, .3, .4, and .5 of our theorem are true. For $X \cap Y = Y \cap X$, $X = X \cap X$, and $(X \cap Y) \cap Z = X \cap (Y \cap Z)$, and $X = -(-X)$.

We next notice that in any topological space we have $(X \leftrightarrow X) = \text{V}$. For $(X \leftrightarrow X) = (X \to X) \cap (X \to X) = \text{V} \cap \text{V} = \text{V}$.

From this it follows that part .9 of our theorem is true. For in a topological space we have ${}^{**}X = {}^*X$.

We next notice that if $X \subset Y$, then $(X \to Y) = \text{V}$. For, if $X \subset Y$, then $X \cap -Y = \Lambda$, so that $(X \to Y) = -{}^*(X \cap -Y) = -{}^*\Lambda = -\Lambda = \text{V}$.

From this it follows that part .2 of our theorem is true. For $X \cap Y \subset X$.

We next notice that, if $X \subset Y$, then ${}^*X \subset {}^*Y$. For if $X \subset Y$ then $Y = X \cup Y$, so that ${}^*Y = {}^*(X \cup Y) = {}^*X \cup {}^*Y$, and hence ${}^*X \subset {}^*Y$.

To see that part .6 of our theorem is true, we notice that $(X \cap -Z) \subset (X \cap -Y) \cup (Y \cap -Z)$, so that ${}^*(X \cap -Z) \subset {}^*[(X \cap -Y) \cup (Y \cap -Z)]$, and hence ${}^*(X \cap -Z) \subset {}^*(X \cap -Y) \cup {}^*(Y \cap -Z)$, or $-[{}^*(X \cap -Y) \cup {}^*(Y \cap -Z)] \subset -{}^*(X \cap -Z)$. Hence by De Morgan's law, $[-{}^*(X \cap -Y) \cap -{}^*(Y \cap -Z)] \subset -{}^*(X \cap -Z)$, or $[(X \to Y) \cap (Y \to Z)] \subset (X \to Z)$. Hence we have $[(X \to Y) \cap (Y \to Z)] \to (X \to Z) = \text{V}$, as was to be shown.

To see that .7 is true, we notice that $-Y \subset -X \cup (X \cap -Y)$ and that $(X \cap -Y) \subset {}^*(X \cap -Y)$. Hence $-Y \subset -X \cup {}^*(X \cap -Y)$. Therefore $X \cap -{}^*(X \cap -Y) \subset Y$, or $[X \cap (X \to Y)] \subset Y$, or finally $[X \cap (X \to Y)] \to Y = \text{V}$, as was to be shown.

To see that .8 is true, we notice that $X \cap Y \subset X$, so that ${}^*(X \cap Y) \subset {}^*X$, or $[{}^*(X \cap Y) \to {}^*X] = \text{V}$, as was to be shown.

THEOREM 15. Let α be a sentence of S4, and let A be the corresponding topological expression. Then, if α is provable in S4, the equation $A = \text{V}$ is true for every topological space in the wider sense.

Proof. I shall prove this theorem by a mathematical induction on the number, n, of applications of Lewis's rules in the proof of α.

If $n=0$, then α is a primitive sentence of S4. But we see from Theorem 14 that the topological formulas, corresponding to the primitive sentences of S4, are true in every topological space.

Suppose that the topological formula corresponding to any sentence of S4 which is provable in k steps, is true in every topological space. I shall show that the topological formula corresponding to any sentence of S4 which is provable in $k+1$ steps, is true in every topological space.

Let α be a sentence of S4 which is provable in $k+1$ steps. Then there are just four possible cases: (1) there is a β and a γ of S4, which are provable in k steps, and such that $\alpha=\beta\cdot\gamma$; (2) there is a β such that β and $\beta ⥽ \alpha$ are provable in k steps; (3) there is a β which is provable in k steps, and which is such that α results from β by a substitution; (4) there is a β, a γ, and a δ such that β and $\gamma\equiv\delta$ are provable in k steps, and such that α results from β by replacing γ by δ.

