Matching in General Graphs: Theoretical Foundations

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Outline

- Motivation
- Basic Definitions
- Maximum Matchings
- Perfect Matchings
- Concluding Remarks

Motivation

Our goal is to prove the following theorem (by Petersen):

Theorem. Every bridgeless cubic graph has a 1-factor.

This theorem ensures that the dual graph of a 4-8 *surface* mesh, which is a bridgeless cubic graph, admits a perfect matching .

Basic Definitions

From now on, we let $G = (V, E)$ (or simply G) denote a *simple* graph with vertex set $V = V(G)$ and edge set $E = E(G)$.

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. If $V' \subseteq V$ and $E' \subseteq E$ then G' is a **subgraph** of G (and G is a **supergraph** of G'), written as $G' \subseteq G$. Less formally, we say G **contains** G'.

Basic Definitions

If $G' \subseteq G$ and G' contains *all* edges $xy \in E$ with $x, y \in V'$, then G' is a **vertex-induced subgraph** of G. We say that V' **induces** or **spans** G' in G, and write $G' = G[V']$. In general, for any $U \subseteq V$, $G[U]$ denotes the graph on U whose edges are precisely the edges of G with both ends in U.

Let E' be a non-empty subset of E. Then, the subgraph $G' = (V', E')$, where $V' \subseteq V$ is exactly the set of end vertices of the edges in E', is a called **edge-induced subgraph** of G. We denote the subgraph G' induced by the set E' by $G[E']$.

Basic Definitions

If H is a subgraph of G (*induced* or *not*), we denote H by $G[H]$.

If $G' \subseteq G$, then G' is a **spanning** subgraph of G if V' spans all of G, i.e., if $V' = V$ (note that G' does not have to be *induced*).

Basic Definitions

A **path** is a non-empty graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, \ldots, x_k\} \quad \text{and} \quad E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\},$$

where the x_i are all distinct. The vertices x_0 and x_k are **linked** by P and are called its **ends**; the vertices $x_1, \ldots, x_{k-1}$ are the **inner** vertices of P. The number of edges of a path is its **length**. We often refer to a path by the natural sequence of its vertices, say $P = x_0x_1 \cdots x_k$, and call P a **path from** x_0 **to** x_k (as well as **between** x_0 **and** x_k).

Basic Definitions

If $P = x_0 x_1 \cdots x_{k-1}$ is a path and $k \geq 3$, then the graph $C = P + x_{k-1}x_0$ is called a **cycle**. As with paths, we often denote a cycle by its cyclic sequence of vertices (for instance, the above cycle C can be written as the sequence $x_0 \cdots x_{k-1}x_0$).

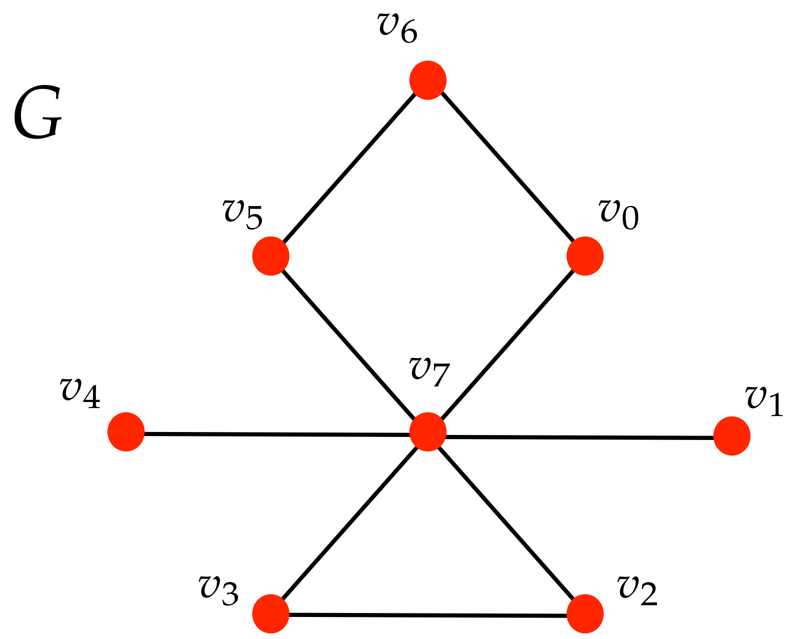
The **length** of a cycle is its number of edges (or vertices).

The cycle of length k is called a ***k*-cycle**.

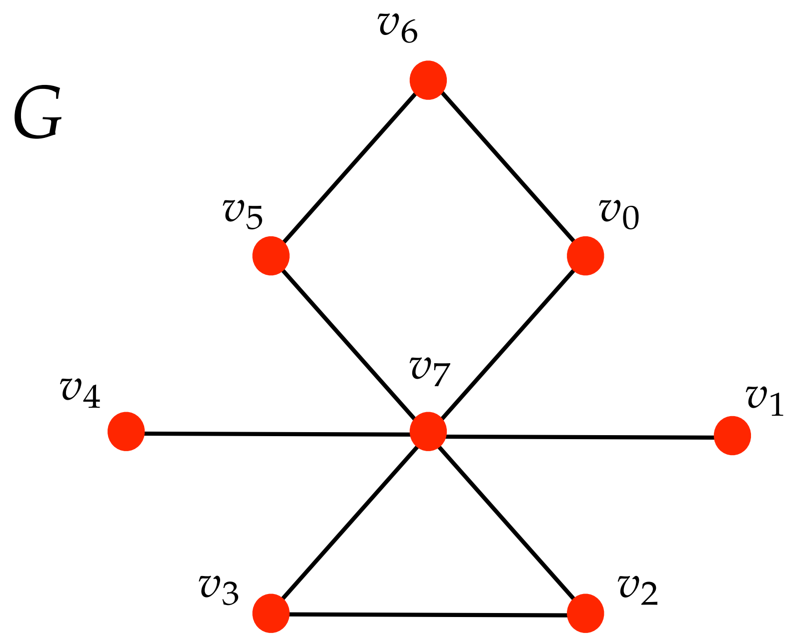
Basic Definitions

A subset M of E is called a **matching** if no two edges of M are adjacent, i.e., if no two edges of M have a vertex in common.

Basic Definitions

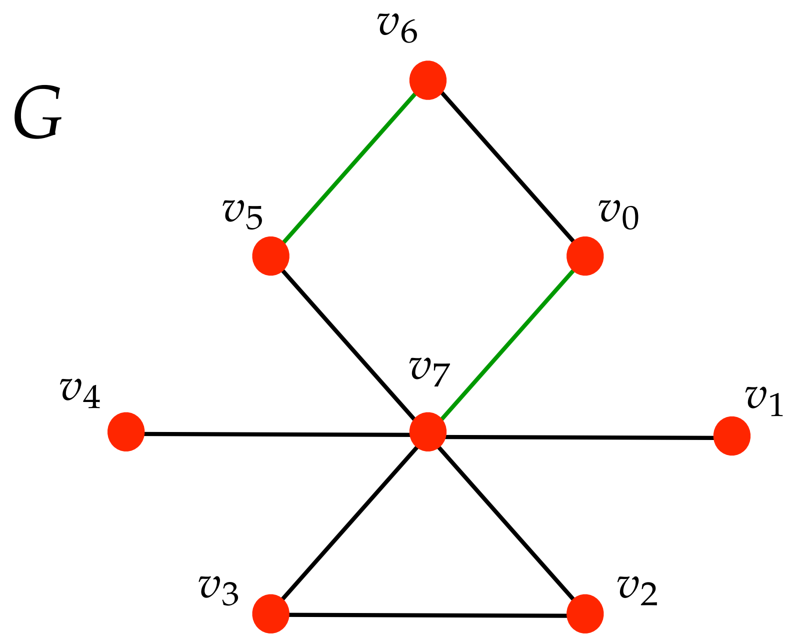


Basic Definitions



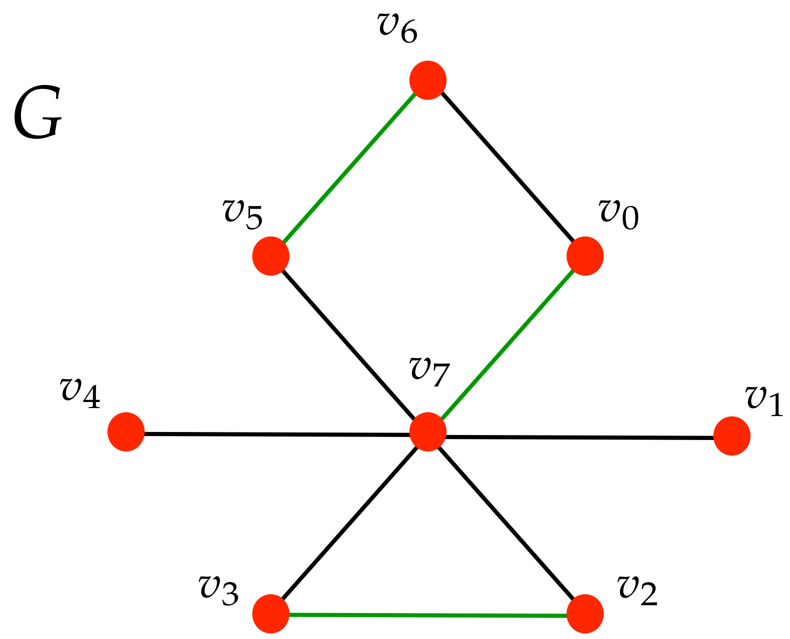
$$M = \emptyset$$

Basic Definitions



$$M = \{\{v_5, v_6\}, \{v_0, v_7\}\}$$

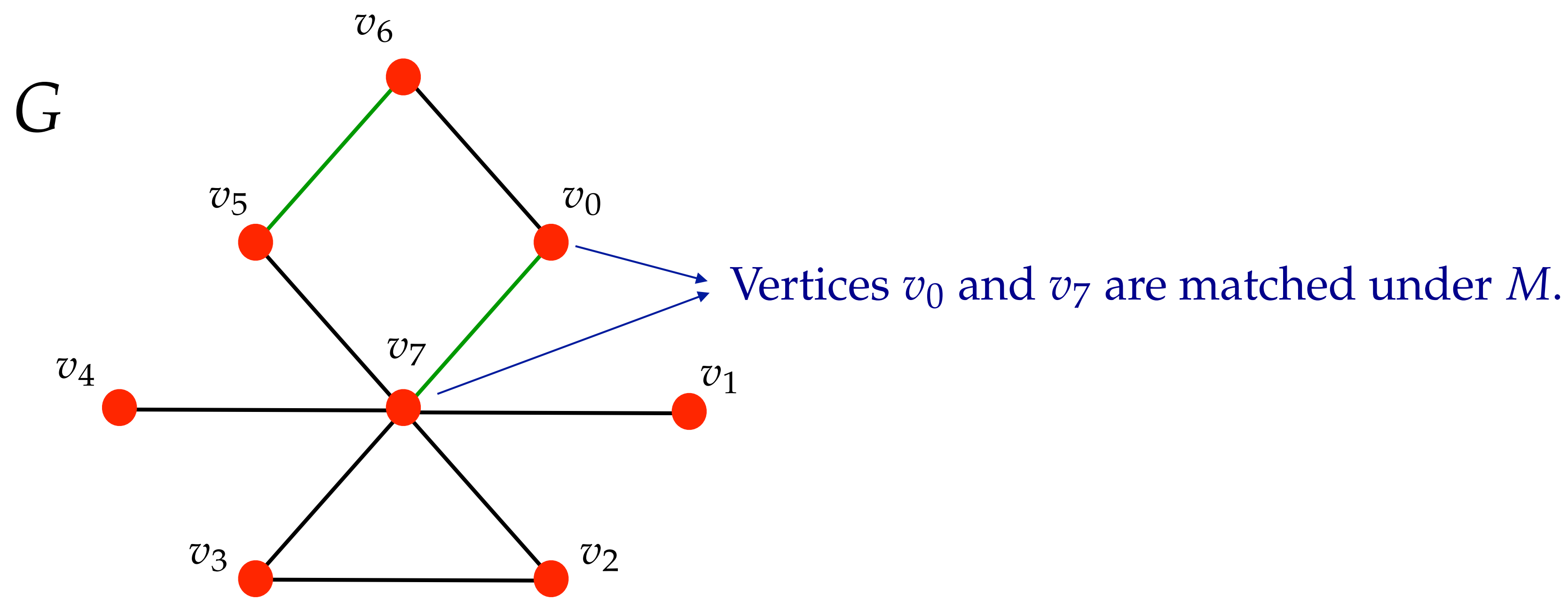
Basic Definitions



$$M = \{\{v_5, v_6\}, \{v_0, v_7\}, \{v_2, v_3\}\}$$

Basic Definitions

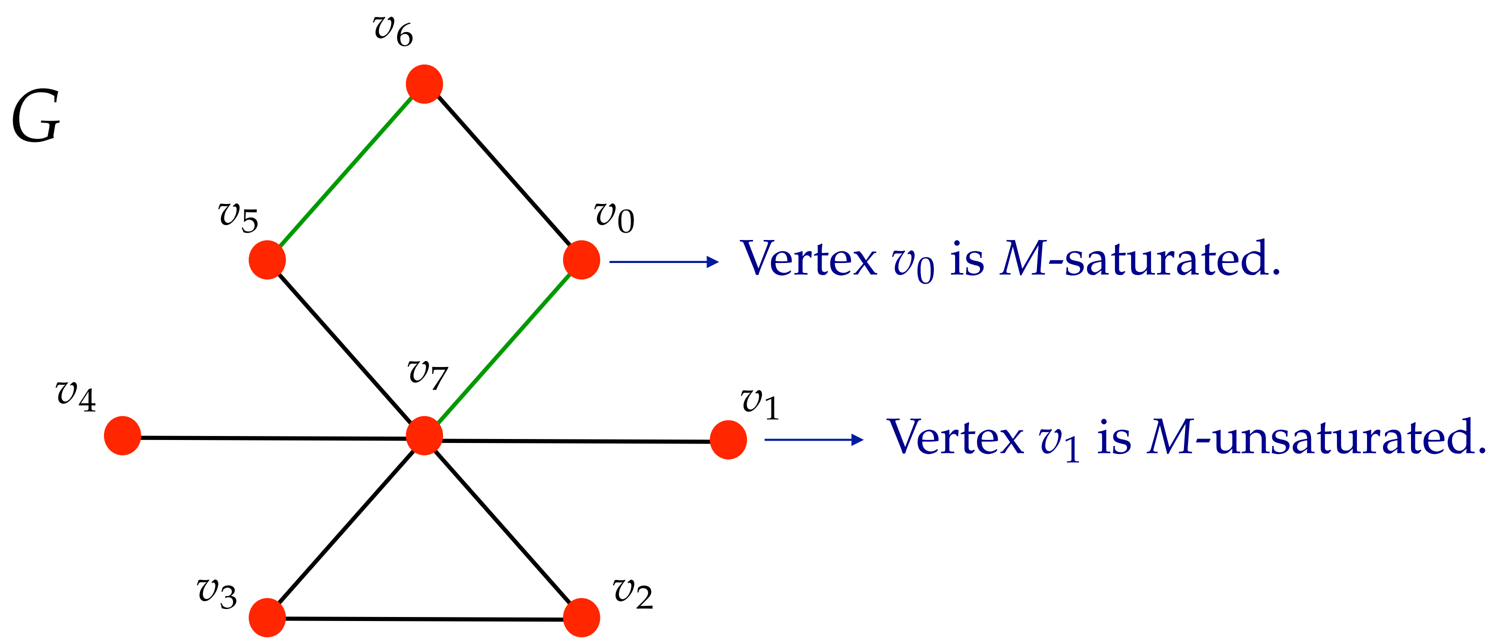
The two ends of an edge in M are said to be **matched under** M.



$M = \{\{v_5, v_6\}, \{v_0, v_7\}\}$

Basic Definitions

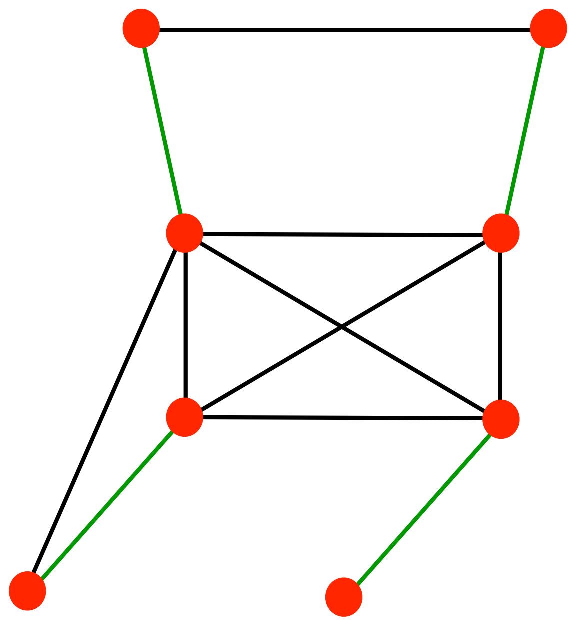
A matching M **saturates** a vertex v, and v is said to be M-**saturated**, if some edge of M is incident with v; otherwise, v is M-**unsaturated**.



$M = \{\{v_5, v_6\}, \{v_0, v_7\}\}$

Basic Definitions

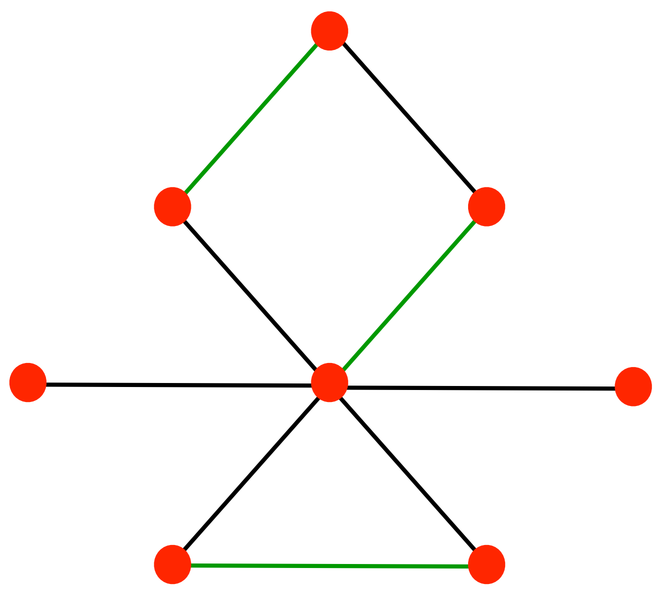
If every vertex is *M*-saturated, the matching is **perfect**.