In what follows, let A, B, C, and D be the topological expressions corresponding respectively to α, β, γ, and δ.

In the first case, since β and γ are provable in k steps, we see by the induction hypothesis that the equations $B=V$ and $C=V$ are true in every topological space. Hence the equation $B\cap C = V$ is true in every topological space. But clearly $B\cap C = A$. Hence $A=V$ is true in every topological space, as was to be shown.

In the second case, we see by the induction hypothesis that the equations $B=V$ and $(B\rightarrow A)=V$ are true in every topological space. Since $B\rightarrow A = V$, we have $-{*}(B\cap A) = V$, and hence ${*}(B\cap -A) = \Lambda$. But $(B\cap -A) \subset {*}(B\cap -A)$, so that $B\cap -A = \Lambda$, and hence $B\subset A$. Since $B=V$, we therefore see that $A=V$.

In the third case, we see that A results from B by a substitution. Hence, since by the induction hypothesis $B=V$, we see that $A=V$.

In the fourth case, we see that the equations $B=V$ and $(C\leftrightarrow D)=V$ are true in every topological space. Since $(C\leftrightarrow D) = (C\rightarrow D)\cap(D\rightarrow C)$, we see that $C\rightarrow D = V$ and $D\rightarrow C = V$. Thus $-{*}(C\cap -D) = V$, so that ${*}(C\cap -D) = \Lambda$, and hence $C\cap -D = \Lambda$, or $C\subset D$. In a similar way we have $D\subset C$. Hence $C=D$. Since α results from β by replacing γ by δ, we see that A results from B by replacing C by D. Hence, since $B=V$ and $C=D$, we have $A=V$, as was to be shown.

This completes the proof of our theorem.

THEOREM 16. Let α be a sentence of S4, and let A be the corresponding topological expression. Then, if α is not provable in S4, there exists a finite topological space in the wider sense, for which $A=V$ is not true.

Proof. Since α is not provable in S4, we see by Theorem 13 that there exists a finite normal S4-matrix $\mathfrak{M} = (K, D, -, {*}, \times)$ which does not satisfy α.

We now define a new finite matrix $\mathfrak{M}_1 = (K_1, D_1, -_1, {*}_1, \times_1)$ as follows. Let K_1 be the class of all sets of atoms of K. We define D_1 to be that subclass of K_1 such that $\{p_1, \cdots, p_n\}$ belongs to D_1 if and only if $p_1+\cdots+p_n$ is an element of D. If x is any element of K_1, let $-_1\, x$ be the complement of x. If x and y are any elements of K_1, let $x \times_1 y$ be the intersection of x and y. If

$\{p_1, \cdots, p_m\}$ and $\{q_1, \cdots, q_n\}$ are any elements of K_1, we set $*_1\{p_1, \cdots, p_m\} = \{q_1, \cdots, q_n\}$ if and only if $*(p_1 + \cdots + p_m) = q_1 + \cdots + q_n$.

It is easily seen that $\mathfrak{M}_1$ is isomorphic to $\mathfrak{M}$, and hence that $\mathfrak{M}_1$ does not satisfy the sentence α.

If V is the Boolean unit-element of K_1, we see by Theorems 10 and 9.2 that V is a topological space in the wider sense with respect to $*_1$.

Since $\mathfrak{M}_1$ does not satisfy α, there are elements $x_1, \cdots, x_r$ of K_1 such that, when the sentential variables in α are replaced by $x_1, \cdots, x_r$, and the symbols "$\sim$", "$\Diamond$", and "$\cdot$" in α are replaced respectively by "$-_1$", "$*_1$", and "$\times_1$", an element is obtained which is not in D_1. Let z be this element. Since, by Theorem 2.8, $V \epsilon D_1$, we see that $z \neq V$. Thus we see that, when the set-variables in the equation $A = V$ are replaced by $x_1, \cdots, x_r$, a false statement is obtained. Thus the equation $A = V$ is not true in the finite topological space $(K_1, *_1)$, as was to be shown.