A perfect matching

Basic Definitions

M is a **maximum matching** if G has no matching M' with $|M'| > |M|$.

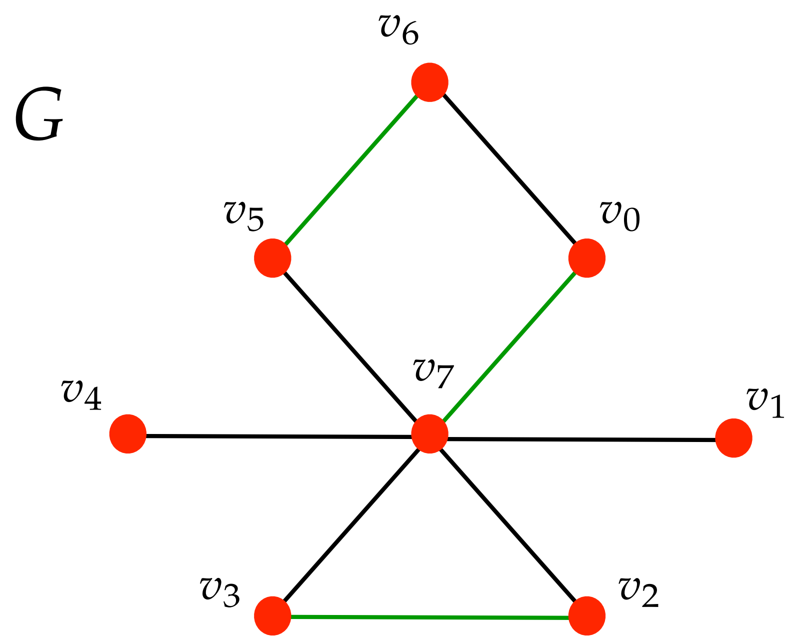


A maximum matching

Clearly, every perfect matching is a maximum matching.

Maximum Matchings

Let M be a matching in G. Then, an M**-alternating path in** G is a path whose edges are alternately in the sets $E - M$ and M.

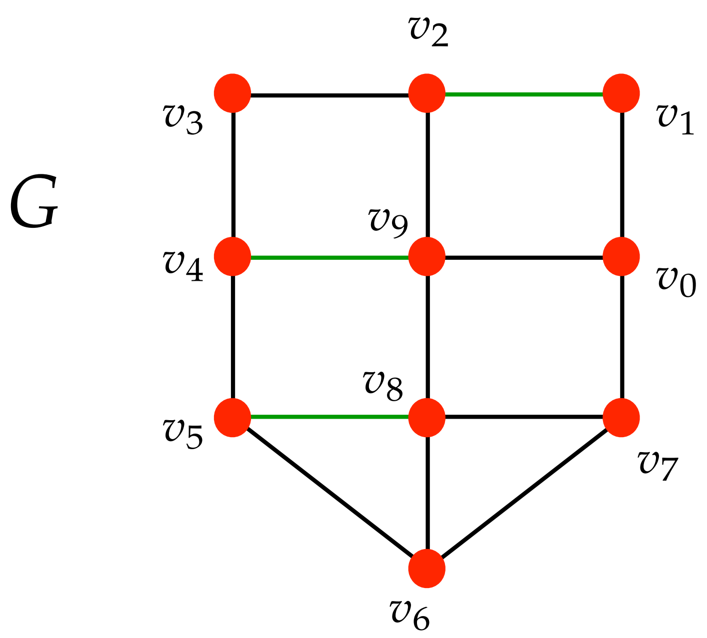


$$M = \{\{v_5, v_6\}, \{v_0, v_7\}, \{v_2, v_3\}\}$$

$v_4 v_7 v_0 v_6 v_5$ is an M-alternating path.

Maximum Matchings

An M**-augmenting path in** G is an M-alternating path in G whose origin and terminus vertices are M-unsaturated.



$v_0v_1v_2v_9v_4v_5v_8v_7$ is an M-augmenting path.

$M = \{\{v_1, v_2\}, \{v_4, v_9\}, \{v_5, v_8\}\}$

Maximum Matchings

Lemma 1

Let M_1 and M_2 be two matchings in a graph G, and let H be the subgraph of G induced by the set of edges

$$M_1 \ominus M_2 = (M_1 - M_2) \cup (M_2 - M_1),$$

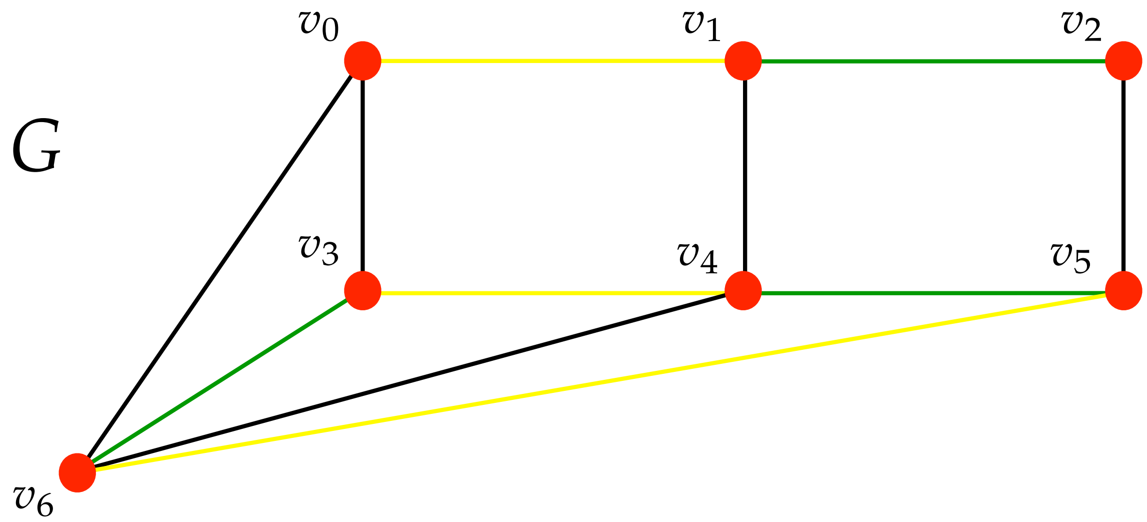
Then each connected component of H is of one of the following two types:

(1) a cycle of even length whose edges are alternately in M_1 and M_2,

(2) a path whose edges are alternately in M_1 and M_2 and whose end vertices are unsaturated in one of the two matchings.

Maximum Matchings

Example:

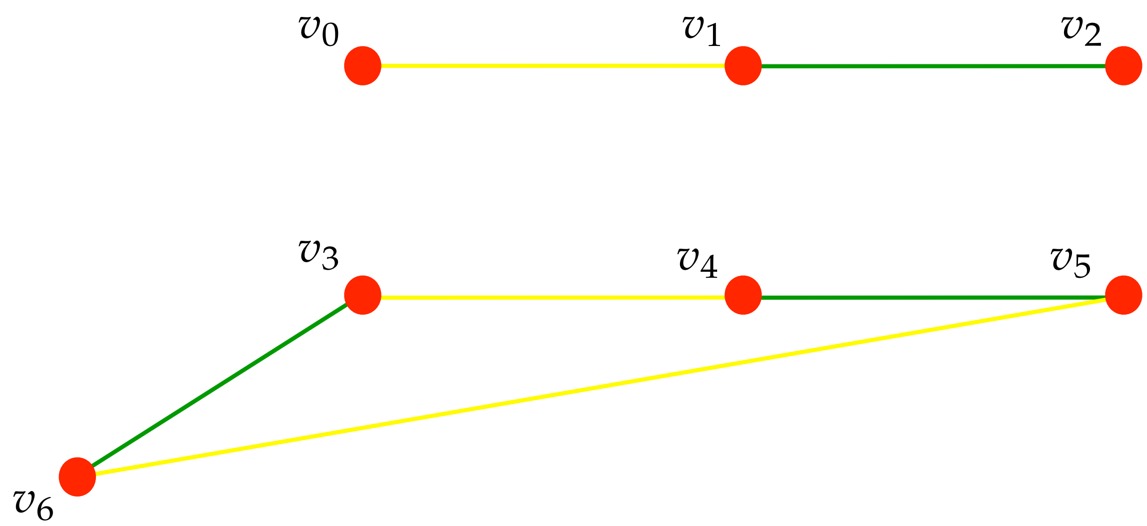


$M_1 = \{\{v_1, v_2\}, \{v_4, v_5\}, \{v_3, v_6\}\}$ and $M_2 = \{\{v_0, v_1\}, \{v_3, v_4\}, \{v_5, v_6\}\}$

$$M_1 \ominus M_2 = \{\{v_1, v_2\}, \{v_4, v_5\}, \{v_3, v_6\}, \{v_0, v_1\}, \{v_3, v_4\}, \{v_5, v_6\}\}$$