THEOREM 17. Let α be a sentence of S4, and let A be the corresponding topological expression. Then a necessary and sufficient condition that the equation $A = V$ be true for every topological space in the wider sense, is that α be provable in S4.

Proof. The condition is necessary by Theorem 16, and sufficient by Theorem 15.

As mentioned earlier, the theorem just proved provides a decision procedure for topological equations. The theorem has also some other interesting consequences; in order to bring these to light, it is convenient to prove the following corollary.

THEOREM 18. Let α and β be sentences of S4, and let A and B be the corresponding topological expressions. Then a necessary and sufficient condition that the inclusion $A \subset B$ be true for every topological space in the wider sense is that $\alpha \prec \beta$ be provable in S4. And a necessary and sufficient condition that the equation $A = B$ be true for every topological space in the wider sense, is that $\alpha \equiv \beta$ be provable in S4.

Proof. To say that $A \subset B$, is equivalent to saying that $A \cap -B = \Lambda$, which is in turn equivalent to saying that $*(A \cap -B) = \Lambda$, and hence to $-*(A \cap -B) = V$. By Theorem 17, this last is equivalent to saying that $\sim\Diamond(\alpha \cdot \sim\beta)$ is provable in S4, or that $\alpha \prec \beta$ is provable in S4.

To say that $A = B$, is equivalent to saying that both $A \subset B$ and $B \subset A$, and hence, by the first paragraph, to saying that both $\alpha \prec \beta$ and $\beta \prec \alpha$ are provable in S4. But to say that both $\alpha \prec \beta$ and $\beta \prec \alpha$ are provable in S4 is equivalent to saying that $\alpha \equiv \beta$ is provable in S4.

From Theorem 18 we see that the results obtained by Parry[15] regarding the number of modalities in S4, are equivalent to the results obtained by Kuratowski[16] regarding the number of distinct functions definable in topology by means of the operations of complement and closure. Thus there are fourteen

[15] See W. T. Parry, *Modalities in the Survey system of strict implication*, this JOURNAL, vol. 4 (1939), pp. 137–154.

[16] Loc. cit., page 186.

modalities in S4, just as there are fourteen distinct functions of the given sort definable in topology. And the relations of implication holding among the fourteen modalities are, of course, parallel to the relations of inclusion holding among the fourteen functions.

It is also seen from Kuratowski's results that there are an infinite number of distinct modal functions of one variable in S4. For Kuratowski shows by an example that if we set $\phi(A) = A \cap *(*A \cap -A)$, then no two members of the infinite sequence of functions, $\phi(A)$, $\phi(\phi(A))$, $\phi(\phi(\phi(A)))$, $\cdots$, are identical. From this we conclude, by means of Theorem 18, that there is provable in S4 no equivalence between two members of the following sequence of sentences:

$$\phi(p),\ \phi(\phi(p)),\ \phi(\phi(\phi(p))),\ \cdots, \quad \text{where} \quad \phi(p) = p \cdot \Diamond(\Diamond p \cdot \sim p).$$

It follows *a fortiori*, that there are also an infinite number of modal functions of one variable in the weaker systems S3, S2, and S1.

It is easily seen that, if there exists a finite characteristic matrix for a system of sentential calculus, then there are also only a finite number of non-equivalent functions of one variable in the system. Thus the result of the preceding paragraph implies that there is no finite characteristic matrix for any one of the systems S4, S3, S2, and S1; this result has previously been obtained by Dugundji[17] in another way.

I shall now prove a theorem which formulates a decision procedure for topological equations independently of the Lewis calculus.