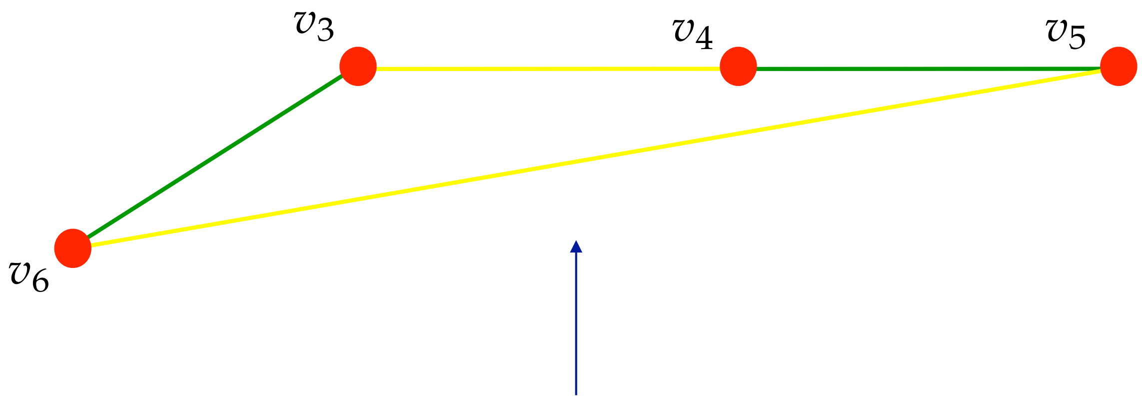
Maximum Matchings

The subgraph H:



Maximum Matchings

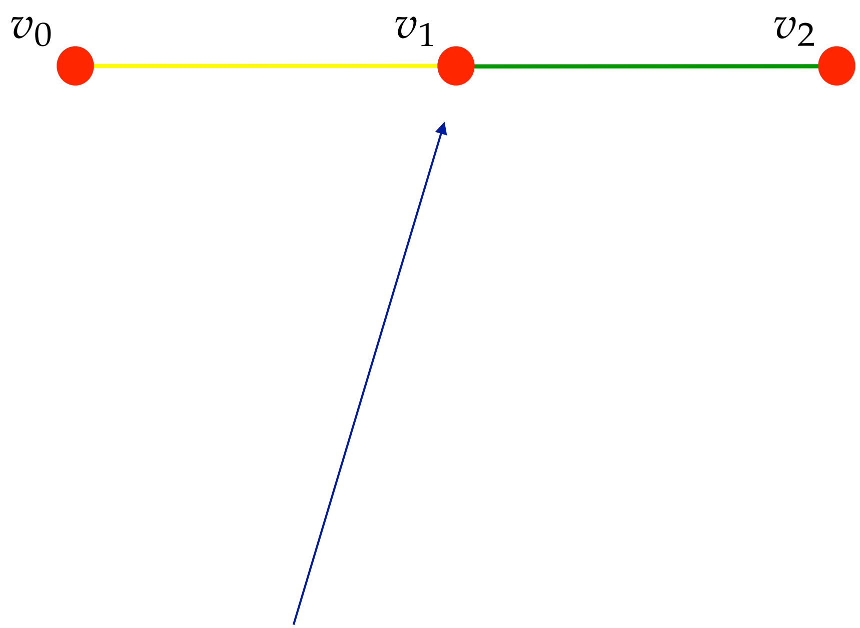
The subgraph H:



A cycle of even length whose edges are alternately in M_1 and M_2.

Maximum Matchings

The subgraph H:



a path whose edges are alternately in M_1 and M_2 and whose end vertices are unsaturated in one of the two matchings (in this example, vertex v_0 is M_1-unsaturated and vertex v_2 is M_2-unsaturated).

Maximum Matchings

Proof:

Let v be any vertex of H. Then either

(a) v is an end vertex of an edge in $M_1 - M_2$ and also of an edge in $M_2 - M_1$, or

(b) v is an end vertex of an edge in one of $M_1 - M_2$ and $M_2 - M_1$ but not both.

In case (a), vertex v has degree 2 in H.

Why?

In case (b), vertex v has degree 1 in H.

So, every vertex of H has either degree 1 or 2.

Maximum Matchings

proof?

Key observation:

If G is a connected graph in which every vertex has degree either 1 or 2 then G is either a path or a cycle. So, every component of H is either a path or a cycle.

Since no two adjacent edges of H can belong to the same matching (otherwise, it would not be a matching), the edges of every cycle or path in H alternate in M_1 and M_2.

Consequently, cycles must have even length.

If v is the end of a path in H, then v has degree 1 in H. So, there is exactly one edge, say e, in $M_1 \ominus M_2$ whose one end vertex is v. Without loss of generality, assume that $e \in M_1$. Since $e \in M_1 \ominus M_2$, we know that $e \notin M_2$. Furthermore, there is no edge $e' \in M_2$, with $e' \neq e$, such that v is an end of e'. This is because $e' \notin M_1$ (as M_1 is a matching). But, this means that $e' \in M_1 \ominus M_2$, and thus vertex v would have degree 2, which contradicts the fact that v is the end of a path in H.

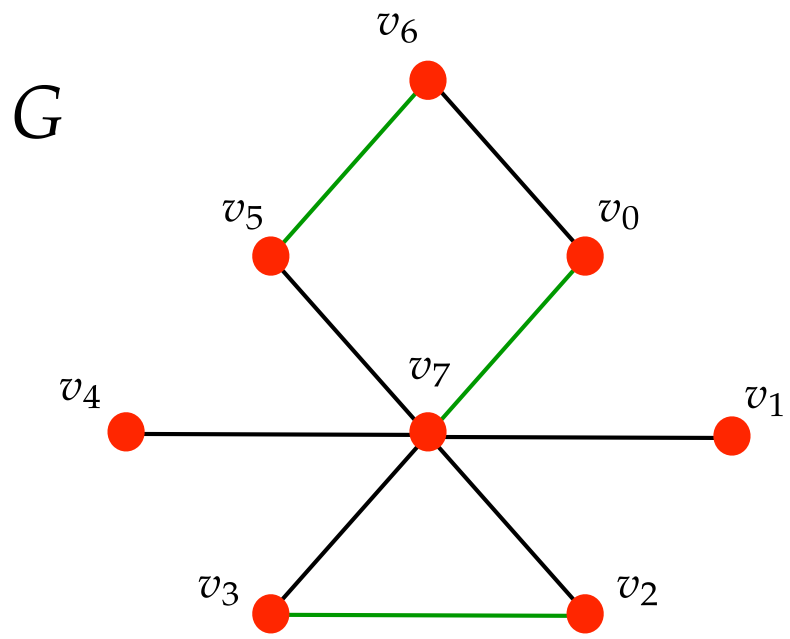
Maximum Matchings

Theorem 1 [Claude Berge, 1957]

A matching M in a graph G is a maximum matching if and only if G contains no M-augmenting path.

Maximum Matchings

Example:



$$M = \{\{v_5, v_6\}, \{v_0, v_7\}, \{v_2, v_3\}\}$$

Maximum Matchings

The **transfer along the augmenting path** strategy

Let M be any matching in the graph G.

Refer to the edges in M as **dark** edges and the edges in $E - M$ as **light** edges.

Let P be an M-alternating path in G.

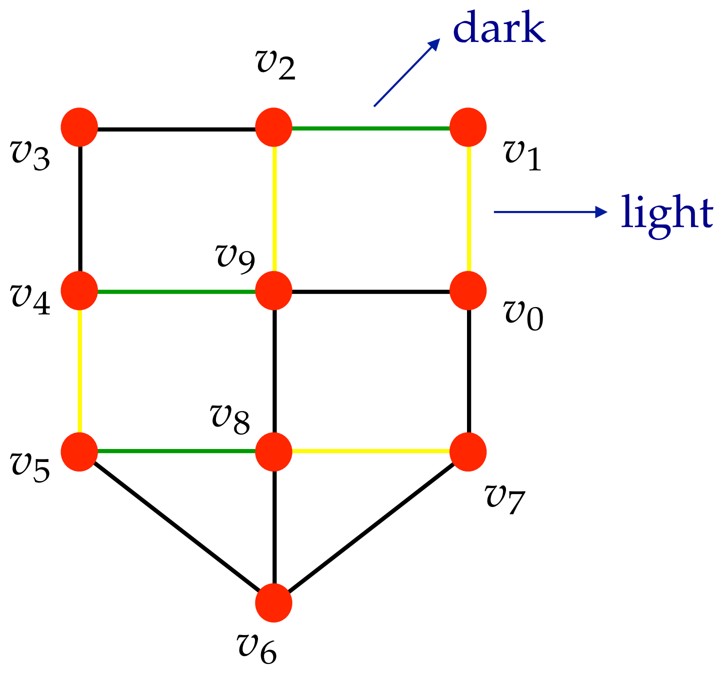
So, the path P alternates dark and light edges.

If we further assume that path P is M-augmenting, i.e., the origin and terminus of P are not M-saturated, then the first and last edge of P must be light edges.

Maximum Matchings

So, the sequence of edges of P is of the form

$$\text{light, dark, light, } \ldots, \text{ dark, light}$$



$\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_9\}, \{v_9, v_4\}, \{v_4, v_5\}, \{v_5, v_8\}, \{v_8, v_7\}$

Maximum Matchings

So, P has an odd number of edges, say $2m + 1$, m of which are dark and $m + 1$ are light.

By assumption, the origin and terminus of P are M-unsaturated.

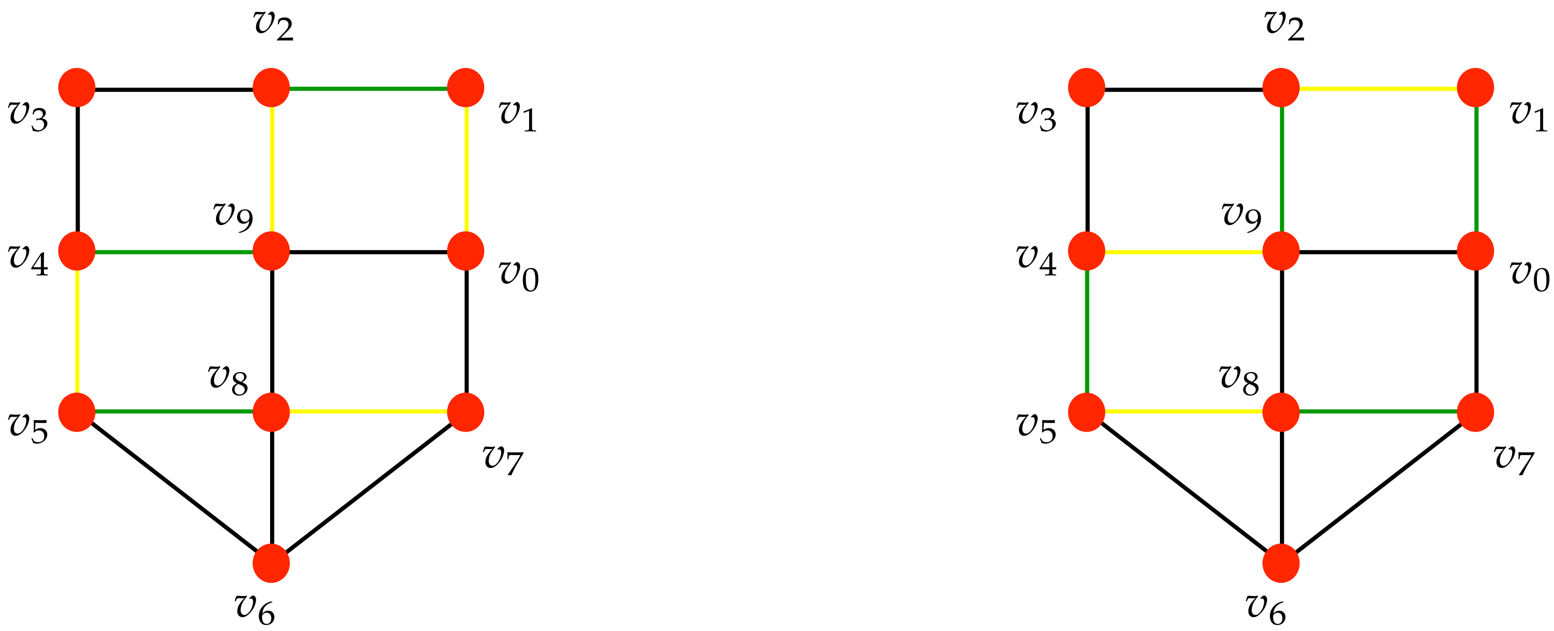
All other vertices of P are M-saturated (by the dark edges).

So, any edge of M that is not in P is not incident to any vertex of P.

Let M' be the set of all dark edges of M not in P and also all light edges of P. We have that M' is a new matching in G with one more edge than matching M.

Maximum Matchings

The **transfer along the augmenting path** strategy



Maximum Matchings

Proof:

(Only if)

Let M be a maximum matching in G.

If there is an M-augmenting path P in G, then we can transfer along P to produce a new matching M' in G which has one more edge then M has. But, this is impossible since M is maximum. So, graph G has no M-augmenting path.

Maximum Matchings

(If)

Suppose that M is a matching in G such that there is no M-augmenting path in G.

We wish to show that M is a maximum matching.

Let M' be any maximum matching of G. Then, we wish to show that

$$|M| = |M'| .$$

Let H be the subgraph induced by the set of edges

$$M \ominus M' = (M - M') \cup (M' - M) .$$

Maximum Matchings

By Lemma 1, the connected components of H are either

(a) cycles of even length whose edges are alternately in M and M', or

(b) paths whose edges are alternately in M and M' and whose end vertices are unsaturated in one of the two matchings.

Since all cycles of H have even length, each such a cycle has the same number of edges from M and M'. The same can be said of each path in H of even length.

So, let us consider a path in H with odd length.

Since the length of the path is odd, the origin and terminus of the path are both M-unsaturated or both M'-unsaturated, as the first and last edges of the path must belong to the same matching. So, the path is either M-augmenting or M'-augmenting.

Maximum Matchings

But, by assumption, matching M has no augmenting path.

Likewise, matching M' has no augmenting path either, as M' is maximum.

So, no component of H can have a path of odd length, which implies that all paths and cycles in H have even length. As a result, we must have $|M - M'| = |M' - M|$.

But,

$$|M| = |M - M'| + |M \cap M'| \quad \text{and} \quad |M'| = |M' - M| + |M' \cap M| \,.$$

So,

$$|M| = |M'| \,.$$

Perfect Matchings

We now turn our attention to perfect matchings.

Recall that a *non-empty* graph G is called **connected** if any two of its vertices are linked by a path in G. If $U \subseteq V(G)$ and $G[U]$ is connected, we also say that U is connected (in G).

A maximal connected subgraph of G is called a **component** of G.

A component, being connected, is always non-empty.

Perfect Matchings

Let G be a graph. Then, if all the vertices of G have the same degree k, then G is said to be ***k*-regular** or simply **regular**.

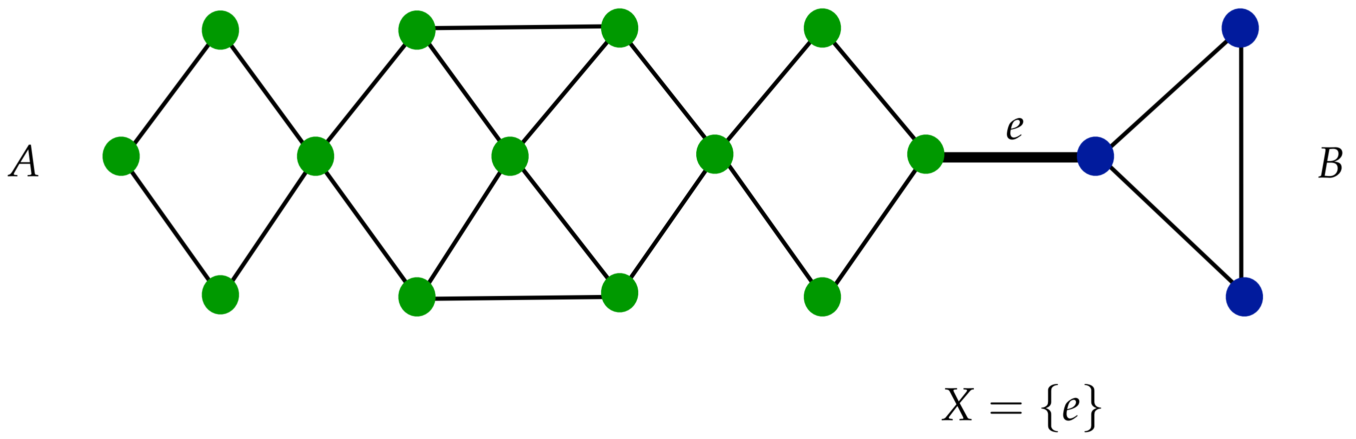
A 3-regular graph is called **cubic**.

A k-regular spanning subgraph is called a ***k*-factor**.

From the above definition of k-factor, we note that a subgraph $H \subseteq G$ is a 1-factor if and only if $E(H)$ is a matching of V.