THEOREM 19. Let A be a topological expression which contains just r topological sub-expressions. Then a necessary and sufficient condition that the equation $A = \mathrm{V}$ be true in every topological space (in the wider sense) is that it be true in every finite topological space (in the wider sense) which contains at most 2^r points.

Proof. If $A = \mathrm{V}$ is true in every topological space, then clearly it is true in every finite topological space which contains at most 2^r elements.

If $A = \mathrm{V}$ is not true in every topological space, then, by Theorem 17, α is not provable in S4, where α is the sentence corresponding to A. By Theorem 13 we then see that there is a normal S4-matrix $\mathfrak{M} = (K, D, -, *, \times)$, with at most 2^{2^r} elements, which does not satisfy α. As in the proof of Theorem 16, we see that V is a topological space with respect to *, and that the equation $A = \mathrm{V}$ is not true in this space. Since K contains at most 2^{2^r} elements, it is seen that V contains at most 2^r elements. Thus the equation $A = \mathrm{V}$ is false for some topological space with not more than 2^r elements, as was to be shown.

I shall now show that my results regarding topology are still true when a certain additional postulate is assumed.

I shall take a *topological space in the narrower sense* to be a topological space (in the wider sense) which satisfies the additional postulate:

T7 If α is a finite subset of S, then $*\alpha = \alpha$.

The following theorem can be regarded as a lemma for Theorem 21.

THEOREM 20. Let S be a finite set which is a topological space in the wider sense with respect to *. Then there exists an infinite set S_1, and an operation $*_1$, such that S_1 is a topological space in the narrower sense with respect to $*_1$,

[17] Loc. cit.

and such that there is an isomorphism between the set of subsets of S and a certain collection of subsets of S_1.

Proof. Let $\alpha_1, \cdots, \alpha_n$ be the subsets of S which contain just one element. Let $B_1, \cdots, B_n$ be a sequence of n mutually exclusive infinite sets. Let $S_1 = B_1 \cup \cdots \cup B_n$.

We now define a closure operation $*_1$ over S_1 as follows:

(A) If γ is a finite set such that, for some $i \leqq n$, we have $\gamma \subset B_i$, then we set $*_1 \gamma = \gamma$.

(B) If γ is an infinite set such that, for some $i \leqq n$, we have $\gamma \subset B_i$, and if $*\alpha_i = \alpha_{i_1} \cup \cdots \cup \alpha_{i_r}$, then we set $*_1 \gamma = B_{i_1} \cup \cdots \cup B_{i_r}$.

(C) If γ is a set which is not contained in any B_i for $i \leqq n$, then we set $*_1 \gamma = *_1(\gamma \cap B_1) \cup *_1(\gamma \cap B_2) \cup \cdots \cup *_1(\gamma \cap B_n)$.

It can now be shown that S_1 is a topological space in the narrower sense. Moreover, if we let each subset $\alpha_{i_1} \cup \cdots \cup \alpha_{i_r}$ of S correspond to the subset $B_{i_1} \cup \cdots \cup B_{i_r}$ of S_1, then an isomorphism is established between the set of all subsets of S, and a certain set of subsets of S_1. I shall omit this proof, however, since it is long without being especially difficult.

The following theorem provides us, in view of Theorem 19, with a method for deciding whether an arbitrary given equation is true in every topological space in the narrower sense.

THEOREM 21. A necessary and sufficient condition that a topological equation be true in every topological space in the narrower sense, is that it be true in every topological space in the wider sense.

Proof. If an equation is true in every topological space in the wider sense, then clearly it is also true in every topological space in the narrower sense, since, by definition, a topological space in the narrower sense is also a topological space in the wider sense.

Suppose, on the other hand, that an equation is not true in every topological space in the wider sense. Then, by Theorem 19, it is false for some finite topological space in the wider sense. By Theorem 20 we then see that it is false in some (infinite) topological space in the narrower sense, as was to be shown.

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