Perfect Matchings

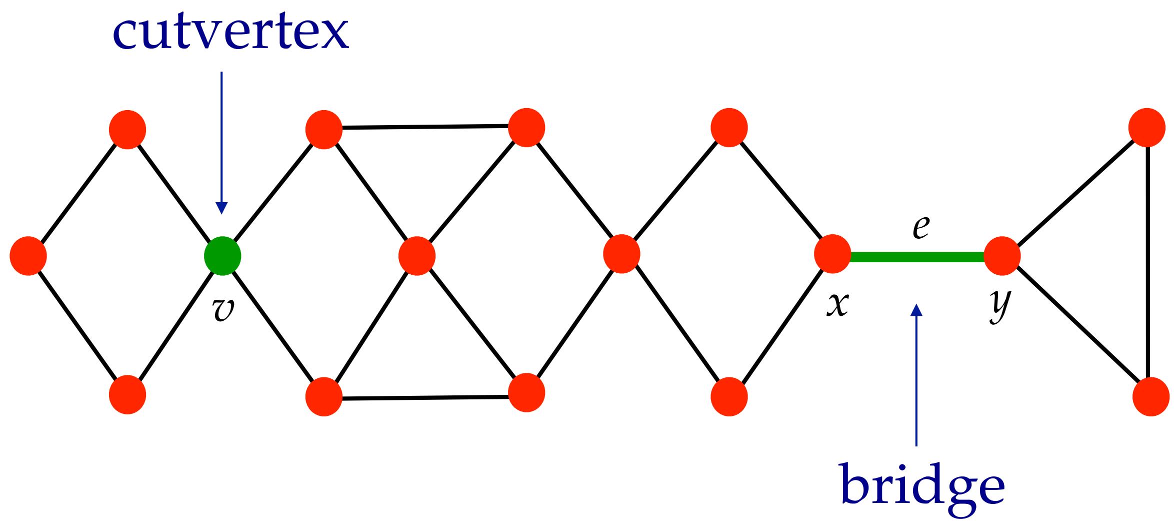
If $A, B \subseteq V$ e $X \subseteq V \cup E$ are such that every $A - B$ path in G (i.e., a path from a vertex in A to a vertex in B) contains a vertex or an edge from X, we say that X **separates** the sets A and B in G.



Perfect Matchings

More generally, we say that X **separates** G, and call X a **separating set** in G, if X separates two vertices of $G - X$ in G.

A vertex which separates two other vertices of the same component is a **cutvertex**, and an edge separating its ends is a **bridge**.



Perfect Matchings

Theorem 2 [Petersen, 1891]

Every bridgeless cubic graph has a 1-factor.

This theorem can be proved as corollary of a theorem by Tutte:

Theorem 3 [Tutte, 1947]

A graph G has a 1-factor if and only if $q(G - S) \leq |S|$, for all $S \subset V(G)$, where $q(G - S)$ denotes the number of components of the graph $G - S$ with an odd number of vertices.

Perfect Matchings

Proof (of Petersen's theorem):

We show that any bridgeless cubic graph G satisfies Tutte's condition.

Let S be any subset of $V(G)$.

If G has no component with an odd number of vertices, then $q(G - S) = 0$, and hence $q(G - S) \leq |S|$, for all $S \subseteq V$. So, Tutte's theorem tells that G has a 1-factor.

So, let us assume that $G - S$ has at least one odd component.

Let C be any odd component of $G - S$.

Perfect Matchings

Since G is cubic, the degrees (in G) of the vertices in C sum to an odd number, i.e.,

$$\sum_{v \in V(C)} d(v) = 3 \cdot |V(C)|$$

is odd, as $|V(C)|$ is odd. Because C is a component of $G - S$, every edge incident to a vertex in C is either an edge with both ends in $V(C)$ or with an end in $V(C)$ and the other one in S. So, the above sum also satisfies the following:

$$\sum_{v \in V(C)} d(v) = m_{SC} + 2 \cdot m_C \,,$$

where m_{SC} is the number of edges with an end in $V(C)$ and another in S, and m_C is the number of edges with both ends in $V(C)$. This means that m_{SC} is also odd.

Perfect Matchings

The previous remark implies that there is an odd number of edges with an end in S and another end in $V(C)$. Actually, there are at least 3 edges, as G is bridgeless!

So, the total number of edges with an end in S and another end in $G - S$ is at least $3 \cdot q(G - S)$. But, because G is cubic, this number is also no larger than $3 \cdot |S|$. So,

$$q(G - S) \leq |S| \,,$$

as required.

We have achieved our goal of proving Petersen's theorem using Tutte's theorem. But, for the sake of completeness, let us go over the proof of Tutte's theorem (*this is actually my excuse to show a beautiful proof given by Lovasz in 1973*).

Perfect Matchings

Theorem 3 [Tutte, 1947]

A graph G has a 1-factor if and only if $q(G-S) \leq |S|$, for all $S \subset V(G)$, where $q(G-S)$ denotes the number of components of the graph $G-S$ with an odd number of vertices.

Perfect Matchings

Before we go over the proof, we note that if G has a 1-factor then

- G must have an even number of vertices.
- G cannot have an isolated vertex (i.e., with degree 0).

In addtion, if G is a complete graph then G has a 1-factor whenever G has an even number of vertices, and a maximum matching of size $|V| - 1$ if its number of vertices is odd.

Why?

Perfect Matchings

Proof:

(Only if)

Suppose that G has a 1-factor.

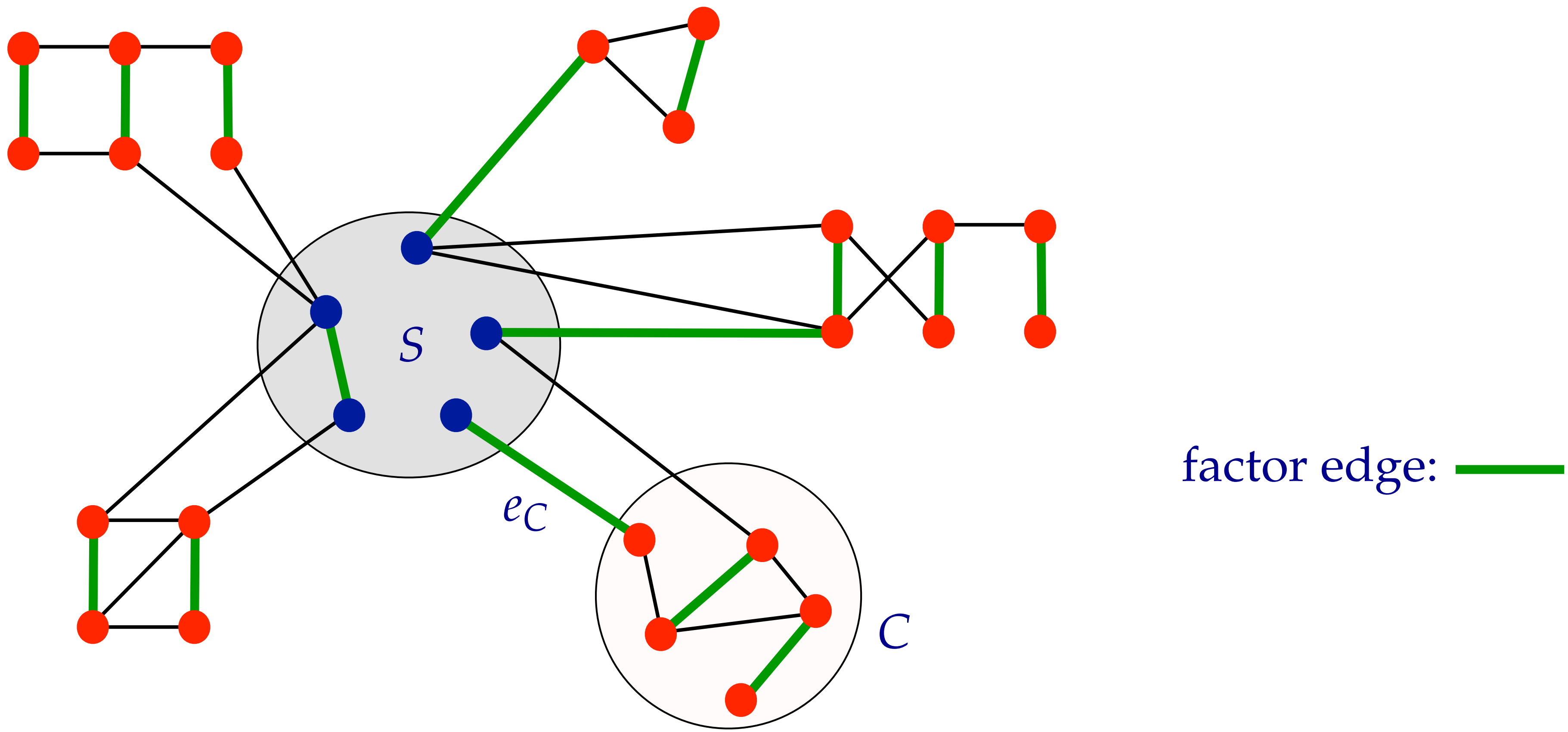
We wish to show that $q(G - S) \leq |S|$, for every $S \subseteq V$.

Let C be an odd component of $G - S$.

Since G has a 1-factor and since C has an odd number of components, any 1-factor of G will have exactly one edge connecting a vertex of C to a vertex of S.

Let e_C be such an edge.

Perfect Matchings



Perfect Matchings

Since e_C belongs to a 1-factor, there is no other edge (of the same 1-factor) that shares a vertex with e_C. So, the $|S|$ is no smaller than the number of factor edges with an end vertex in S and the other end vertex in an odd component of $G - S$.

But, we just noted that each odd component, C, of $G - S$ has exactly one factor edge with an end vertex in S and the other end vertex in C. So, we must have

$$q(G - S) \leq |S| .$$

Perfect Matchings

Proof:

(If)

Now, suppose that $q(G - S) \leq |S|$, for every $S \subseteq V$.

We must show that G has a 1-factor.

For that, we will use an argument given by the brilliant Hungarian mathematician László Lovász, in May 1973. This argument is quite elaborated, but beautiful!

The main ingredient of Lovász's proof is the notion of *saturated* graphs.

Perfect Matchings

First, note that *G must contain an even number of vertices.*

This is because $q(G - S) \leq |S|$, for every $S \subseteq V$. So, if we take $S = \emptyset$, we have that

$$q(G) = q(G - S) \leq |S| = 0\,.$$

So, G has no odd components. This means that every component of G has an even number of vertices, which in turn implies that G has an even number of vertices.

To prove that G does possess a 1-factor, we proceed by contradiction.

So, *assume* that G does not possess a 1-factor.

So, G has an even number of vertices, but does not have a 1-factor.

Perfect Matchings

Let $G^\star$ be a graph obtained from G by iteratively inserting edges in G (but no vertex!) while the resulting graph does not possess a 1-factor. So, graph $G^\star$ is such that $G^\star + xy$ has a 1-factor, for any two non-adjacent vertices $x, y \in V(G^\star)$.

Note that $V(G^\star) = V(G)$. So, $|V(G^\star)|$ is also even.

Note also that $G^\star$ is edge-maximal with respect to the property of not having a 1-factor.

But, graph $G^\star$ is not necessarily unique, and it is really immaterial which of the possible edge-maximal graphs w.r.t. the property of not having a 1-factor we get.

Perfect Matchings

Finally, it is possible that $G = G^\star$; that is, G might be edge-maximal w.r.t to the property of not having a 1-factor. Of course, we will show that this is impossible.

We claim that $q(G^\star - S) \leq q(G - S)$, for all $S \subseteq V(G)$.

To see why, let G' be a graph, S be any subset of $V(G')$, and x, y be any two non-adjacent vertices of G'. Then, consider the insertion of the edge xy into G'.

If xy connects two odd components of $G' - S$ then $q(G' - S)$ goes down by 2, as an even component is formed. If xy connects an odd component and an even component of $G' - S$ then $q(G' - S)$ remains unchanged. This is also true if xy connects an even or an odd component of $G' - S$ with S, or two vertices of S.

Perfect Matchings

By hypothesis, $q(G - S) \leq |S|$, for all $S \subseteq V(G)$. So, $q(G^\star - S) \leq |S|$, for all $S \subseteq V(G^\star)$.

Lovász showed that the condition $q(G^\star - S) \leq |S|$, for all $S \subseteq V(G^\star)$, implies that $G^\star$ has a 1-factor. But then graph G is the edge-maximal graph with no 1-factor.

Well, this is the same as saying that $G^\star = G$.

However, we know that $q(G - S) \leq |S|$, for all $S \subseteq V(G^\star)$, which means that G must have a 1-factor for the same exact reason $G^\star$ does (using Lovász's proof).

So, everything goes down to prove that $G^\star$ has a 1-factor.

Perfect Matchings

Let S denote the subset of $V(G^\star)$ such that each vertex of S is joined to *every* other vertex of $G^\star$. In particular, we might have $S = \emptyset$. Singling out this particular subset for special consideration enables us to conclude that $G^\star$ has a 1-factor.

Lovázs noticed (and then showed it) that every component of $G^\star - S$ is complete.

But, what does it have to do with the fact that $G^\star$ has a 1-factor?

The fact that every component of $G^\star - S$ is complete implies that the vertices of each *even* component can be matched up completely with edges from the component itself. The same is true for all but one vertex of each *odd* component.

Perfect Matchings

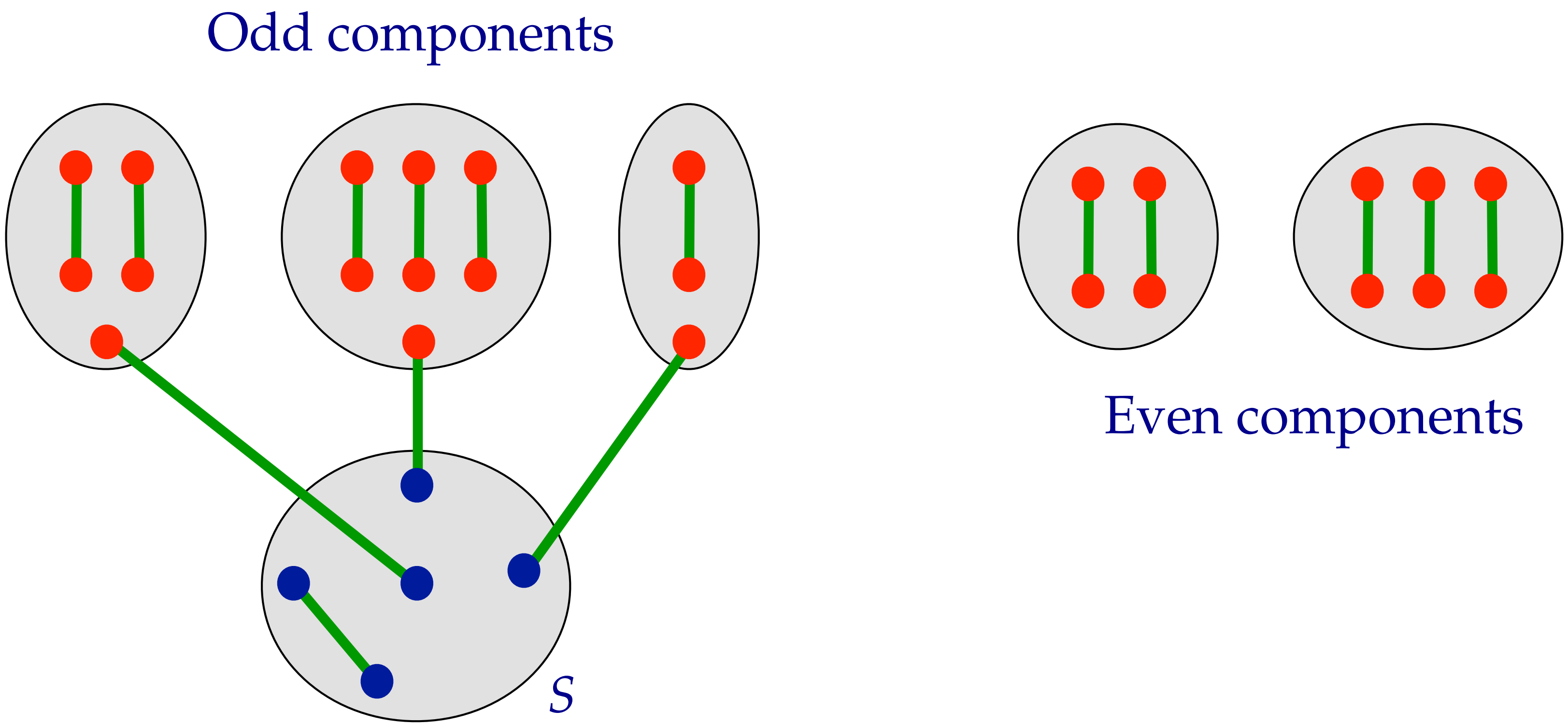
But the unmatched vertex of an odd component can be matched up with a vertex of S, as every vertex of S is connected to every other vertex of $G^\star$, by hypothesis.

Since $q(G^\star - S) \leq |S|$, we can actually pair up each unmatched vertex (from each odd component) with a distinct vertex of S. So, we are left with the unmatched vertices of S, as all vertices of $G^\star - S$ and their mates in S are now paired up.

Now, we use the facts that $V(G^\star) = V(G)$ is even and each vertex in S is connected to every vertex in $G^\star$ to conclude that there is an even number of unpaired vertices in S, and that they can be paired up with independent edges from S only.

So, we have that $G^\star$ possesses a perfect matching, and thus a 1-factor.

Perfect Matchings



Perfect Matchings

So, we are left with the task of proving that each component of $G^\star - S$ is complete.

Indeed, we must show that for every component C of the graph $G^\star - S$ and for every two vertices, x and y, in C, there exists an edge, xy, connecting x to y.

Again, we proceed by contradiction.

Let C be a component of $G^\star - S$, and assume that C is *not* complete.

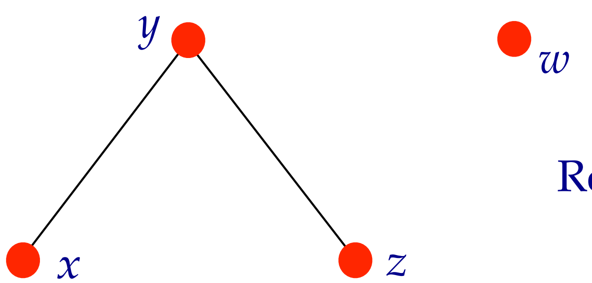
We claim that C has at least three vertices. Indeed, if C had less than three vertices, then C would have only one or only two vertices (a component cannot be empty), which would imply that C is complete, as C must be a connected graph.

Perfect Matchings

Since C is not complete and C has at least three vertices, there are two vertices, x and x', in C such that there is no edge xx' in $E(G^\star)$. But, because C is connected, there must be a path between x and x' in C. So, we can pick the first three vertices, say x, y, and z, in a *shortest* path from x to x' in C (of course, $x' \neq y$).

Since we picked the shortest path, we have that $xy, yz \in E(G^\star)$ and $xz \notin E(G^\star)$.

Since $y \notin S$, there must be a vertex $w \in (G^\star - S)$ such that $yw \notin E(G^\star)$.



Recall that each vertex in S is connected to every vertex in $V(G^\star)$.

Perfect Matchings

Since $G^\star$ is edge-maximal and contains no 1-factor, $G^\star + e$ has a 1-factor, for all $e \notin G^\star$.

This is the first time we use the edge-maximality of $G^\star$ in this proof!

So, consider the perfect matchings M_1 and M_2 in $G^\star + xz$ and $G^\star + yw$, respectively.

Denote by H the subgraph of $G^\star \cup \{xz, yw\}$ induced by $M_1 \ominus M_2$.

What can we say about H?

Do you remember Lemma 1?

Perfect Matchings

Lemma 1

Let M_1 and M_2 be two matchings in a graph G, and let H be the subgraph of G induced by the set of edges

$$M_1 \ominus M_2 = (M_1 - M_2) \cup (M_2 - M_1)\,,$$

Then each connected component of H is of one of the following two types:

(1) a cycle of even length whose edges are alternately in M_1 and M_2,

(2) a path whose edges are alternately in M_1 and M_2 and whose end vertices are unsaturated in one of the two matchings.

Perfect Matchings

Since M_1 and M_2 are perfect matchings, no component of H can be a path whose end points are unsaturated vertices in one of the matchings. So, they are all cycles.

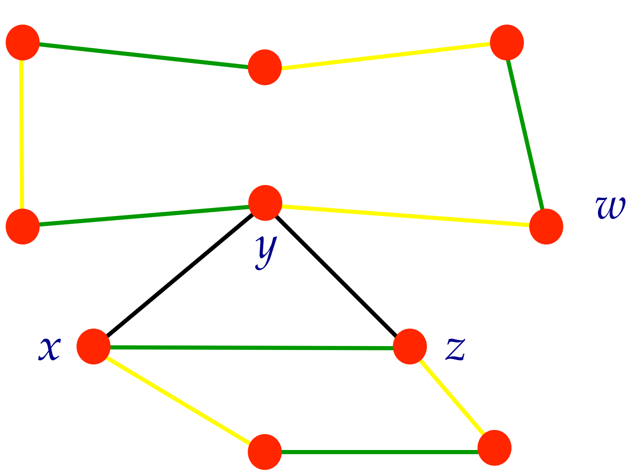
Since each cycle (component of H) alternates edges from M_1 and M_2, the length of such a cycle is always even, and hence H is a disjoint union of even cycles.

Perfect Matchings

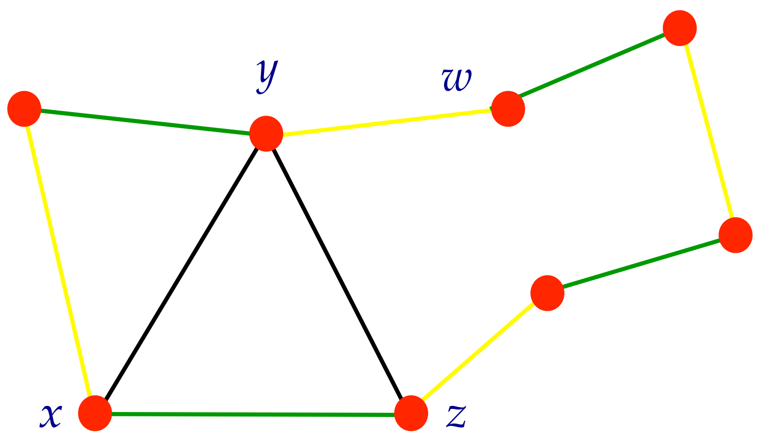
We distinguish two cases:

1) xz and yw are in distinct cycles of H.

2) xz and yw are in the same cycle of H.



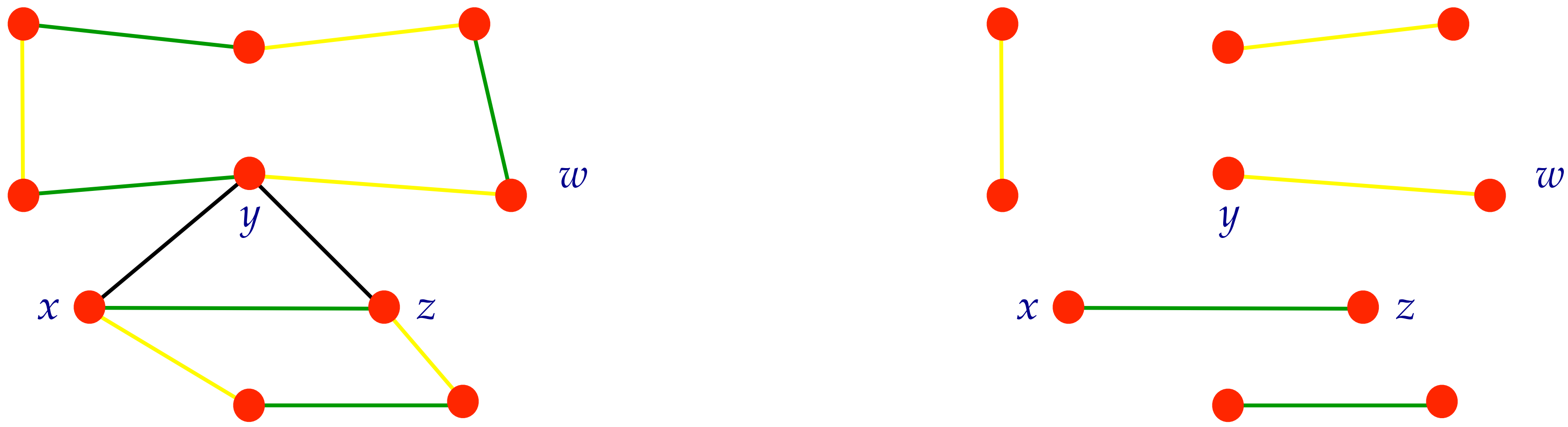
Case 1



Case 2

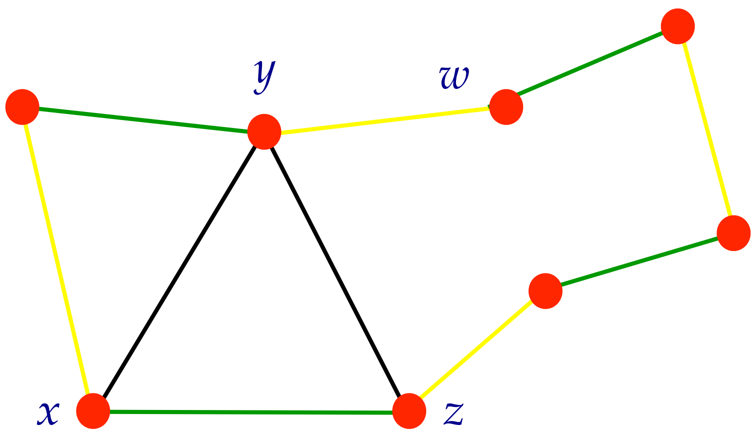
Perfect Matchings

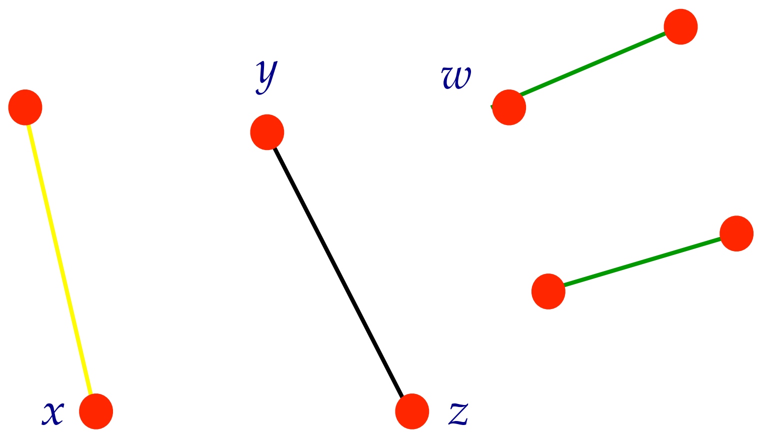
Consider case 1. If w is in the cycle B of H, the edges of M_1 in B, together with the edges of M_2 *not* in B, constitute a *perfect* matching of $G^\star$. So, $G^\star$ has a 1-factor.



Perfect Matchings

Consider case 2. By symmetry of x and z, we may assume that the vertices x, y, w, and z occur in that order on C. Then the edges of M_1 in the section $yw \ldots z$ of C, together with the edge yz and the edges of M_2 not in the section $yw \ldots z$ of C, constitute a *perfect* matching in $G^\star$. So, we can conclude that $G^\star$ has a 1-factor.





Perfect Matchings

In either case, we get that $G^\star$ has a 1-factor, which contradicts our assumption. So, each component of $G^\star - S$ is complete, which ends Lovász's proof of Tutte's theorem.

Concluding Remarks

We showed a fundamental result from Graph Theory, which seems to be closely related to the problem of converting a triangle surface mesh into a quadrilateral surface mesh.

Our next lecture will give an overview of some algorithms for maximum matchings in general graphs. After that, we will turn our attention to the problem of finding maximum weighted matching in general graphs and related algorithms.

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Questions?